Auxiliary relations and sandwich theorems

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Below let $\mathbb{I} = [0,1]$. A well-known topological theorem due to Katětov states:

Suppose (X, τ) *is a normal topological space, and let* $f : X \to \mathbb{I}$ *be upper semicontinuous,* $g : X \to \mathbb{I}$ *be lower semicontinuous, and* $f \leq g$*. Then there is a continuous* $h : X \to \mathbb{I}$ *such that* $f \leq h \leq g$*.*

Recall that $f : X \to \mathbb{I}$ is upper semicontinuous if f is continuous from (X, τ) to $(\mathbb{I}, \omega)$; lower semicontinuous if continuous from (X, τ) to $(\mathbb{I}, \sigma)$.

It is natural to try to extend this theorem to completely regular spaces, and for this audience, also to remove symmetry.

This is done by first following Urysohn in noting that a topological space is normal if and only if the relation on the poset $(2^X, \subseteq)$ defined by: $A \lhd B \Leftrightarrow$ for some closed C and open T, $A \subseteq C \subseteq T \subseteq B$, is interpolative (satisfies $\lhd_2$ below). This relation gives rise to the topology (in a sense to be made precise below) if the space is also T_1: for each x, $\mathsf{cl}\{x\} = \{x\}$.

Given a poset $(P, \leq)$, a transitive relation $\lhd$ on P is a *Urysohn relation* if it satisfies $\lhd_{1-2}$ below, and is an *auxiliary relation* if it satisfies $\lhd_{1-4}$ below:

- $\lhd_1$ If $a \lhd b$ then $a \leq b$.
- $\lhd_2$ $a \lhd b \Rightarrow$ for some c, $a \lhd c \lhd b$.
- $\lhd_3$ $c \leq a \lhd b \leq d \;\Rightarrow\; c \lhd d$.
- $\lhd_4$ $a, c \lhd b \Rightarrow$ for some d, $a, c \leq d \lhd b$;

The auxiliary relation $\lhd$ is *dualizable* if $\lhd^*$ is an auxiliary relation on $(L, \geq)$. This happens iff $\lhd$ also satisfies:

- $\lhd_{4d}$ $b \lhd a, c \Rightarrow$ for some d, $b \lhd d \leq a, c$;

Using, for $A \subseteq P$, $p \in P$, the notation:

$\Uparrow p = \{q \mid p \lhd q\}$, $\Uparrow A = \bigcup_{p \in A} \Uparrow p$,

and similar conventions with $\Downarrow$ (for $\rhd = \lhd^{-1}$), $\uparrow$, (for $\leq$) and $\downarrow$ (for $\geq$), $\lhd_4$ says that each $\Downarrow p$ is directed by $\lhd$; $\lhd_{4d}$ requires that each $\Uparrow p$ is directed by $\rhd$.

An auxiliary relation $\lhd$, is *approximating* if each $p = \bigvee \Downarrow p$, *dually approximating* if each $p = \bigwedge \Uparrow p$.

A poset $(P, \leq, \lhd)$ with auxiliary relation *has enough bounds* if $(P, \leq)$ has suprema for pairs that are bounded above in $(P, \leq)$ and infima for pairs that are bounded below in $(P, \leq)$, and has suprema for the countable nonempty sets that are bounded above in $(P, \lhd)$.

We have the following key but somewhat technical lemma:

1 Lemma. *Let* $(P, \leq, \lhd)$ *be a poset with approximating auxiliary relation that has enough bounds, and let* M *be a countable dense subset of* $\mathbb{I}$*. Suppose* $F, G : M \to P$ *are order preserving and whenever* $p < q$ *then* $F(p) \lhd G(q)$*. There is then an* $H : \mathbb{I} \to P$ *which is order preserving, is such that whenever* $p < u < v < q$, $p, q \in M$, $u, v \in \mathbb{I}$ *then* $F(p) \lhd H(u)$, $H(u) \lhd H(v)$, $H(v) \lhd G(q)$*, and for each* $D \subseteq \mathbb{I}$, $H(\bigwedge D) = \bigwedge H[D]$*.*

