A domain of spacetime intervals in general relativity

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Abstract

We prove that a globally hyperbolic spacetime with its causality relation is a bicontinuous poset whose interval topology is the manifold topology. This implies that from only a countable dense set of events and the causality relation, it is possible to reconstruct a globally hyperbolic spacetime in a purely order theoretic manner. The ultimate reason for this is that globally hyperbolic spacetimes belong to a category that is equivalent to a special category of domains called *interval domains.* We obtain a mathematical setting in which one can study causality independently of geometry and differentiable structure, and which also suggests that spacetime emanates from something discrete.

1 Introduction

It has been known for some time that the topology of spacetime could be characterized purely in terms of causality. But what is causality? To illustrate what we are asking, the physical idea 'rate of change' is formalized

Dagstuhl Seminar Proceedings 04351
Spatial Representation: Discrete vs. Continuous Computational Models
http://drops.dagstuhl.de/opus/volltexte/2005/135

mathematically by 'derivative', etc. What is the mathematical formalization of 'causality' – assuming it has one at all?

In this paper, we prove that the causality relation is much more than a relation – it turns a globally hyperbolic spacetime into what is known as a *bicontinuous poset.* The order on a bicontinuous poset allows one to define an intrinsic topology called *the interval topology.* On a globally hyperbolic spacetime, the interval topology is the manifold topology.

This directly implies that a globally hyperbolic spacetime can be reconstructed in a purely order theoretic manner, beginning from only a countable dense set of events and the causality relation. The ultimately reason for this is that the category of globally hyperbolic posets, which contains the globally hyperbolic spacetimes, is *equivalent* to a very special category of domains called *interval domains.*

Domains were discovered in computer science by Scott [8] for the purpose of providing a semantics for the lambda calculus. They are partially ordered sets which carry intrinsic (order theoretic) notions of completeness and approximation. From a certain viewpoint, then, the fact that the category of globally hyperbolic posets is equivalent to the category of interval domains is surprising, since globally hyperbolic spacetimes are usually not order theoretically complete. This equivalence also explains why spacetime can be reconstructed order theoretically from a countable dense set: each ω-continuous domain is the ideal completion of a countable abstract basis, i.e., the interval domains associated to globally hyperbolic spacetimes are the systematic 'limits' of discrete sets. This may be relevant to the development of a foundation for quantum gravity, an idea we discuss at the end.

But, with all speculation aside, the importance of these results and ideas is that they suggest an abstract formulation of causality – a setting where one can study causality independently of geometry and differentiable structure.

2 Domains, continuous posets and topology

A *poset* is a partially ordered set, i.e., a set together with a reflexive, antisymmetric and transitive relation.

Definition 2.1 Let $(P, \sqsubseteq)$ be a partially ordered set. A nonempty subset $S \subseteq P$ is *directed* if $(\forall x, y \in S)(\exists z \in S)\, x, y \sqsubseteq z$. The *supremum* of $S \subseteq P$ is the least of all its upper bounds provided it exists. This is written $\bigsqcup S$.

These ideas have duals that will be important to us: A nonempty $S \subseteq P$ is *filtered* if $(\forall x, y \in S)(\exists z \in S)\, z \sqsubseteq x, y$. The *infimum* $\bigwedge S$ of $S \subseteq P$ is the greatest of all its lower bounds provided it exists.

Definition 2.2 For a subset X of a poset P, set

$$\uparrow\!X := \{y \in P : (\exists x \in X)\, x \sqsubseteq y\} \;\;\&\;\; \downarrow\!X := \{y \in P : (\exists x \in X)\, y \sqsubseteq x\}.$$

We write $\uparrow x = \uparrow\{x\}$ and $\downarrow x = \downarrow\{x\}$ for elements $x \in X$.

A partial order allows for the derivation of several intrinsically defined topologies. Here is our first example.

Definition 2.3 A subset U of a poset P is *Scott open* if

(i) U is an upper set: $x \in U \;\&\; x \sqsubseteq y \Rightarrow y \in U$, and

(ii) U is inaccessible by directed suprema: For every directed $S \subseteq P$ with a supremum,
$$\bigsqcup S \in U \Rightarrow S \cap U \neq \emptyset.$$

The collection of all Scott open sets on P is called the *Scott topology.*

Definition 2.4 A *dcpo* is a poset in which every directed subset has a supremum. The *least element* in a poset, when it exists, is the unique element $\perp$ with $\perp \sqsubseteq x$ for all x.

The set of *maximal elements* in a dcpo D is

$$\max(D) := \{x \in D : \uparrow\!x = \{x\}\}.$$

Each element in a dcpo has a maximal element above it.

Definition 2.5 For elements x, y of a poset, write $x \ll y$ iff for all directed sets S with a supremum,

$$y \sqsubseteq \bigsqcup S \Rightarrow (\exists s \in S)\, x \sqsubseteq s.$$

We set $\mathord{\downarrow\!\!\downarrow} x = \{a \in D : a \ll x\}$ and $\mathord{\uparrow\!\!\uparrow} x = \{a \in D : x \ll a\}$.

For the symbol "$\ll$," read "approximates."

Definition 2.6 A *basis* for a poset D is a subset B such that $B \cap \mathord{\downarrow\!\!\downarrow} x$ contains a directed set with supremum x for all $x \in D$. A poset is *continuous* if it has a basis. A poset is *ω-continuous* if it has a countable basis.

Continuous posets have an important property, they are *interpolative.*

Proposition 2.7 *If $x \ll y$ in a continuous poset P, then there is $z \in P$ with $x \ll z \ll y$.*

This enables a clear description of the Scott topology,

Theorem 2.8 *The collection $\{\twoheaduparrow x : x \in D\}$ is a basis for the Scott topology on a continuous poset.*

And also helps us give a clear definition of the *Lawson topology.*

Definition 2.9 The *Lawson topology* on a continuous poset P has as a basis all sets of the form $\twoheaduparrow x \backslash \uparrow F$, for $F \subseteq P$ finite.

The next idea is fundamental to the present work:

Definition 2.10 A continuous poset P is *bicontinuous* if

- For all $x, y \in P$, $x \ll y$ iff for all filtered $S \subseteq P$ with an infimum,

$$\bigwedge S \sqsubseteq x \Rightarrow (\exists s \in S)\, s \sqsubseteq y,$$

  and

- For each $x \in P$, the set $\twoheaduparrow x$ is filtered with infimum x.

Example 2.11 $\mathbb{R}$, $\mathbb{Q}$ are bicontinuous.

Definition 2.12 On a bicontinuous poset P, sets of the form

$$(a, b) := \{x \in P : a \ll x \ll b\}$$

form a basis for a topology called *the interval topology.*

The proof uses interpolation and bicontinuity. A bicontinuous poset P has $\twoheaduparrow x \neq \emptyset$ for each x, so it is rarely a dcpo. Later we will see that on a bicontinuous poset, the Lawson topology is contained in the interval topology (causal simplicity), the interval topology is Hausdorff (strong causality), and $\leq$ is a closed subset of P^2.

Definition 2.13 A *continuous dcpo* is a continuous poset which is also a dcpo. A *domain* is a continuous dcpo.

Example 2.14 Let X be a locally compact Hausdorff space. Its *upper space*

$$\mathbf{U}X = \{\emptyset \neq K \subseteq X : K \text{ is compact}\}$$

ordered under reverse inclusion

$$A \sqsubseteq B \Leftrightarrow B \subseteq A$$

is a continuous dcpo:

- For directed $S \subseteq \mathbf{U}X$, $\bigsqcup S = \bigcap S$.
- For all $K, L \in \mathbf{U}X$, $K \ll L \Leftrightarrow L \subseteq \text{int}(K)$.
- $\mathbf{U}X$ is ω-continuous iff X has a countable basis.

It is interesting here that the space X can be recovered from $\mathbf{U}X$ in a purely order theoretic manner:

$$X \simeq \max(\mathbf{U}X) = \{\{x\} : x \in X\}$$

where $\max(\mathbf{U}X)$ carries the relative Scott topology it inherits as a subset of $\mathbf{U}X$. Several constructions of this type are known.

The next example is due to Scott[8]; it will be good to keep in mind when studying the analogous construction for globally hyperbolic spacetimes.

