# Deadlocks and Dihomotopy in Mutual Exclusion Models

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## 1 Introduction: Mutual Exclusion Models

Already in 1968, E.W. Dijkstra [Dij68] proposed to apply a geometric point of view in the consideration of coordination situations in *concurrency*. His progress graphs were the basis of the Higher Dimensional Automata (HDA) introduced by V. Pratt[Pra91] and developed in the thesis of É. Goubault[Gou95] and in later research (cf. [FGR99]).

In this abstract, we stick to a simple continuous geometric model. A system of $n$ concurrent processes will be represented as a subset of Euclidean space $I\!R^n$ with the usual partial order. Each coordinate axis corresponds to one of the processes performing a linear programme[1]; a state of the system is a point in $I\!R^n$ with its $i$th coordinate describing "local time" of the $i$th processor. A run of a concurrent program is modelled by a *continuous increasing* path – time increases for every participating processor – between two states.

Shared resources can often only be used by one or a limited number of processors at the same time. As a consequence, certain *hyperrectangles* – corresponding to conflict in the access to such a resource – have to be removed from the model; together, they form the *forbidden region*.

The resulting *mutual exclusion models* are more general than those modelling *semaphore* programs. They allow us to consider also $k$-semaphores, where a shared object may be accessed by $k$, but not by $k+1$ processors.

To get more formal, let $I = [0,1]$ denote the unit interval, and let $\boldsymbol{I}^n \subset I\!R^n$ denote the unit hypercube. An (open) isothetic hyperrectangle is a subset

$$R = (a_1, b_1) \times \cdots \times (a_n, b_n) \subset I^n;$$

closed or half-open coordinate intervals are exceptionally allowed in the forms $[0, b)$, $(a, 1]$, resp. $[0, 1]$. The forbidden region $F = \bigcup_1^r R^i$ is then a finite union of $n$-hyperrectangles $R^i = (a_1^i, b_1^i) \times \cdots \times (a_n^i, b_n^i)$, and the *state space* has the form $X = I^n \setminus F$. We assume that $\mathbf{0} = (0, \ldots, 0)$ and $\mathbf{1} = (1, \ldots, 1)$ are *not* contained in the forbidden region $F$.

[1] More general progams can be included by replacing an axis by a graph and the state space by a product of graphs, cf. [FS00].

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In this abstract, we address two questions: How can one use the geometric/combinatorial description of the forbidden region to

1. detect *deadlocks* and associated *unsafe*, resp. *unreachable* regions? This is a survey of the results obtained in [FGR98].
2. obtain information on the number of "essentially different" schedules between two states; these results are new and will be explained in Sect. 4.

## 2 Deadlock Detection in Mutual Exclusion Models

The "Swiss flag" example from Fig. 1 below (the forbidden region is dashed) conveys the idea, that deadlocks – with no possible legal move – in such mutual exclusion models are associated to $n$-dimensional "lower corners" below the forbidden region.

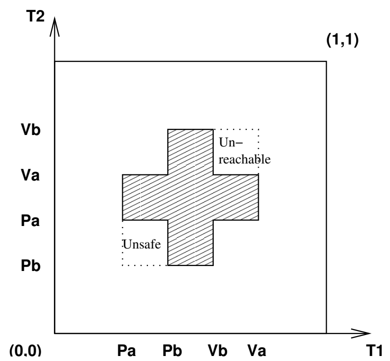


**Fig. 1.** "Swiss flag"

To make this formal, we call a continuous path[2] $\alpha : \boldsymbol{I} \rightarrow X \subset \boldsymbol{I}^n$ from $\mathbf{x} = \alpha(0)$ to $\mathbf{y} = \alpha(1)$ a *dipath* (directed path) if every composition $pr_i \circ \alpha$ is increasing. We introduce a new partial order $\preceq$ on $X$ by

$$\mathbf{x} \preceq \mathbf{y} \Leftrightarrow \text{ there is a dipath } \alpha \text{ from } \mathbf{x} \text{ to } \mathbf{y} \text{ in } X.$$

As can be seen e.g. in Fig. 1, this partial order is in general finer than the one $X$ inherits from the usual partial order on $I\!R^n$.

An element $\mathbf{x} \in X$ is called *admissible* if $\mathbf{x} \preceq \mathbf{1}$ and *unsafe* else. An element $\mathbf{y} \in X$ is called *reachable* if $\mathbf{0} \preceq y$ and *unreachable* else. An element $\mathbf{x} \in X$ is called a *deadlock* if $\mathbf{x} \preceq \mathbf{y} \Rightarrow \mathbf{y} = \mathbf{x}$; cf. Fig. 1.