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Proof: We first define by recursion $H_0 : M \to P$ such that if $p < q$ then $F(p) \lhd H_0(q)$, $H_0(p) \lhd H_0(q)$ and $H_0(p) \lhd G(q)$. Next we define H on $\mathbb{I}$ by $H(u) = \bigwedge\{H_0(q) \mid u < q \in M\}$. Then for $p, q \in M$, $v \in \mathbb{I}$, if $p < v < q$ then $F(q) \lhd H(v) \lhd G(q)$. Finally, if $D \subseteq \mathbb{I}$ then $H(\bigwedge D) = \bigwedge H[D]$. $\square$

Let P, Q be posets, $l : P \to Q$, $u : Q \to P$ be order preserving maps. Then u is an *upper adjoint* for l if for each $p \in P$, $q \in Q$, $q \leq u(p) \Leftrightarrow l(q) \leq p$. In this case, l is a *lower adjoint* for u.

If $f : X \to Y$ is any function then the inverse image map $f^{\leftarrow}$ of f is an upper adjoint to its image function $f^{\rightarrow}$ between the posets $(2^X, \subseteq)$ and $(2^Y, \subseteq)$ (that is, if $A \subseteq X$, $B \subseteq Y$, then $A \subseteq f^{\leftarrow}(B) \Leftrightarrow f^{\rightarrow}(A) \subseteq B$, where $f^{\leftarrow}(B) = \{x \mid f(x) \in B\}$ and $f^{\rightarrow}(A) = \{f(x) \mid x \in A\}$). We are interested in a slight generalization of this: if P, Q are posets, then $f^{\leftarrow}$ is an upper adjoint to $\downarrow f^{\rightarrow} : (D(P), \subseteq) \to (D(Q), \subseteq)$; here $D(X)$ is the set of lower subsets of P (those for which $\downarrow L = L$), and $(\downarrow\ f^{\rightarrow})(L) = \downarrow (f(L))$.

The following useful observations on adjunctions are gathered in [G&], section 0.3: A function from one poset to another has at most one upper adjoint. Each function with an upper adjoint preserves $\bigvee$; as a partial converse, if the domain is a complete lattice, then each function that preserves $\bigvee$ has an upper adjoint. Results on adjoints are easily dualizable, since clearly if u is an upper adjoint for l regarded as a map from $(P, \leq_P)$ to $(Q, \leq_Q)$ then l is one for u, seen as a map from $(Q, \leq_Q^{-1})$ to $(P, \leq_P^{-1})$.

Let $(P, \leq_P, \lhd_P)$, $(Q, \leq_Q, \lhd_Q)$ be posets with approximating auxiliary relations. Let Q^P denote the set of maps $f : P \to Q$, with upper adjoints (denoted, $f^{\uparrow} : Q \to P$), and on it let $\leq_{Q^P}$ be the pointwise order and $\lhd_{Q^P}$ be the relation defined by $f \lhd_{Q^P} g$ if whenever $q \lhd_Q r$ then $f^{\uparrow}(q) \lhd_P g^{\uparrow}(r)$.

2 Theorem. *Let $(P, \leq, \lhd)$ be a poset with approximating auxiliary relation that has enough bounds. Suppose $f, g \in \mathbb{I}^P$ and $f \lhd_{\mathbb{I}^P} g$. Then for some $h : P \to \mathbb{I}$, $f \lhd_{\mathbb{I}^P} h \lhd_{\mathbb{I}^P} h \lhd_{\mathbb{I}^P} g$. In particular, $\lhd_{\mathbb{I}^P}$ is a Urysohn relation on $\mathbb{I}^P$.*

Proof: Given such $f, g : P \to \mathbb{I}$, with an eye on Lemma 1, let $M = \mathbb{Q} \cap \mathbb{I}$ and define $F, G : (M, \leq) \to (P, \leq_{\lhd})$ by $F = f^{\uparrow}|M$, $G = g^{\uparrow}|M$. Then M, F, G satisfy its hypotheses, so there is an order preserving $H : (\mathbb{I}, \leq) \to (P, \leq_{\lhd})$ such that whenever $p < u < v < q$, $p, q \in M$, $u, v \in \mathbb{I}$ then $F(p) \lhd H(u) \lhd H(v) \lhd G(q)$, and for each $D \subseteq \mathbb{I}$, $H(\bigwedge D) = \bigwedge H[D]$. By the dual of the comments on adjoints, H thus has a lower adjoint, $h : (P, \leq_{\lhd}) \to (\mathbb{I}, \leq)$, so H is the upper adjoint $H = h^{\uparrow}$, to h. Thus if $u < v$ then $h^{\uparrow}(u) = H(u) \lhd_P H(v) = h^{\uparrow}(v)$, so $h \lhd_{\mathbb{I}^P} h$.