Example 2.15 The collection of compact intervals of the real line

$$\mathbf{I}\mathbb{R} = \{[a, b] : a, b \in \mathbb{R} \ \&\ a \leq b\}$$

ordered under reverse inclusion

$$[a, b] \sqsubseteq [c, d] \Leftrightarrow [c, d] \subseteq [a, b]$$

is an ω-continuous dcpo:

- For directed $S \subseteq \mathbf{I}\mathbb{R}$, $\bigsqcup S = \bigcap S$,
- $I \ll J \Leftrightarrow J \subseteq \text{int}(I)$, and
- $\{[p, q] : p, q \in \mathbb{Q} \ \&\ p \leq q\}$ is a countable basis for $\mathbf{I}\mathbb{R}$.

The domain $\mathbf{I}\mathbb{R}$ is called the *interval domain.*

We also have $\max(\mathbf{I}\mathbb{R}) \simeq \mathbb{R}$ in the Scott topology. Approximation can help explain why:

Example 2.16 A basic Scott open set in $\mathbf{I}\mathbb{R}$ is

$$\Uparrow[a, b] = \{x \in \mathbf{I}\mathbb{R} : x \subseteq (a, b)\}.$$

We have not considered algebraic domains here, though should point out to the reader that algebraic models of globally hyperbolic spacetime are easy to construct.

3 The causal structure of spacetime

A *manifold* $\mathcal{M}$ is a locally Euclidean Hausdorff space that is connected and has a countable basis. A connected Hausdorff manifold is paracompact iff it has a countable basis. A *Lorentz metric* on a manifold is a symmetric, nondegenerate tensor field of type $(0, 2)$ whose signature is $(- + ++)$.

Definition 3.1 A *spacetime* is a real four-dimensional smooth manifold $\mathcal{M}$ with a Lorentz metric g_{ab}.

Let $(\mathcal{M}, g_{ab})$ be a time orientable spacetime. Let $\Pi^+_\leq$ denote the future directed causal curves, and $\Pi^+_<$ denote the future directed time-like curves.

Definition 3.2 For $p \in \mathcal{M}$,

$$I^+(p) := \{q \in \mathcal{M} : (\exists \pi \in \Pi^+_<)\, \pi(0) = p, \pi(1) = q\}$$

and

$$J^+(p) := \{q \in \mathcal{M} : (\exists \pi \in \Pi^+_\leq)\, \pi(0) = p, \pi(1) = q\}$$

Similarly, we define $I^-(p)$ and $J^-(p)$.

We write the relation J^+ as

$$p \sqsubseteq q \equiv q \in J^+(p).$$

The following properties from [4] are very useful:

Proposition 3.3 *Let $p, q, r \in \mathcal{M}$. Then*

(i) *The sets $I^+(p)$ and $I^-(p)$ are open.*

(ii) $p \sqsubseteq q$ *and* $r \in I^+(q) \Rightarrow r \in I^+(p)$

(iii) $q \in I^+(p)$ *and* $q \sqsubseteq r \Rightarrow r \in I^+(p)$

(iv) $\mathrm{Cl}(I^+(p)) = \mathrm{Cl}(J^+(p))$ *and* $\mathrm{Cl}(I^-(p)) = \mathrm{Cl}(J^-(p))$.

We always assume the chronology conditions that ensure $(\mathcal{M}, \sqsubseteq)$ is a partially ordered set. We also assume *strong causality* which can be characterized as follows [7]:

Theorem 3.4 *A spacetime $\mathcal{M}$ is strongly causal iff its Alexandroff topology is Hausdorff iff its Alexandroff topology is the manifold topology.*

The Alexandroff topology on a spacetime has $\{I^+(p) \cap I^-(q) : p, q \in \mathcal{M}\}$ as a basis [7].

4 Global hyperbolicity

Penrose has called *globally hyperbolic* spacetimes "the physically reasonable spacetimes [9]." In this section, $\mathcal{M}$ denotes a globally hyperbolic spacetime, and we prove that $(\mathcal{M}, \sqsubseteq)$ is a bicontinuous poset.

Definition 4.1 A spacetime $\mathcal{M}$ is *globally hyperbolic* if it is strongly causal and if $\uparrow a \cap \downarrow b$ is compact in the manifold topology, for all $a, b \in \mathcal{M}$.

Lemma 4.2 *If (x_n) is a sequence in $\mathcal{M}$ with $x_n \sqsubseteq x$ for all n, then*

$$\lim_{n\to\infty} x_n = x \;\Rightarrow\; \bigsqcup_{n\geq 1} x_n = x.$$

Proof. Let $x_n \sqsubseteq y$. By global hyperbolicity, $\mathcal{M}$ is causally simple, so the set $J^-(y)$ is closed. Since $x_n \in J^-(y)$, $x = \lim x_n \in J^-(y)$, and thus $x \sqsubseteq y$. This proves $x = \bigsqcup x_n$. $\square$

Lemma 4.3 *For any $x \in \mathcal{M}$, $I^-(x)$ contains an increasing sequence with supremum x.*

Proof. Because $x \in \mathrm{Cl}(I^-(x)) = J^-(x)$ but $x \notin I^-(x)$, x is an *accumulation point* of $I^-(x)$, so for every open set V with $x \in V$, $V \cap I^-(x) \neq \emptyset$. Let (U_n) be a countable basis for x, which exists because $\mathcal{M}$ is locally Euclidean. Define a sequence (x_n) by first choosing

$$x_1 \in U_1 \cap I^-(x) \neq \emptyset$$

and then whenever

$$x_n \in U_n \cap I^-(x)$$

we choose

$$x_{n+1} \in (U_n \cap I^+(x_n)) \cap I^-(x) \neq \emptyset.$$

By definition, (x_n) is increasing, and since (U_n) is a basis for x, $\lim x_n = x$. By Lemma 4.2, $\bigsqcup x_n = x$. $\square$

Proposition 4.4 *Let $\mathcal{M}$ be a globally hyperbolic spacetime. Then*

$$x \ll y \Leftrightarrow y \in I^+(x)$$

for all $x, y \in \mathcal{M}$.

Proof. Let $y \in I^+(x)$. Let $y \sqsubseteq \bigsqcup S$ with S directed. By Prop. 3.3(iii),

$$y \in I^+(x) \;\&\; y \sqsubseteq \bigsqcup S \;\Rightarrow\; \bigsqcup S \in I^+(x)$$

Since $I^+(x)$ is manifold open and $\mathcal{M}$ is locally compact, there is an open set $V \subseteq \mathcal{M}$ whose closure $\mathrm{Cl}(V)$ is compact with $\bigsqcup S \in V \subseteq \mathrm{Cl}(V) \subseteq I^+(x)$. Then, using approximation on the upper space of $\mathcal{M}$,

$$\mathrm{Cl}(V) \ll \left\{\bigsqcup S\right\} = \bigcap_{s \in S} [s, \bigsqcup S]$$

where the intersection on the right is a filtered collection of nonempty compact sets by directedness of S and global hyperbolicity of $\mathcal{M}$. Thus, for some $s \in S$, $[s, \bigsqcup S] \subseteq \mathrm{Cl}(V) \subseteq I^+(x)$, and so $s \in I^+(x)$, which gives $x \sqsubseteq s$. This proves $x \ll y$.

Now let $x \ll y$. By Lemma 4.3, there is an increasing sequence (y_n) in $I^-(y)$ with $y = \bigsqcup y_n$. Then since $x \ll y$, there is n with $x \sqsubseteq y_n$. By Prop. 3.3(ii),

$$x \sqsubseteq y_n \;\&\; y_n \in I^-(y) \;\Rightarrow\; x \in I^-(y)$$

which is to say that $y \in I^+(x)$. $\square$

Theorem 4.5 *If $\mathcal{M}$ is globally hyperbolic, then $(\mathcal{M}, \sqsubseteq)$ is a bicontinuous poset with $\ll\, = I^+$ whose interval topology is the manifold topology.*

Proof. By combining Lemma 4.3 with Prop. 4.4, $\mathord{\downarrow\!\!\!\downarrow} x$ contains an increasing sequence with supremum x, for each $x \in \mathcal{M}$. Thus, $\mathcal{M}$ is a continuous poset.