To formulate results, we need to introduce $k$-element intersections of the hyperrectangles $R_i$ forming part of the forbidden region $F = \bigcup_1^r R^i$: For any

[2] In the rest of this note we distinguish between the interval $\boldsymbol{I}$ with the *usual* order as partial order and the interval $I$ with the equality relation as partial order.

non-empty index set $J = \{i_1, \dots, i_k\} \subset \{1, \dots, r\}$ let $R^J = R^{i_1} \cap \dots \cap R^{i_k}$. Unless $R^J = \emptyset$, it is again a hyperrectangle $R^J = (a_1^J, b_1^J) \times \dots \times (a_n^J, b_n^J)$ with $a_j^J = \max\{a_j^i \mid i \in J\}$ and $b_j^J = \min\{b_j^i \mid i \in J\}$. The minimal vertex of $R^J$ is given by $\mathbf{a}^J = (a_1^J, \dots, a_n^J)$. Moreover, let $\tilde{a}_j^J$ denote the "second largest" of the j-th coordinates $a_j^i$; we consider also the "corner" $Us^J = (\tilde{a}_1^J, a_1^J] \times \cdots (\tilde{a}_n^J, a_n^J] \subset X$.

**Proposition 1.** *1. An element $\mathbf{x} \neq \mathbf{1}$ in the interior of $\boldsymbol{I}^n$ is a deadlock if and only if there is an $n$-*element *index set $J = \{i_1, \dots, i_n\}$ with $R^J \neq \emptyset$ and $\mathbf{x} = \mathbf{a}^J = \min R^J$.*
*2. If $\mathbf{x} = \min R^J$ is a deadlock, then all elements of the $n$-hyperrectangle $Us^J$ are unsafe.*

A simple trick allows to detect deadlock points that are contained in the boundary of $I^n$ as well; cf. [FGR98] and also Sect. 6. In a similar way, one can find an unreachable region $Ur^J$ "above" the maximal element of an $n$-intersection $R^J$.

In [FGR98], we describe a fast incremental algorithm, that detects the *entire* unsafe region (consisting of all unsafe elements in $X$) in few steps – usually, many (discrete) states are detected in one single step. One has to take into account the (order) combinatorics of intersections of forbidden hyperrectangles and of those hyperrectangles that have found to be unsafe in previous steps. An implementation of this algorithm can be found on the URL `http://www.ens.fr/~goubault`.

## 3 The Dihomotopy Concept

An execution of a concurrent proces corresponds to a dipath (cf. Sect. 2) in the state space $X$. The most interesting dipaths are those starting at $\mathbf{0}$ and terminating at $\mathbf{1}$ (a complete run), but also dipaths starting and/or terminating at other elements need to be considered; both for practical purposes in state space analysis and as intermediate steps in theoretical calculations.

Many executions will "automatically" be equivalent; this means that all conceivable concurrent calculations along the corresponding schedules/paths yield the same result. In geometric language, this is the case when the dipaths corresponding to the executions are *dihomotopic*. Dihomotopy is a modification of the notion *homotopy* – which is fundamental and well-studied in Algebraic Topology – taking into account partial order. There are several definitions for dihomotopy, all of which are equivalent in the case of our simple partially ordered state space, cf. [Faj03]. We need to work with two of these definitions:

**Definition 1.** *Two continuous dipaths $\alpha_0, \alpha_1 : \boldsymbol{I} \to X$ from $\mathbf{x} \in X$ to $\mathbf{y} \in X$ are called* dihomotopic *if there exists a continuous 1-parameter deformation* (dihomotopy) $H : \boldsymbol{I} \times I \to X$ *with $H(0,t) = \mathbf{x}, H(1,t) = \mathbf{y}$ for all $t \in I$ and $H(s,0) = \alpha_0(s), H(s,1) = \alpha_1(s)$ for all $s \in \boldsymbol{I}$ and such that for all $t$, the "interpolating" paths $\alpha_s : t \mapsto H(s,t)$ are* dipaths.

Only the last requirement is characteristic for a dihomotopy compared to a homotopy (with fixed ends). Examples (cf. [FGR99] or Example 1 below) show, that dihomotopy in general is a finer relation than homotopy of dipaths. It is important to notice, that dihomotopy in general does *not* satisfy a cancellation property: $\alpha * \beta_1$ dihomotopic to $\alpha * \beta_2$ does not always imply that $\beta_1$ is dihomotopic to $\beta_2$. Examples are given in [FGR99]; you may also construct one from Example 1 below.