Since $f^{\uparrow}, g^{\uparrow} : \mathbb{I} \to P$, have lower adjoints they are order preserving, so if $u < v$, $u, v \in \mathbb{I}$ then for some $p \in M$, $u < p < v$, thus $f^{\uparrow}(u) \leq f^{\uparrow}(p) = F(p) \lhd H(v) = h^{\uparrow}(v)$ so $f \lhd_{\mathbb{I}^P} h$ and $h^{\uparrow}(u) = H(u) \lhd G(p) = g^{\uparrow}(p) \leq g^{\uparrow}(v)$, so $h \lhd_{\mathbb{I}^P} g$. $\square$

3 Lemma. *(a) Given a Urysohn relation $\lhd$ on $(2^X, \subseteq)$, define $\tau_{\lhd}$ to be the collection of subsets of X such that $T \in \tau_{\lhd}$ if for each $x \in T$, there is some finite set F of subsets of X such that $\{x\} \lhd B$ for each $B \in F$, and $\bigcap F \subseteq T$. Then $\tau_{\lhd}$ is a topology on X.*

(b) If $f : (X, \lhd_X) \to (Y, \lhd_Y)$ and $f^{\rightarrow} \lhd_{Q^P} f^{\rightarrow}$, then $f : (X, \tau_{\lhd_X}) \to (Y, \tau_{\lhd_Y})$ is continuous.

Proof: We bother only with (b). If $f^{\rightarrow} \lhd_{Q^P} f^{\rightarrow}$ and $x \in f^{\leftarrow}[T]$, $T \in \tau_{\lhd_Y}$ then for some finite set F of subsets of Y, $\{f(x)\} \lhd_Y B$ for each $B \in F$ and $\bigcap F \subseteq T$, so by the definition of $\lhd_{Q^P}$, $\{x\} \subseteq f^{\leftarrow}[\{f(x)\}] \lhd_Y f^{\leftarrow}[B]$ for each $B \in F$, and $\bigcap_{B \in F} f^{\leftarrow}[B] = f^{\leftarrow}[\bigcap F] \subseteq f^{\leftarrow}[T]$; by the arbitrary nature of $x \in f^{\leftarrow}[T]$, this shows that $f^{\leftarrow}[T] \in \tau_{\lhd_X}$, therefore f is continuous. □

4 Definition. Given a Urysohn relation $\lhd$ on $(2^X, \subseteq)$, its *Urysohn dual* of $\lhd$ is denoted by $\lhd^u$ and defined by $A \lhd^u B \Leftrightarrow X \setminus B \lhd X \setminus A$.

It is simple to see that $(\lhd^u)^u = \lhd$.

5 Theorem. *Let $\lhd_X$, $\lhd_Y$ be Urysohn relations on X, Y, respectively, and let $f, g : X \to Y$. If $f^{\rightarrow} \lhd_{Y^X} g^{\rightarrow}$ then $g^{\rightarrow} \lhd^u_{Y^X} f^{\rightarrow}$. In particular, if $f^{\rightarrow} \lhd_{Y^X} f^{\rightarrow}$ then $f : (X, \tau_{\lhd_X}, \tau_{\lhd^u_X}) \to (Y, \tau_{\lhd_Y}, \tau_{\lhd^u_Y})$ is pairwise continuous.*