For the bicontinuity, Lemmas 4.2, 4.3 and Prop. 4.4 have "duals" which are obtained by replacing 'increasing' by 'decreasing', I^+ by I^-, J^- by J^+, etc. For example, the dual of Lemma 4.3 is that I^+ contains a *decreasing* sequence with *infimum* x. Using the duals of these two lemmas, we then give an alternate characterization of $\ll$ in terms of infima:

$$x \ll y \equiv (\forall S) \bigwedge S \sqsubseteq x \;\Rightarrow\; (\exists s \in S)\, s \sqsubseteq y$$

where we quantify over *filtered* subsets S of $\mathcal{M}$. These three facts then imply that $\mathord{\uparrow\!\!\!\uparrow} x$ contains a decreasing sequence with inf x. But because $\ll$ can be phrased in terms of infima, $\mathord{\uparrow\!\!\!\uparrow} x$ itself must be filtered with inf x.

Finally, $\mathcal{M}$ is bicontinuous, so we know it has an interval topology. Because $\ll = I^+$, the interval topology is the one generated by the timelike causality relation, which by strong causality is the manifold topology. $\square$

Bicontinuity, as we have defined it here, is really quite a special property, and some of the nicest posets in the world are not bicontinuous. For example, the powerset of the naturals $\mathcal{P}\omega$ is not bicontinuous, because we can have $F \ll G$ for G finite, and $F = \bigcap V_n$ where all the V_n are infinite.

5 Causal simplicity

It is also worth pointing out that causal simplicity has a characterization in order theoretic terms.

Definition 5.1 A spacetime $\mathcal{M}$ is *causally simple* if $J^+(x)$ and $J^-(x)$ are closed for all $x \in \mathcal{M}$.

Theorem 5.2 *Let $\mathcal{M}$ be a spacetime and $(\mathcal{M}, \sqsubseteq)$ a continuous poset with $\ll = I^+$. The following are equivalent:*

(i) *$\mathcal{M}$ is causally simple.*

(ii) *The Lawson topology on $\mathcal{M}$ is a subset of the interval topology on $\mathcal{M}$.*

Proof (i) $\Rightarrow$ (ii): We want to prove that

$$\{\Uparrow\!x \cap \uparrow\!F : x \in \mathcal{M} \ \&\ F \subseteq \mathcal{M} \text{ finite}\} \subseteq \text{int}_{\mathcal{M}}.$$

By strong causality of $\mathcal{M}$ and $\ll = I^+$, $\text{int}_{\mathcal{M}}$ is the manifold topology, and this is the crucial fact we need as follows. First, $\Uparrow\!x = I^+(x)$ is open in the manifold topology and hence belongs to $\text{int}_{\mathcal{M}}$. Second, $\uparrow\!x = J^+(x)$ is closed in the manifold topology by causal simplicity, so $\mathcal{M}\backslash\!\uparrow\!x$ belongs to $\text{int}_{\mathcal{M}}$. Then $\text{int}_{\mathcal{M}}$ contains the basis for the Lawson topology given above.

(ii) $\Rightarrow$ (i): First, since $(\mathcal{M}, \sqsubseteq)$ is continuous, its Lawson topology is Hausdorff, so $\text{int}_{\mathcal{M}}$ is Hausdorff since it contains the Lawson topology by assumption. Since $\ll = I^+$, $\text{int}_{\mathcal{M}}$ is the Alexandroff topology, so Theorem 3.4 implies $\mathcal{M}$ is strongly causal.

Now, Theorem 3.4 also tells us that $\text{int}_{\mathcal{M}}$ is the manifold topology. Since the manifold topology $\text{int}_{\mathcal{M}}$ contains the Lawson by assumption, and since

$$J^+(x) = \uparrow\!x \quad\text{and}\quad J^-(x) = \downarrow\!x$$

are both Lawson closed (the second is Scott closed), each is also closed in the manifold topology, which means $\mathcal{M}$ is causally simple. $\Box$

Note in the above proof that we have assumed causally simplicity implies strong causality. If we are wrong about this, then (i) above should be replaced with 'causal simplicity+strong causality'.

6 Global hyperbolicity in the abstract

There are two elements which make the topology of a globally hyperbolic spacetime tick. They are:

(i) A bicontinuous poset $(X, \leq)$.

(ii) The intervals $[a, b] = \{x : a \leq x \leq b\}$ are compact in the interval topology on X.

From these two we can deduce some aspects we already know as well as some new ones. In particular, bicontinuity ensures that the topology of X, the interval topology, is implicit in $\leq$. We call such posets *globally hyperbolic.*

Theorem 6.1 *A globally hyperbolic poset is locally compact Hausdorff.*

(i) *The Lawson topology is contained in the interval topology.*

(ii) *Its partial order $\leq$ is a closed subset of X^2.*

(iii) *Each directed set with an upper bound has a supremum.*

(iv) *Each filtered set with a lower bound has a infimum.*

Proof. First we show that the Lawson topology is contained in the interval topology. Sets of the form $\twoheaduparrow x$ are open in the interval topology. To prove $X \backslash \uparrow x$ is open, let $y \in X \backslash \uparrow x$. Then $x \not\sqsubseteq y$. By bicontinuity, there is b with $y \ll b$ such that $x \not\sqsubseteq b$. For any $a \ll y$,

$$y \in (a, b) \subseteq X \backslash \uparrow x$$

which proves the Lawson topology is contained in the interval topology. Because the Lawson topology is always Hausdorff on a continuous poset, X is Hausdorff in its interval topology.

Let $x \in U$ where U is open. Then there is an open interval $x \in (a, b) \subseteq U$. By continuity of $(X, \leq)$, we can interpolate twice, obtaining a closed interval $[c, d]$ followed by another open interval we call V. We get

$$x \in V \subseteq [c, d] \subseteq (a, b) \subseteq U.$$

The closure of V is contained in $[c, d]$: X is Hausdorff so compact sets like $[c, d]$ are closed. Then $\mathrm{Cl}(V)$ is a closed subset of a compact space $[c, d]$, so it must be compact. This proves X is locally compact.

To prove $\leq$ is a closed subset of X^2, let $(a,b) \in X^2 \backslash \leq$. Since $a \not\leq b$, there is $x \ll a$ with $x \not\leq b$ by continuity. Since $x \not\leq b$, there is y with $b \ll y$ and $x \not\leq y$ by bicontinuity. Now choose elements 1 and 2 such that $x \ll a \ll 1$ and $2 \ll b \ll y$. Then

$$(a,b) \in (x,1) \times (2,y) \subseteq X^2 \backslash \leq .$$

For if $(c,d) \in (x,1) \times (2,y)$ and $c \leq d$, then $x \leq c \leq 1$ and $2 \leq d \leq y$, and since $c \leq d$, we get $x \leq y$, a contradiction. This proves $X^2 \backslash \leq$ is open.

Given a directed set $S \subseteq X$ with an upper bound x, if we fix any element $1 \in S$, then the set $\uparrow 1 \cap S$ is also directed and has a supremum iff S does. Then we can assume that S has a least element named $1 \in S$. The inclusion $f : S \to X :: s \mapsto s$ is a net and since S is contained in the compact set $[1,x]$, f has a convergent subnet $g : I \to S$. Then $T := g(I) \subseteq S$ is directed and cofinal in S. We claim $\bigsqcup T = \lim T$.

First, $\lim T$ is an upper bound for T. If there were $t \in T$ with $t \not\sqsubseteq \lim T$, then $\lim T \in X \backslash \uparrow t$. Since $X \backslash \uparrow t$ is open, there is $\alpha \in I$ such that

$$(\forall \beta \in I) \alpha \leq \beta \Rightarrow g(\beta) \in X \backslash \uparrow t.$$

Let $u = g(\alpha)$ and $t = g(\gamma)$. Since I is directed, there is $\beta \in I$ with $\alpha, \gamma \leq \beta$. Then

$$g(\beta) \in X \backslash \uparrow t \;\;\&\;\; t = g(\gamma) \leq g(\beta)$$

where the second inequality follows from the fact that subnets are monotone by definition. This is a contradiction, which proves $t \sqsubseteq \lim T$ for all t.

To prove $\bigsqcup T = \lim T$, let u be an upper bound for T. Then $t \sqsubseteq u$ for all t. However, if $\lim T \not\leq u$, then $\lim T \in X \backslash \downarrow u$, and since $X \backslash \downarrow u$ is open, we get that $T \cap (X \backslash \downarrow u) \neq \emptyset$, which contradicts that u is an upper bound for T. (Equivalently, we could have just used the fact that $\leq$ is closed.)