In the case of the state space of a mutual exclusion model (more generally, for a cubical complex), one may restrict attention to dipaths on the *1-skeleton* of $X$ and to *combinatorial* dihomotopies [FGR99]. To explain these terms in our simple case, one considers the projections of all hyperrectangles within the forbidden region to the coordinate axes. These axes are then subdivided into intervals. The 1-skeleton corresponding to the subdivision consists of the line sections parallel to one of the axes and constant at one of the subdivision points for all other directions. A (locally serial) dipath along this 1-skeleton proceeds at every time along one of these line sections. An *elementary* dipath proceeding "one step" parallel to the $x_i$-axis will be denoted $\sigma_i$ (This notation is not unambiguous, but good enough for our purposes). Two such dipaths $\sigma_i, \sigma_j$ can be concatenated to yield $\sigma_i * \sigma_j$ if the target of the first agrees with the source of the second.

**Definition 2.** *1. Two dipaths $\alpha_0 = \sigma_i * \sigma_j$ and $\alpha_1 = \sigma_j * \sigma_i$ in $X$ with the same source* $\mathbf{x}$ *and target* $\mathbf{y}$ *are called* elementarily dihomotopic *if the 2-dimensional rectangle with lower vertex in* $\mathbf{x}$ *and upper vertex in* $\mathbf{y}$ *is contained in $X$.*

*2. Dihomotopy is obtained from elementary dihomotopy via concatenation and reflexive and transitive closure.*

An elementary dihomotopy (given by such a rectangle in the state space) reflects the fact that the result of the compound execution of $\sigma_i$ and $\sigma_j$ is independent of the order in which these are performed (even after a possible subdivision into smaller parts).

In the "Swiss flag" example from Fig. 1 in Sect. 2, there are two dihomotopy classes of dipaths connecting $\mathbf{0}$ and $\mathbf{1}$. A complete classification algorithm for dipaths up to dihomotopy in *2*-dimensional models had been given in [Rau00].

## 4 Dihomotopy and Deadlocks in Mutual Exclusion Models

The purpose of this section is to make a link between the detection of deadlocks and unsafe regions in mutual exclusion models and the occurence of non-dihomotopic dipaths in such models. It had been conjectured for a long time, that $(n-1)$ intersecting $n$-rectangles should likewise give rise to non-trivial non-local dihomotopy.[3] We discuss here when and why this in fact is the case.

[3] Even a single $n$-rectangle in the forbidden region creates dihomotopy, but only between points that are "sufficiently close" to that $n$-rectangle, cf. the discussion in dimension 3 in [FGHR04].

Forgetting about the last coordinate (processor) amounts to projecting the forbidden hyperrectangles and the forbidden region under $\pi : I\!R^n \to I\!R^{n-1}$, $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_{n-1})$, arriving at a forbidden region $\bar{F} = \pi(F)$ and a state space $\bar{X} = \boldsymbol{I}^{n-1} \setminus \bar{F}$ (different from $\pi(X)$, in general!)

Let us compare the forbidden regions in $X$ and in $\bar{X}$: Consider an $(n-1)$-element index set $J$ with non-empty intersection hyperrectangle $R^J \subset F$. If the participating hyperrectangles intersect *generically* – in particular, if $R^J \neq R^K$ for every smaller index set $K \subset J$ – then every of the $(n-1)$ hyperrectangles $R_i$ will "contribute" at least one coordinate to the minimum $\mathbf{a}^J$ of $R^J$ – and similarly to its maximum $\mathbf{b}^J$. We may then suppose without restriction, that

$$a_1^J = a_1^1, \dots, a_{n-2}^J = a_{n-2}^{n-2}, a_{n-1}^J = a_{n-1}^{n-1}, a_n^J = a_n^{n-1}.$$

The $(n-1)$ hyperrectangles $\pi(R^i)$ in $\boldsymbol{I}^{n-1}$ intersect in $\pi(R^J) = \pi(R)^J$, a hyperrectangle with minimal vertex $\pi(\mathbf{a}^J) = (a_1^J, \dots, a_{n-1}^J)$, which is a *deadlock* for the model space $\bar{X}$. The intersection $\pi(R^J)$ gives furthermore rise to an unsafe region $Us(\pi(R^J)) \subset \bar{X}$. In a similar way, we can consider the projection $\pi' : I\!R^n \to I\!R^{n-1}, (x_1, \dots, x_n) \mapsto (x_1, \dots, x_{n-2}, x_n)$, giving rise to the deadlock $\pi'(\mathbf{a}^J) = (a_1^J, \dots, a_{n-2}^J, a_n^J)$ and the unsafe region $Us(\pi'(R^J)) \subset \boldsymbol{I}^{n-1} \setminus \pi'(F)$.