Proof: Let $f^{\rightarrow} \lhd g^{\rightarrow}$ and assume $A \lhd^u_Y B$. Then $Y \setminus B \lhd_Y Y \setminus A$, so $X \setminus f^{\leftarrow}[B] = f^{\leftarrow}[Y \setminus B] \lhd_X g^{\leftarrow}[Y \setminus A] = X \setminus g^{\leftarrow}[A]$, so $g^{\leftarrow}[A] \lhd^u_{Y^X} f^{\leftarrow}[B]$. In particular, if $f^{\rightarrow} \lhd f^{\rightarrow}$ then $f^{\rightarrow} \lhd^u f^{\rightarrow}$, so by Lemma 3(b), f is continuous both from $(X, \tau_{\lhd_X})$ to $(Y, \tau_{\lhd_Y})$ and from $(X, \tau_{\lhd^u_X})$ to $(Y, \tau_{\lhd^u_Y})$, showing the theorem. □

A bitopological space (X, τ, τ^*) is *completely regular* if whenever $x \in U \in \tau$, then there is a pairwise continuous f from (X, τ, τ^*) to $(\mathbb{I}, \sigma, \omega)$ such that $f(x) = 1$ and $f(y) = 0$ whenever $y \notin T$. For any property Q of bitopological spaces, (X, τ, τ^*) is *pairwise* Q if (X, τ, τ^*) and its *bitopological dual,* (X, τ^*, τ) are both Q. In [Ko], it is shown that (X, τ, τ^*) is pairwise completely regular if and only if there is a Urysohn relation $\lhd$ on 2^X such that $\tau = \tau_{\lhd}$ and $\tau^* = \tau_{\lhd^u}$; further, the Urysohn relation can be taken to be a dualizable auxiliary relation which is approximating and dually approximating. The above easily yields the fact that a topological space (X, τ) is completely regular if and only if there is a Urysohn relation $\lhd$ on 2^X such that $\tau = \tau_{\lhd} = \tau_{\lhd^u}$, since (X, τ) is completely regular if and only if (X, τ, τ) is pairwise completely regular. Also, f is pairwise continuous from (X, τ, τ) to $(\mathbb{I}, \sigma, \omega)$ if and only if, f is pairwise continuous from (X, τ) to $(\mathbb{I}, us)$, where us is the usual topology on the unit interval. (This Urysohn relation can be extended to a proximity, giving the classical characterization of complete regularity.)

For any poset $(P, \leq)$ with approximating auxiliary relation $\ll$, the pseudoScott topology ρ, is that generated by the set of $\Uparrow p$, $p \in P$. Also define $\lhd_{\ll}$ on 2^P by $A \lhd_{\ll} B \Leftrightarrow (\exists r, s \in P)(r \ll s \& A \subseteq \uparrow s \& \Uparrow r \subseteq B)$. Then $\lhd_{\ll}$ is a Urysohn relation, $\tau_{\lhd_{\ll}} = \rho$ and $\tau_{\lhd^u_{\ll}} = \omega$. Thus the bitopological space (P, ρ, ω) is pairwise completely regular (much of this is in [FK]). In particular, if $(P, \leq)$ is a continuous poset and $\ll$ is its way-below relation, then $\rho = \sigma$, its Scott topology. Now we apply this to $\mathbb{I}$, $\ll$ ($\ll$ is $<$ except that $0 \ll 0$), and by Theorems 2 and 5 we have:

6 Corollary. *Let P be a continuous poset and $f, g : P \to \mathbb{I}$ be such that whenever $r, s \in \mathbb{I}$, $r < s$ there are $p, q \in P$ such that $q \ll p$, $\uparrow p \subseteq f^{\leftarrow}[\uparrow s]$ and $\uparrow r \subseteq f^{\leftarrow}[\uparrow q]$. Then there is an $h : P \to \mathbb{I}$ such that $f \leq h \leq g$ and h is pairwise continuous from (P, σ, ω) to $(\mathbb{I}, \sigma, \omega)$. Thus (P, σ, ω) is pairwise completely regular. Also, if $f : P \to \mathbb{I}$ is is the directed sup of $\Downarrow_{\mathbb{I}^P} f$ then it is the directed sup of $\{h \in \Downarrow_{\mathbb{I}^P} f \mid h$ is pairwise continuous from (P, σ, ω) to $(\mathbb{I}, \sigma, \omega)\}$.*

The same holds for posets with approximating auxiliary relation and (P,ρ,ω).