Now we prove $\bigsqcup S = \lim T$. Let $s \in S$. Since T is cofinal in S, there is $t \in T$ with $s \leq t$. Hence $s \leq t \leq \lim T$, so $\lim T$ is an upper bound for S. To finish, any upper bound for S is one for T so it must be above $\lim T$. Then $\bigsqcup S = \lim T$.

Given a filtered set S with a lower bound x, we can assume it has a greatest element 1. The map $f : S^* \to S :: x \mapsto x$ is a net where the poset S^* is obtained by reversing the order on S. Since $S \subseteq [x,1]$, f has a convergent subnet g, and now the proof is simply the dual of the suprema case. $\Box$

Globally hyperbolic posets share a remarkable property with metric spaces, that separability and second countability are equivalent.

Proposition 6.2 *Let $(X, \leq)$ be a bicontinuous poset. If $C \subseteq X$ is a countable dense subset in the interval topology, then*

(i) *The collection*
$$\{(a_i, b_i) : a_i, b_i \in C, a_i \ll b_i\}$$
is a countable basis for the interval topology. Thus, separability implies second countability, and even complete metrizability if X is globally hyperbolic.

(ii) *For all $x \in X$, $\mathord{\downarrow\!\!\downarrow} x \cap C$ contains a directed set with supremum x, and $\mathord{\uparrow\!\!\uparrow} x \cap C$ contains a filtered set with infimum x.*

Proof. (i) Sets of the form $(a, b) := \{x \in X : a \ll x \ll b\}$ form a basis for the interval topology. If $x \in (a, b)$, then since C is dense, there is $a_i \in (a, x) \cap C$ and $b_i \in (x, b) \cap C$ and so $x \in (a_i, b_i) \subseteq (a, b)$.

(ii) Fix $x \in X$. Given any $a \ll x$, the set (a, x) is open and C is dense, so there is $c_a \in C$ with $a \ll c_a \ll x$. The set $S = \{c_a \in C : a \ll x\} \subseteq \mathord{\downarrow\!\!\downarrow} x \cap C$ is directed: If $c_a, c_b \in S$, then since $\mathord{\downarrow\!\!\downarrow} x$ is directed, there is $d \ll x$ with $c_a, c_d \sqsubseteq d \ll x$ and thus $c_a, c_b \sqsubseteq c_d \in S$. Finally, $\bigsqcup S = x$: Any upper bound for S is also one for $\mathord{\downarrow\!\!\downarrow} x$ and so above x by continuity. The dual argument shows $\mathord{\uparrow\!\!\uparrow} x \cap C$ contains a filtered set with inf x. $\Box$

Globally hyperbolic posets are very much like the real line. In fact, a well-known domain theoretic construction pertaining to the real line extends in perfect form to the globally hyperbolic posets:

Theorem 6.3 *The closed intervals of a globally hyperbolic poset X*

$$\mathbf{I}X := \{[a, b] : a \leq b \ \&\ a, b \in X\}$$

ordered by reverse inclusion

$$[a, b] \sqsubseteq [c, d] \equiv [c, d] \subseteq [a, b]$$

form a continuous domain with

$$[a, b] \ll [c, d] \equiv a \ll c \ \&\ d \ll b.$$

The poset X has a countable basis iff $\mathbf{I}X$ is ω-continuous. Finally,

$$\max(\mathbf{I}X) \simeq X$$

where the set of maximal elements has the relative Scott topology from $\mathbf{I}X$.

Proof. If $S \subseteq \mathbf{I}X$ is a directed set, we can write it as

$$S = \{[a_i, b_i] : i \in I\}.$$

Without loss of generality, we can assume S has a least element $1 = [a, b]$. Thus, for all $i \in I$, $a \leq a_i \leq b_i \leq b$. Then $\{a_i\}$ is a directed subset of X bounded above by b, $\{b_i\}$ is a filtered subset of X bounded below by a. We know that $\bigsqcup a_i = \lim a_i$, $\bigwedge b_i = \lim b_i$ and that $\leq$ is closed. It follows that

$$\bigsqcup S = \left[\bigsqcup a_i, \bigwedge b_i\right].$$

For the continuity of $\mathbf{I}X$, consider $[a, b] \in \mathbf{I}X$. If $c \ll a$ and $b \ll d$, then $[c, d] \ll [a, b]$ in $\mathbf{I}X$. Then

$$[a, b] = \bigsqcup\{[c, d] : c \ll a \;\&\; b \ll d\} \tag{1}$$

a supremum that is directed since X is bicontinuous. Suppose now that $[x, y] \ll [a, b]$ in $\mathbf{I}X$. Then using (1), there is $[c, d]$ with $[x, y] \sqsubseteq [c, d]$ such that $c \ll a$ and $b \ll d$ which means $x \sqsubseteq c \ll a$ and $b \ll d \sqsubseteq y$ and thus $x \ll a$ and $b \ll y$. This completely characterizes the $\ll$ relation on $\mathbf{I}X$, which now enables us to prove $\max(\mathbf{I}X) \simeq X$, since we can write

$$\twoheaduparrow[a, b] \cap \max(\mathbf{I}X) = \{\{x\} : x \in X \;\&\; a \ll x \ll b\}$$

and $\twoheaduparrow[a, b]$ is a basis for the Scott topology on $\mathbf{I}X$.

Finally, if X has a countable basis, then it has a countable dense subset $C \subseteq X$, which means $\{[a_n, b_n] : a_n \ll b_n, a_n, b_n \in C\}$ is a countable basis for $\mathbf{I}X$ by Prop. 6.2(ii). $\square$

The endpoints of an interval $[a, b]$ form a two element list $x : \{1, 2\} \to X$ with $a = x(1) \leq x(2) = b$. We call these *formal intervals.* They determine the information in an interval as follows:

Corollary 6.4 *The formal intervals ordered by*

$$x \sqsubseteq y \equiv x(1) \leq y(1) \;\&\; y(2) \leq x(2)$$

form a domain isomorphic to $\mathbf{I}X$*.*

This observation – that spacetime has a canonical domain theoretic model – has at least two important applications, one of which we now consider. We prove that from only a countable set of events and the causality relation, one can reconstruct spacetime in a purely order theoretic manner. Explaining this requires domain theory.

7 Spacetime from discrete causality

Recall from the appendix on domain theory that an *abstract basis* is a set $(C, \ll)$ with a *transitive* relation that is *interpolative* from the $-$ *direction*:

$$F \ll x \Rightarrow (\exists y \in C)\, F \ll y \ll x,$$

for all finite subsets $F \subseteq C$ and all $x \in F$. Suppose, though, that it is also interpolative from the $+$ *direction*:

$$x \ll F \Rightarrow (\exists y \in C)\, x \ll y \ll F.$$

Then we can define a new abstract basis of *intervals*

$$\mathrm{int}(C) = \{(a, b) : a \ll b\} = \ll \subseteq C^2$$

whose relation is

$$(a, b) \ll (c, d) \equiv a \ll c \ \&\ d \ll b.$$

Lemma 7.1 *If $(C, \ll)$ is an abstract basis that is $\pm$ interpolative, then $(\mathrm{int}(C), \ll)$ is an abstract basis.*

Proof. Let $F = \{(a_i, b_i) : 1 \leq i \leq n\} \ll (a, b)$. Let $A = \{a_i\}$ and $B = \{b_i\}$. Then $A \ll a$ and $b \ll B$ in C. Since C lets us interpolate in both directions, we get (x, y) with $F \ll (x, y) \ll (a, b)$. Transitivity is inherited from C. $\Box$

Let $\mathbf{I}C$ denote the ideal completion of the abstract basis $\mathrm{int}(C)$.

Theorem 7.2 *Let C be a countable dense subset of a globally hyperbolic spacetime $\mathcal{M}$ and $\ll = I^+$ be timelike causality. Then*

$$\max(\mathbf{I}C) \simeq \mathcal{M}$$

where the set of maximal elements have the Scott topology.