**Lemma 1.** *Let* $\mathbf{x}, \mathbf{y} \in X$ *satisfy*

$$(x_1, \dots, x_{n-1}) \in Us(\pi(R)^J) \text{ or } (x_1, \dots, x_{n-2}, x_n) \in Us(\pi'(R)^J), \mathbf{x} \preceq \mathbf{a}^J, \mathbf{b}^J \preceq \mathbf{y}.$$

*Then there exist at least* two non-dihomotopic *dipaths from* $\mathbf{x}$ *to* $\mathbf{y}$ *in* $X$*.*

An instructive example is given by two dipaths from $\mathbf{a}^J$ to $\mathbf{b}^J$: While all the other coordinates remain fixed, for the first dipath we let first $a_{n-1}$ grow to $b_{n-1}$ and after that $a_n$ to $b_n$; in the second, the $n$th coordinate grows before the $(n-1)$st. Remark that there are no *upward* restrictions for the *end* point $\mathbf{y}$. There is a similar result for non-dihomotopic dipaths with the only restriction that $(n-1)$ of the *end* point's coordinates are contained in the *unreachable* region of a projected arrangement of $(n-1)$ hyperrectangles.

A *single* arrangement of $(n-1)$ intersecting hyperrectangles will in general not lead to non-dihomotopic dipaths from $\mathbf{0}$ to $\mathbf{1}$. This can be seen e.g. for the state space with a *single* wedge (cf. Example 1 below) as the forbidden region. We have to consider (at least) *two* arrangements consisting of $(n-1)$ intersecting $n$-rectangles each within the forbidden region $F$; as usual, $X = \boldsymbol{I}^n \setminus F$. Suppose that those intersect in $n$-rectangles $R^J = \pi(R^J) \times (a_n, b_n)$, resp. $R^K = \pi(R^K) \times (c_n, d_n)$ such that $a_n < d_n$.

An application of Marco Grandis' version of the van Kampen theorem for the fundamental category in directed homotopy [Gra03] yields

**Proposition 2.** *Suppose that the intersection hyperrectangle*

$$C = Ur(\pi(R)^J) \cap Us(\pi(R)^K) \subset \bar{X} \subset \boldsymbol{I}^{n-1}$$

*is* disconnected (no continuous path!) *from both* $\mathbf{0}$ *and* $\mathbf{1}$*. If there is a dipath* $\alpha : \boldsymbol{I} \to X$*, such that* $a_n < \alpha_n(t) < d_n \Rightarrow \pi(\alpha((t)) \in C$*, then there are at least* *two* non-dihomotopic *dipaths from* $\mathbf{0}$ *to* $\mathbf{1}$ *in* $X$*.*

From an application point ov view, this implies the existence of different terminating schedules that can yield different results of distributed calculations.

*Example 1.* The situation from Prop. 2 arises in 3 dimensions, when the forbidden region is a *cylinder* (with a quadrangle as cross-section). More strikingly, there are state spaces with *trivial fundamental group*, that allow non-dihomotopic dipaths: It suffices to consider a forbidden region consisting of two “wedges”, one behind the other and not connected to each other; one of them yields a deadlock after projection (to the “front”) and the other unreachable points; cf. Fig. 2 below.

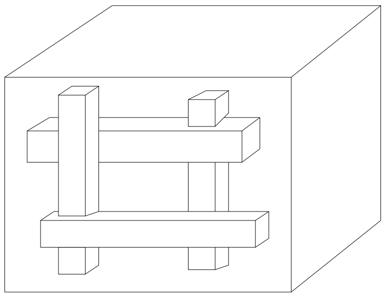

**Fig. 2.** Two wedges

## 5 Trivial dihomotopy for models with less complicated constraints

In contrast, for a model space with a less complicated forbidden region, we can show by a simple essentially combinatorial argument and using the characterisation of dihomotopy from Def. 2:

**Proposition 3.** *For a model space $X$ with the property that $R^J = \emptyset$ for all index sets $J$ of cardinality $n-1$,* every *two dipaths from* $\mathbf{0}$ *to* $\mathbf{1}$ *are dihomotopic to each other.*

A similar result holds also in the classical non-directed case: Using duality and Čech-type cohomology, it is easy to see that the complement of a forbidden region with $R^J = \emptyset$ for all index sets $J$ of cardinality $n-1$ has a trivial first *homology* group.

From an application point of view, the criterion from Prop. 3 is easy to check and ensures that *all* runs in such a distributed calculation will yield the *same* result. This should also be interesting for data base scheduling; compare [Gun94] and [FGR99], Sect. 8.