A bitopological space (X,τ,τ^*) is *normal* if whenever $C\subseteq U$, C is τ^*-closed and U τ-open, then there is a τ^*-closed D and a τ-open V such that $C\subseteq V\subseteq D\subseteq U$. It's easy to check that a bitopological space is normal if and only if the binary relation $\lhd_{\mathcal{N}}$ on 2^X defined by $A\lhd_{\mathcal{N}} B \Leftrightarrow \mathsf{cl}_{\tau^*}A\subseteq \mathsf{int}_\tau B$, is a Urysohn relation; in this case it is in fact a dualizable auxiliary relation, and if $\leq_\tau^{-1}=\leq_{\tau^*}$ then $\tau=\tau_{\lhd_{\mathcal{N}}}$ and $\tau^*=\tau_{\lhd^u_{\mathcal{N}}}$. It is *joincompact* if it is pairwise completely regular and $\tau\vee\tau^*$ is compact and T_0. Each joincompact bitopological space is normal, and the specializations of the topologies are inverse to each other.

7 Corollary. *(a) Let* (X,τ,τ^*) *be a normal bitopological space,* f *be continuous from* (X,τ) *to* $(\mathbb{I},\omega)$, g *be continuous from* (X,τ^*) *to* $(\mathbb{I},\sigma)$, *and* $f\leq g$. *Then for some pairwise continuous* h *from* (X,τ,τ^*) *to* $(\mathbb{I},\sigma,\omega)$, $f\leq h\leq g$.

(b) Let P *be a Scott domain and* $f,g:P\to\mathbb{I}$ *be such that* $f\leq g$, f *is continuous from* (P,ω) *to* $(\mathbb{I},\omega)$, *and* g *is continuous from* (P,σ) *to* $(\mathbb{I},\sigma)$. *Then there is an* $h:P\to\mathbb{I}$ *such that* $f\leq h\leq g$ *and* h *is pairwise continuous from* (P,σ,ω) *to* $(\mathbb{I},\sigma,\omega)$.

In particular, for each $f:P\to\mathbb{I}$, f *is Scott continuous if and only if,* f *is the directed sup of* $\{h\leq f \mid h$ *is pairwise continuous from* (P,σ,ω) *to* $(\mathbb{I},\sigma,\omega)\}$.

Proof: (a) Let $\lhd=\lhd_{\mathcal{N}}$; then $\tau=\tau_\lhd$ and $\tau^*=\tau_{\lhd^u}$. Notice that $f\lhd_{\mathbb{I}^X} g$, for if $A\lhd_\ll B$ there is $r\ll s$ so that $A\subseteq\uparrow s\&\Uparrow r\subseteq B$, so $\mathsf{cl}_\omega A\subseteq \mathsf{int}_\sigma B$ (if $r=0$ then $\mathbb{I}=\Uparrow r$). By continuity of f from (X,τ) to $(\mathbb{I},\omega)$ and g from (X,τ^*) to $(\mathbb{I},\sigma)$, we have that $\mathsf{cl}_{\tau^*}f^{\leftarrow}[A]\subseteq \mathsf{int}_\tau g^{\leftarrow}[B]$, so $f^{\leftarrow}[A]\lhd_{\mathcal{N}} g^{\leftarrow}[B]$. Thus $f^{\rightarrow}\lhd_{\mathbb{I}^X} g^{\rightarrow}$, so by Theorem 2, there is an $h:P\to\mathbb{I}$ such that $f\lhd_{\mathbb{I}^P} h\lhd_{\mathbb{I}^P} h\lhd_{\mathbb{I}^P} g$. By Theorem 5, h is pairwise continuous from (X,τ,τ^*) to $(\mathbb{I},\sigma,\omega)$.

(b) For a Scott domain P, the Lawson topology, $\sigma\vee\omega$ is compact (see [G&]) and and (P,σ,ω) is joincompact (see [Ko], 3.2), therefore normal ([Ko], 3.6), and the specializations of the topologies are inverse to each other. That Scott continuous functions are the directed sup of the pairwise continuous functions below them follows from (a) and Corollary 6. Conversely, any sup of functions continuous from any topological space to $(\mathbb{I},\sigma)$ is such a function. $\square$

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