Proof. Because $\mathcal{M}$ is bicontinuous, the sets $\Uparrow\! x$ and $\Downarrow\! x$ are filtered and directed respectively. Thus $(C, \ll)$ is an abstract basis for which $(\mathrm{int}(C), \ll)$ is also an abstract basis. Because C is dense, $(\mathrm{int}(C), \ll)$ is a basis for the domain $\mathbf{I}\mathcal{M}$. But, the ideal completion of any basis for $\mathbf{I}\mathcal{M}$ must be isomorphic to $\mathbf{I}\mathcal{M}$. Thus, $\mathbf{I}C \simeq \mathbf{I}\mathcal{M}$, and so $\mathcal{M} \simeq \max(\mathbf{I}\mathcal{M}) \simeq \max(\mathbf{I}C)$. $\Box$

In "ordering the order" I^+, taking its completion, and then the set of maximal elements, we recover spacetime by reasoning only about the causal relationships between a countable dense set of events. We should say a bit more too.

Theorem 7.2 is very different from results like "Let $\mathcal{M}$ be a certain spacetime with relation $\leq$. Then the interval topology is the manifold topology." Here we identify, in abstract terms, a beautiful process by which a countable set with a causality relation determines a space. The process is entirely order theoretic in nature, spacetime is not required to understand or execute it (i.e., if we put $C = \mathbb{Q}$ and $\ll = <$, then $\max(\mathbf{I}C) \simeq \mathbb{R}$). In this sense, our understanding of the relation between causality and the topology of spacetime is now explainable independently of geometry.

Last, notice that if we naively try to obtain $\mathcal{M}$ by taking the ideal completion of $(S, \sqsubseteq)$ or $(S, \ll)$ that it will not work: $\mathcal{M}$ is not a dcpo. Some *other* process is necessary, and the *exact* structure of globally hyperbolic spacetime allows one to carry out this alternative process. Ideally, one would now like to know what constraints on C in general imply that $\max(\mathbf{I}C)$ is a manifold.

8 Spacetime as a domain

The category of globally hyperbolic posets is naturally isomorphic to a special category of domains called interval domains.

Definition 8.1 An *interval poset* is a poset D that has two functions $\mathrm{left} : D \to \max(D)$ and $\mathrm{right} : D \to \max(D)$ such that

(i) Each $x \in D$ is an "interval" with $\mathrm{left}(x)$ and $\mathrm{right}(x)$ as endpoints:

$$(\forall x \in D)\, x = \mathrm{left}(x) \sqcap \mathrm{right}(x),$$

(ii) The union of two intervals with a common endpoint is another interval: For all $x, y \in D$, if $\mathrm{right}(x) = \mathrm{left}(y)$, then

$$\mathrm{left}(x \sqcap y) = \mathrm{left}(x) \;\;\&\;\; \mathrm{right}(x \sqcap y) = \mathrm{right}(y),$$

(iii) Each point $p \in \uparrow\! x \cap \max(D)$ of an interval $x \in D$ determines two subintervals, $\mathrm{left}(x) \sqcap p$ and $p \sqcap \mathrm{right}(x)$, with endpoints:

$$\mathrm{left}(\mathrm{left}(x) \sqcap p) = \mathrm{left}(x) \quad \& \quad \mathrm{right}(\mathrm{left}(x) \sqcap p) = p$$

$$\mathrm{left}(p \sqcap \mathrm{right}(x)) = p \quad \& \quad \mathrm{right}(p \sqcap \mathrm{right}(x)) = \mathrm{right}(x)$$

Notice that a nonempty interval poset D has $\max(D) \neq \emptyset$ by definition. With interval posets, we only assume that infima indicated in the definition exist; in particular, we do not assume the existence of all binary infima.

Definition 8.2 For an interval poset $(D, \text{left}, \text{right})$, the relation $\leq$ on $\max(D)$ is

$$a \leq b \equiv (\exists\, x \in D)\, a = \text{left}(x) \;\&\; b = \text{right}(x)$$

for $a, b \in \max(D)$.

Lemma 8.3 $(\max(D), \leq)$ *is a poset.*

Proof. Reflexivity: By property (i) of an interval poset, $x \sqsubseteq \text{left}(x), \text{right}(x)$, so if $a \in \max(D)$, $a = \text{left}(a) = \text{right}(a)$, which means $a \leq a$. Antisymmetry: If $a \leq b$ and $b \leq a$, then there are $x, y \in D$ with $a = \text{left}(x) = \text{right}(y)$ and $b = \text{right}(x) = \text{left}(y)$, so this combined with property (i) gives

$$x = \text{left}(x) \sqcap \text{right}(x) = \text{right}(y) \sqcap \text{left}(y) = y$$

and thus $a = b$. Transitivity: If $a \leq b$ and $b \leq c$, then there are $x, y \in D$ with $a = \text{left}(x)$, $b = \text{right}(x) = \text{left}(y)$ and $c = \text{right}(y)$, so property (ii) of interval posets says that for $z = x \sqcap y$ we have

$$\text{left}(z) = \text{left}(x) = a \;\;\&\;\; \text{right}(z) = \text{right}(y) = c$$

and thus $a \leq c$. $\square$

An interval poset D is the set of intervals of $(\max(D), \leq)$ ordered by reverse inclusion:

Lemma 8.4 *If D is an interval poset, then*

$$x \sqsubseteq y \equiv (\text{left}(x) \leq \text{left}(y) \leq \text{right}(y) \leq \text{right}(x))$$

Proof ($\Rightarrow$) Since $x \sqsubseteq y \sqsubseteq \text{left}(y)$, property (iii) of interval posets implies $z = \text{left}(x) \sqcap \text{left}(y)$ is an "interval" with

$$\text{left}(z) = \text{left}(x) \;\&\; \text{right}(z) = \text{left}(y)$$

and thus $\text{left}(x) \leq \text{left}(y)$. The inequality $\text{right}(y) \leq \text{right}(x)$ follows similarly. The inequality $\text{left}(y) \leq \text{right}(y)$ follows from the definition of $\leq$.

($\Leftarrow$) Applying the definition of $\leq$ and properties (ii) and (i) of interval posets to $\text{left}(x) \leq \text{left}(y) \leq \text{right}(x)$, we get $x \sqsubseteq \text{left}(y)$. Similarly, $x \sqsubseteq \text{right}(y)$. Then $x \sqsubseteq \text{left}(y) \sqcap \text{right}(y) = y$. $\square$

Corollary 8.5 *If D is an interval poset,*

$$\phi : D \to \mathbf{I}(\max(D), \leq) :: x \mapsto [\mathrm{left}(x), \mathrm{right}(x)]$$

is an order isomorphism.

In particular,

$$p \in \uparrow x \cap \max(D) \equiv \mathrm{left}(x) \leq p \leq \mathrm{right}(x)$$

in any interval poset.

Definition 8.6 If $(D, \mathrm{left}, \mathrm{right})$ is an interval poset,

$$[p, \cdot] := \mathrm{left}^{-1}(p) \quad \text{and} \quad [\cdot, q] := \mathrm{right}^{-1}(q)$$

for any $p, q \in \max(D)$.

Definition 8.7 An *interval domain* is an interval poset $(D, \mathrm{left}, \mathrm{right})$ where D is a continuous dcpo such that

(i) If $p \in \twoheaduparrow x \cap \max(D)$, then

$$\twoheaduparrow(\mathrm{left}(x) \sqcap p) \neq \emptyset \quad \& \quad \twoheaduparrow(p \sqcap \mathrm{right}(x)) \neq \emptyset.$$

(ii) For all $x \in D$, the following are equivalent:

(a) $\twoheaduparrow x \neq \emptyset$
(b) $(\forall y \in [\,\mathrm{left}(x), \cdot\,])(\ y \sqsubseteq x \ \Rightarrow\ y \ll \mathrm{right}(y) \text{ in } [\,\cdot, \mathrm{right}(y)\,]\ )$
(c) $(\forall y \in [\cdot, \mathrm{right}(x)])(\ y \sqsubseteq x \ \Rightarrow\ y \ll \mathrm{left}(y) \text{ in } [\,\mathrm{left}(y), \cdot\,]\ )$

(iii) Invariance of endpoints under suprema:

(a) For all directed $S \subseteq [p, \cdot]$

$$\mathrm{left}(\bigsqcup S) = p \quad \& \quad \mathrm{right}(\bigsqcup S) = \mathrm{right}(\bigsqcup T)$$

for any directed $T \subseteq [q, \cdot]$ with $\mathrm{right}(T) = \mathrm{right}(S)$.