## 6 Concluding Remarks. Future Work

As said earlier, there may be non-trivial dihomotopy between intermediate states although dihomotopy between the initial and the terminal state is trivial. But it is not difficult to find criteria for when this happens and when not: If one considers dihomotopy between $\mathbf{y} \in X$ and $\mathbf{z} \in X$, one looks at a restricted state space, i.e., what is left in the hypercube with these two points as its lower, resp. upper vertex after removal of the forbidden region $F$. As in [FGR98] or [Rau00], one may instead introduce the $2n$ additional $n$-hyperrectangles given by $x_i < y_i$, resp. $x_i > z_i$, to adjust (enlarge) the forbidden region in the *original* hypercube $\boldsymbol{I}^n$. This yields, in general, more complicated intersections, unsafe regions etc., which are "responsible" for extra "local" dihomotopy.

It should then also be interesting to see how the components of the fundamental category of $X$ from [FGHR04] relate to this approach.

The ultimate goal for the work initiated here is the construction of an algorithm determining the set of dihomotopy classes between two given states, building on the deadlock algorithm from [FGR98] and generalising the algorithm given in [Rau00] in the two-dimensional case. To this end, one has to investigate the "directed combinatorics" between situations as they arise in Prop. 2 more closely. Moreover, the (non-orientable) (non)connectivity of $C$ from Prop. 2 has to be determined algorithmically. But this relies merely on a determination of (non)-reachability between certain deadlocking, resp. unreachable states. Nevertheless, it can be complicated to determine (the existence of) a path connecting a point in central $C$ with $\mathbf{0}$, resp. $\mathbf{1}$, as you can see in Fig. 3 below.

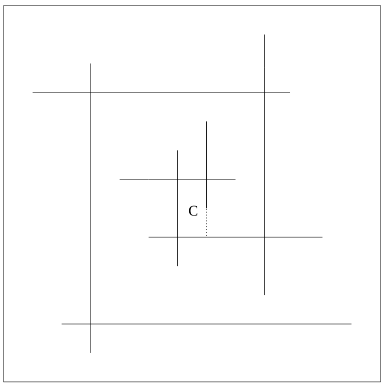


**Fig. 3.** A labyrinth state space

All proofs and an outline of the algorithm above are deferred to a later paper.

## References


[Dij68] E.W. Dijkstra, *Co-operating sequential processes*, Programming Languages (F. Genuys, ed.), Academic Press, New York, 1968, pp. 43–110.

[Faj03] L. Fajstrup, *Dipaths and dihomotopies in a cubical complex*, Tech. Report R-2003-22, Department of Mathematical Sciences, Aalborg University, DK-9220 Aalborg Øst, 2003.

[FGHR04] L. Fajstrup, É. Goubault, E. Haucourt, and M. Raussen, *Components of the fundamental category*, Appl. Categ. Structures **12** (2004), no. 1, 81–108.

[FGR98] L. Fajstrup, É. Goubault, and M. Raussen, *Detecting Deadlocks in Concurrent Systems*, CONCUR '98; Concurrency Theory (Nice, France) (D. Sangiorgi and R. de Simone, eds.), Lect. Notes Comp. Science, vol. 1466, Springer-Verlag, September 1998, 9th Int. Conf., Proceedings, pp. 332 – 347.

[FGR99] ———, *Algebraic topology and concurrency*, Tech. Report R-99-2008, Department of Mathematical Sciences, Aalborg University, DK-9220 Aalborg Øst, June 1999, conditionally accepted for publication in Theoret. Comput. Sci.

[FS00] L. Fajstrup and S. Sokolowski, *Infinitely running concurrents processes with loops from a geometric viewpoint*, Electronic Notes Theor. Comput. Sci. **39** (2000), no. 2, 19 pp., URL: http://www.elsevier.nl/locate/entcs/volume39.html.

[Gou95] É. Goubault, *The Geometry of Concurrency*, Ph.D. thesis, Ecole Normale Superieure, Paris, 1995.

[Gra03] M. Grandis, *Directed Homotopy Theory I. The Fundamental Category*, Cahiers Topologie Géom. Différentielle Catégoriques **44** (2003), 281 – 316.

[Gun94] J. Gunawardena, *Homotopy and concurrency*, Bulletin of the EATCS **54** (1994), 184–193.

[Pra91] V. Pratt, *Modelling concurrency with geometry*, Proc. of the 18th ACM Symposium on Principles of Programming Languages. (1991), 311–322.

[Rau00] M. Raussen, *On the classification of dipaths in geometric models for concurrency*, Math. Structures Comput. Sci. **10** (2000), no. 4, 427–457.