(b) For all directed $S \subseteq [\cdot, q]$

$$\mathrm{left}(\bigsqcup S) = \mathrm{left}(\bigsqcup T) \quad \& \quad \mathrm{right}(\bigsqcup S) = q$$

for any directed $T \subseteq [\cdot, p]$ with $\mathrm{left}(T) = \mathrm{left}(S)$.

(iv) Intervals are compact: For all $x \in D$, $\uparrow x \cap \max(D)$ is Scott compact.

Interval domains are interval posets whose axioms also take into account the completeness and approximation present in a domain: (i) says if a point p belongs to the interior of an interval $x \in D$, the subintervals $\text{left}(x) \sqcap p$ and $p \sqcap \text{right}(x)$ both have nonempty interior; (ii) says an interval has nonempty interior iff all intervals that contain it have nonempty interior locally; (iii) explains the behavior of endpoints when taking suprema.

For a globally hyperbolic $(X, \leq)$, we define $\text{left} : \mathbf{I}X \to \mathbf{I}X :: [a, b] \mapsto [a]$ and $\text{right} : \mathbf{I}X \to \mathbf{I}X :: [a, b] \mapsto [b]$.

Lemma 8.8 *If $(X, \leq)$ is a globally hyperbolic poset, then $(\mathbf{I}X, \text{left}, \text{right})$ is an interval domain.*

In essence, we now prove that this is the only example.

Definition 8.9 The category $\underline{\text{IN}}$ of interval domains and commutative maps is given by

- **objects** Interval domains $(D, \text{left}, \text{right})$.
- **arrows** Scott continuous $f : D \to E$ that commute with left and right, i.e., such that both

$$\begin{tikzcd} D \arrow[r, "\text{left}_D"] \arrow[d, "f"'] & D \arrow[d, "f"] \\ E \arrow[r, "\text{left}_E"'] & E \end{tikzcd}$$

and

$$\begin{tikzcd} D \arrow[r, "\text{right}_D"] \arrow[d, "f"'] & D \arrow[d, "f"] \\ E \arrow[r, "\text{right}_E"'] & E \end{tikzcd}$$

commute.

- **identity** $1 : D \to D$.
- **composition** $f \circ g$.

Definition 8.10 The category $\underline{\mathrm{G}}$ is given by

- **objects** Globally hyperbolic posets $(X, \leq)$.
- **arrows** Continuous in the interval topology, monotone.
- **identity** $1 : X \to X$.
- **composition** $f \circ g$.

It is routine to verify that $\underline{\mathrm{IN}}$ and $\underline{\mathrm{G}}$ are categories.

Proposition 8.11 *The correspondence* $\mathbf{I} : \underline{\mathrm{G}} \to \underline{\mathrm{IN}}$ *given by*

$$(X, \leq) \mapsto (\mathbf{I}X, \mathrm{left}, \mathrm{right})$$

$$(f : X \to Y) \mapsto (\bar{f} : \mathbf{I}X \to \mathbf{I}Y)$$

is a functor between categories.

Proof. The map $\bar{f} : \mathbf{I}X \to \mathbf{I}Y$ defined by $\bar{f}[a, b] = [f(a), f(b)]$ takes intervals to intervals since f is monotone. It is Scott continuous because suprema and infima in X and Y are limits in the respective interval topologies and f is continuous with respect to the interval topology. $\Box$

Now we prove there is also a functor going the other way. Throughout the proof, we use $\bigsqcup$ for suprema in $(D, \sqsubseteq)$ and $\bigvee$ for suprema in $(\max(D), \leq)$.

Lemma 8.12 *Let D be an interval domain with $x \in D$ and $p \in \max(D)$. If $x \ll p$ in D, then* $\mathrm{left}(x) \ll p \ll \mathrm{right}(x)$ *in* $(\max(D), \leq)$.

Proof. Since $x \ll p$ in D, $x \sqsubseteq p$, and so $\mathrm{left}(x) \leq p \leq \mathrm{right}(x)$.

($\Rightarrow$) First we prove $\mathrm{left}(x) \ll p$. Let $S \subseteq \max(D)$ be a $\leq$-directed set with $p \leq \bigvee S$. For $\bar{x} := \phi^{-1}([\mathrm{left}(x), p])$ and $y := \phi^{-1}([\mathrm{left}(x), \bigvee S])$, we have $y \sqsubseteq \bar{x}$. By property (i) of interval domains, $\Uparrow x \neq \emptyset$ implies that $\Uparrow \bar{x} = \Uparrow(\mathrm{left}(x) \sqcap p) \neq \emptyset$, so property (ii) of interval domains says $y \ll \mathrm{right}(y)$ in the poset $[\cdot, \mathrm{right}(y)]$. Then

$$y \ll \mathrm{right}(y) = \bigsqcup_{s \in S} \phi^{-1}[s, \bigvee S]$$

which means $y \sqsubseteq \phi^{-1}[s, \bigvee S]$ for some $s \in S$. So by monotonicity of ϕ, $\mathrm{left}(x) \le s$. Thus, $\mathrm{left}(x) \ll p$ in $(\max(D), \le)$.

Now we prove $p \ll \mathrm{right}(x)$. Let $S \subseteq \max(D)$ be a $\le$-directed set with $\mathrm{right}(x) \le \bigvee S$. For $\bar{x} := \phi^{-1}([p, \mathrm{right}(x)])$ and $y := \phi^{-1}([p, \bigvee S])$, $y \sqsubseteq \bar{x}$, and since $\Uparrow\bar{x} \neq \emptyset$ by property (i) of interval domains, property (ii) of interval domains gives $y \ll \mathrm{right}(y)$ in $[\cdot, \mathrm{right}(y)]$. Then

$$y \ll \mathrm{right}(y) = \bigsqcup_{s \in S} \phi^{-1}[s, \bigvee S]$$

which means $[s, \bigvee S] \subseteq [p, \bigvee S]$ and hence $p \le s$ for some $s \in S$. $\square$

Now we begin the proof that $(\max(D), \le)$ is a globally hyperbolic poset when D is an interval domain.

Lemma 8.13 *Let $p, q \in \max(D)$.*

(i) *If $S \subseteq [p, \cdot]$ is directed, then*

$$\mathrm{right}(\bigsqcup S) = \bigwedge_{s \in S} \mathrm{right}(s).$$

(ii) *If $S \subseteq [\cdot, q]$ is directed, then*

$$\mathrm{left}(\bigsqcup S) = \bigvee_{s \in S} \mathrm{left}(s).$$

Proof. (i) First, $\mathrm{right}(\bigsqcup S)$ is a $\le$-lower bound for $\{\mathrm{right}(s) : s \in S\}$ because

$$\phi(\bigsqcup S) = [\mathrm{left}(\bigsqcup S), \mathrm{right}(\bigsqcup S)] = \bigcap_{s \in S} [p, \mathrm{right}(s)].$$

Given any other lower bound $q \le \mathrm{right}(s)$ for all $s \in S$, the set

$$T := \{\phi^{-1}([q, \mathrm{right}(s)]) : s \in S\} \subseteq [q, \cdot]$$

is directed with $\mathrm{right}(T) = \mathrm{right}(S)$, so

$$q = \mathrm{left}(\bigsqcup T) \le \mathrm{right}(\bigsqcup T) = \mathrm{right}(\bigsqcup S)$$

where the two equalities follow from property (iii)(a) of interval domains, and the inequality follows from the definition of $\le$. This proves the claim.

(ii) This proof is simply the dual of (i), using property (iii)(b) of interval domains. $\square$

Lemma 8.14 *Let D be an interval domain. If $\Uparrow x \neq \emptyset$ in D, then*

$$\bigwedge S \leq \mathrm{left}(x) \Rightarrow (\exists s \in S)\, s \leq \mathrm{right}(x)$$

for any $\leq$-filtered $S \subseteq \max(D)$ with an infimum in $(\max(D), \leq)$.

Proof. Let $S \subseteq \max(D)$ be a $\leq$-filtered set with $\bigwedge S \leq \mathrm{left}(x)$. There is some $[a, b]$ with $x = \phi^{-1}[a, b]$. Setting $y := \phi^{-1}[\bigwedge S, b]$, we have $y \sqsubseteq x$ and $\Uparrow x \neq \emptyset$, so property (ii)(c) of interval domains says $y \ll \mathrm{left}(y)$ in $[\mathrm{left}(y), \cdot]$. Then

$$y \ll \mathrm{left}(y) = \bigsqcup_{s \in S} \phi^{-1}[\bigwedge S, s]$$

where this set is $\sqsubseteq$-directed because S is $\leq$-filtered. Thus, $y \sqsubseteq \phi^{-1}[\bigwedge S, s]$ for some $s \in S$, which gives $s \leq b$. $\square$

Lemma 8.15 *Let D be an interval domain. Then*

(i) *The set $\Downarrow x$ is $\leq$-directed with $\bigvee \Downarrow x = x$.*

(ii) *For all $a, b \in \max(D)$, $a \ll b$ in $(\max(D), \leq)$ iff for all $\leq$-filtered $S \subseteq \max(D)$ with an infimum, $\bigwedge S \leq a \Rightarrow (\exists s \in S)\, s \leq b$.*

(iii) *The set $\Uparrow x$ is $\leq$-filtered with $\bigwedge \Uparrow x = x$.*

Thus, the poset $(\max(D), \leq)$ is bicontinuous.

Proof. (i) By Lemma 8.12, if $x \ll p$ in D, then $\mathrm{left}(x) \ll p$ in $\max(D)$. Then the set

$$T = \{\mathrm{left}(x) : x \ll p \text{ in } D\} \subseteq \Downarrow p$$

is $\leq$-directed. We will prove $\bigvee S = p$. To see this,

$$S = \{\phi^{-1}[\mathrm{left}(x), p] : x \ll p \text{ in } D\}$$

is a directed subset of $[\cdot, p]$, so by Lemma 8.13(ii),

$$\mathrm{left}(\bigsqcup S) = \bigvee T.$$

Now we calculate $\bigsqcup S$. We know $\bigsqcup S = \phi^{-1}[a, b]$, where $[a, b] = \bigcap[\mathrm{left}(x), p]$. Assume $\bigsqcup S \neq p$. By maximality of p, $p \not\sqsubseteq \bigsqcup S$, so there must be an $x \in D$ with $x \ll p$ and $x \not\sqsubseteq \bigsqcup S$. Then $[a, b] \not\subseteq [\mathrm{left}(x), \mathrm{right}(x)]$, so either

$$\mathrm{left}(x) \not\leq a \quad \text{or} \quad b \not\leq \mathrm{right}(x)$$

But, $[a,b] \subseteq [\mathrm{left}(x),p]$ for any $x \ll p$ in D, so we have $\mathrm{left}(x) \leq a$ and $b \leq p \leq \mathrm{right}(x)$, which is a contradiction. Thus,

$$p = \bigsqcup S = \mathrm{left}(\bigsqcup S) = \bigvee T,$$

and since $\mathord{\downarrow\!\!\downarrow} p$ contains a $\leq$-directed set with sup p, $\mathord{\downarrow\!\!\downarrow} p$ itself is $\leq$-directed with $\bigvee \mathord{\downarrow\!\!\downarrow} p = p$. This proves $(\max(D), \leq)$ is a continuous poset.

(ii) $(\Rightarrow)$ Let $a \ll b$ in $\max(D)$. Let $x := \phi^{-1}[a,b]$. We first prove $\mathord{\uparrow\!\!\uparrow} x \neq \emptyset$ using property (ii)(b) of interval domains. Let $y \sqsubseteq x$ with $y \in [a,\cdot]$. We need to show $y \ll \mathrm{right}(y)$ in the poset $[\cdot,\mathrm{right}(y)]$. Let $S \subseteq [\cdot,\mathrm{right}(y)]$ be directed with $\mathrm{right}(y) \sqsubseteq \bigsqcup S$ and hence $\mathrm{right}(y) = \bigsqcup S$ by maximality. Using Lemma 8.13(ii),

$$\mathrm{right}(y) = \bigsqcup S = \mathrm{left}(\bigsqcup S) = \bigvee_{s \in S} \mathrm{left}(s)$$

But $y \sqsubseteq x$, so $b \leq \mathrm{right}(y) = \bigvee_{s \in S} \mathrm{left}(s)$, and since $a \ll b$, $a \leq \mathrm{left}(s)$ for some $s \in S$. Then since for this same s, we have

$$\mathrm{left}(y) = a \leq \mathrm{left}(s) \leq \mathrm{right}(s) = \mathrm{right}(y)$$

which means $y \sqsubseteq s$. Then $y \ll \mathrm{right}(y)$ in the poset $[\cdot,\mathrm{right}(y)]$. By property (ii)(b), we have $\mathord{\uparrow\!\!\uparrow} x \neq \emptyset$, so Lemma 8.14 now gives the desired result.

(ii) $(\Leftarrow)$ First, $S = \{a\}$ is one such filtered set, so $a \leq b$. Let $x = \phi^{-1}[a,b]$. We prove $\mathord{\uparrow\!\!\uparrow} x \neq \emptyset$ using axiom (ii)(c) of interval domains. Let $y \sqsubseteq x$ with $y \in [\cdot,b]$. To prove $y \ll \mathrm{left}(y)$ in $[\mathrm{left}(y),\cdot]$, let $S \subseteq [\mathrm{left}(y),\cdot]$ be directed with $\mathrm{left}(y) \sqsubseteq \bigsqcup S$. By maximality, $\mathrm{left}(y) = \bigsqcup S$. By Lemma 8.13(i),

$$\mathrm{left}(y) = \bigsqcup S = \mathrm{right}(\bigsqcup S) = \bigwedge_{s \in S} \mathrm{right}(s)$$

and $\{\mathrm{right}(s) : s \in S\}$ is $\leq$-filtered. Since $y \sqsubseteq x$,

$$\bigwedge_{s \in S} \mathrm{right}(s) = \mathrm{left}(y) \leq \mathrm{left}(x) = a,$$

so by assumption, $\mathrm{right}(s) \leq b$, for some $s \in S$. Then for this same s,

$$\mathrm{left}(y) = \mathrm{left}(s) \leq \mathrm{right}(s) \leq b = \mathrm{right}(y)$$

which means $y \sqsubseteq s$. Then $y \ll \mathrm{left}(y)$ in $[\mathrm{left}(y),\cdot]$. By property (ii)(c) of interval domains, $\mathord{\uparrow\!\!\uparrow} x \neq \emptyset$. By Lemma 8.12, taking any $p \in \mathord{\uparrow\!\!\uparrow} x$, we get $a = \mathrm{left}(x) \ll p \ll \mathrm{right}(x) = b$.

(iii) Because of the characterization of $\ll$ in (ii), this proof is simply the dual of (i). $\square$

Lemma 8.16 *Let* $(D, \mathrm{left}, \mathrm{right})$ *be an interval domain. Then*

(i) *If* $a \ll p \ll b$ *in* $(\max(D), \leq)$, *then* $\phi^{-1}[a, b] \ll p$ *in* D.

(ii) *The interval topology on* $(\max(D), \leq)$ *is the relative Scott topology* $\max(D)$ *inherits from* D.

Thus, the poset $(\max(D), \leq)$ *is globally hyperbolic.*

Proof. (i) Let $S \subseteq D$ be directed with $p \sqsubseteq \bigsqcup S$. Then $p = \bigsqcup S$ by maximality. The sets $L = \{\phi^{-1}[\mathrm{left}(s), p] : s \in S\}$ and $R = \{\phi^{-1}[p, \mathrm{right}(s)] : s \in S\}$ are both directed in D. For their suprema, Lemma 8.13 gives

$$\mathrm{left}(\bigsqcup L) = \bigvee_{s \in S} \mathrm{left}(s) \;\;\&\;\; \mathrm{right}(\bigsqcup R) = \bigwedge_{s \in S} \mathrm{right}(s)$$

Since $s \sqsubseteq \phi^{-1}[\bigvee_{s \in S} \mathrm{left}(s), \bigwedge_{s \in S} \mathrm{right}(s)]$ for all $s \in S$,

$$p = \bigsqcup S \sqsubseteq \phi^{-1}\left[\bigvee_{s \in S} \mathrm{left}(s), \bigwedge_{s \in S} \mathrm{right}(s)\right],$$

and so

$$\bigvee_{s \in S} \mathrm{left}(s) = p = \bigwedge_{s \in S} \mathrm{right}(s).$$

Since $a \ll p$, there is $s_1 \in S$ with $a \leq \mathrm{left}(s_1)$. Since $p \ll b$, there is $s_2 \in S$ with $\mathrm{right}(s_2) \leq b$, using bicontinuity of $\max(D)$. By the directedness of S, there is $s \in S$ with $s_1, s_2 \sqsubseteq s$, which gives

$$a \leq \mathrm{left}(s_1) \leq \mathrm{left}(s) \leq \mathrm{right}(s) \leq \mathrm{right}(s_2) \leq b$$

which proves $\phi^{-1}[a, b] \sqsubseteq s$.

(ii) Combining (i) and Lemma 8.12,

$$a \ll p \ll b \text{ in } (\max(D), \leq) \;\Leftrightarrow\; \phi^{-1}[a, b] \ll p \text{ in } D.$$

Thus, the identity map $1 : (\max(D), \leq) \to (\max(D), \sigma)$ sends basic open sets in the interval topology to basic open sets in the relative Scott topology, and conversely, so the two spaces are homeomorphic.

Finally, since $\uparrow\! x \cap \max(D) = \{p \in \max(D) : \mathrm{left}(x) \leq p \leq \mathrm{right}(x)\}$, and this set is Scott compact, it must also be compact in the interval topology on $(\max(D), \leq)$, since they are homeomorphic. $\Box$

Proposition 8.17 *The correspondence* max : $\underline{\text{IN}} \to \underline{\text{G}}$ *given by*

$$(D, \text{left}, \text{right}) \mapsto (\max(D), \leq)$$

$$(f : D \to E) \mapsto (f|_{\max(D)} : \max(D) \to \max(E))$$

is a functor between categories.

Proof. First, commutative maps $f : D \to E$ preserve maximal elements: If $x \in \max(D)$, then $f(x) = f(\text{left}_D(x)) = \text{left}_E \circ f(x) \in \max(E)$. By Lemma 8.16(ii), $f|_{\max(D)}$ is continuous with respect to the interval topology. For monotonicity, let $a \leq b$ in $\max(D)$ and $x := \phi^{-1}[a, b] \in D$. Then

$$\text{left}_E \circ f(x) = f(\text{left}_D(x)) = f(a)$$

and

$$\text{right}_E \circ f(x) = f(\text{right}_D(x)) = f(b)$$

which means $f(a) \leq f(b)$, by the definition of $\leq$ on $\max(E)$. $\Box$

Before the statement of the main theorem in this section, we recall the definition of a natural isomorphism.

Definition 8.18 A *natural transformation* $\eta : F \to G$ between functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ is a collection of arrows $(\eta_X : F(X) \to G(X))_{X \in \mathcal{C}}$ such that for any arrow $f : A \to B$ in $\mathcal{C}$,

$$\begin{array}{ccc} F(A) & \xrightarrow{\eta_A} & G(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow[\eta_B]{} & G(B) \end{array}$$

commutes. If each η_X is an isomorphism, η is a *natural isomorphism.*

Categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* when there are functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $\eta : 1_{\mathcal{C}} \to GF$ and $\mu : 1_{\mathcal{D}} \to FG$.

Theorem 8.19 *The category of globally hyperbolic posets is equivalent to the category of interval domains.*

Proof. We have natural isomorphisms

$$\eta : 1_{\underline{\mathrm{IN}}} \to \mathbf{I} \circ \max$$

and

$$\mu : 1_{\underline{\mathrm{G}}} \to \max \circ \mathbf{I}$$

$\square$

This result suggests that questions about spacetime can be converted to domain theoretic form, where we can use domain theory to answer them, and then translate the answers back to the language of physics (and vice-versa).

It also shows that causality between events is equivalent to an order on *regions* of spacetime. Most importanly, we have shown that globally hyperbolic spacetime with causality is equivalent to a structure $\mathbf{I}X$ whose origins are "discrete." This is the formal explanation for why spacetime can be reconstructed from a countable dense set of events in a purely order theoretic manner.

9 Conclusion and future work

It seems that it might be possible to use order as the basis for new and useful causality conditions which generalize globally hyperbolicity. Some possible candidates are to require $(\mathcal{M}, \sqsubseteq)$ a continuous (bicontinuous) poset. Bicontinuity, in particular, has the nice consequence that one does not have to explicitly assume strong causality as one does with global hyperbolicity. Is $\mathcal{M}$ bicontinuous iff it is causally simple?

We have shown that globally hyperbolic spacetimes live in a category that is equivalent to the category of interval domains. Because ω-continuous domains are the ideal completions of countable abstract bases, spacetime can be order theoretically reconstructed from a dense 'discrete' set. (Ideally we would like to remove the requirement that the set be dense by assuming some additional structure and using it to *derive* a dense set.) Thus, with the benefit of the domain theoretic viewpoint, we are able to see that a globally hyperbolic spacetime emanates from something discrete.

It is now natural to ask about the domain theoretic analogue of 'Lorentz metric', and the authors suspect it is related to the study of measurement ([5][6]). After that, we should ask about the domain theoretic analogue of Einstein's equation, etc. Given a reformulation of general relativity in domain theoretic terms, a first step toward a theory of quantum gravity would be to restrict to a countable abstract basis with a measurement. The advantage though of the domain theoretic formulation is that we will know up front how to reconstruct 'classical' general relativity as an order theoretic 'limit' – which is what one is not currently able to do with the standard formulation of general relativity.

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Appendix: Domain theory

A useful technique for constructing domains is to take the *ideal completion* of an *abstract basis.*

Definition 9.1 An *abstract basis* is given by a set B together with a transitive relation $<$ on B which is *interpolative,* that is,

$$M < x \Rightarrow (\,\exists\, y \in B\,)\, M < y < x$$

for all $x \in B$ and all finite subsets M of B.

Notice the meaning of $M < x$: It means $y < x$ for all $y \in M$. Abstract bases are covered in [1], which is where one finds the following.

Definition 9.2 An *ideal* in $(B, <)$ is a nonempty subset I of B such that

(i) I is a lower set: $(\,\forall\, x \in B\,)(\,\forall\, y \in I\,)\, x < y \Rightarrow x \in I$.

(ii) I is directed: $(\,\forall\, x, y \in I\,)(\,\exists\, z \in I\,)\, x, y < z$.

The collection of ideals of an abstract basis $(B, <)$ ordered under inclusion is a partially ordered set called the *ideal completion* of B. We denote this poset by $\bar{B}$.

The set $\{y \in B : y < x\}$ for $x \in B$ is an ideal which leads to a natural mapping from B into $\overline{B}$, given by $i(x) = \{y \in B : y < x\}$.

Proposition 9.3 *If* $(B, <)$ *is an abstract basis, then*

(i) *Its ideal completion* $\bar{B}$ *is a dcpo.*

(ii) *For* $I, J \in \bar{B}$,

$$I \ll J \Leftrightarrow (\,\exists\, x, y \in B\,)\, x < y \;\&\; I \subseteq i(x) \subseteq i(y) \subseteq J.$$

(iii) $\bar{B}$ *is a continuous dcpo with basis* $i(B)$.

If one takes any basis B of a domain D and restricts the approximation relation $\ll$ on D to B, they are left with an abstract basis $(B, \ll)$ whose ideal completion is D. Thus, all domains arise as the ideal completion of an abstract basis.

Appendix: Topology

Nets are a generalization of sequences. Let X be a space.

Definition 9.4 A *net* is a function $f : I \to X$ where I is a directed poset.

A subset J of I is *cofinal* if for all $\alpha \in I$, there is $\beta \in J$ with $\alpha \leq \beta$.

Definition 9.5 A *subnet* of a net $f : I \to X$ is a function $g : J \to I$ such that J is directed and

- For all $x, y \in J$, $x \leq y \Rightarrow g(x) \leq g(y)$
- $g(J)$ is cofinal in I.

Definition 9.6 A net $f : I \to X$ *converges* to $x \in X$ if for all open $U \subseteq X$ with $x \in U$, there is $\alpha \in I$ such that

$$\alpha \leq \beta \Rightarrow f(\beta) \in U$$

for all $\beta \in I$.

A space X is *compact* if every open cover has a finite subcover.

Proposition 9.7 *A space X is compact iff every net $f : I \to X$ has a convergent subnet.*