Some Notes On “When is 0.999... equal to 1?”

Carsten Schneider*

Research Institute for Symbolic Computation
J. Kepler University Linz
A-4040 Linz, Austria
Carsten.Schneider@risc.uni-linz.ac.at

Abstract. In joint work Robin Pemantle and I (2004) consider a doubly infinite sum which is not equal to 1, as first suspected, but evaluates to a sum of products of values of the zeta function. Subsequently, I report on this project.

Keywords. Symbolic summation, computer algebra, proofs, harmonic numbers, zeta-relations

During my talk at the Dagstuhl-Seminar “Mathematics, Algorithms, Proofs” (09.01.-14.01.2005) I presented results from [1], namely a doubly infinite sum which, numerically evaluated at between 0.999 and 1.001, turns out not to equal 1, but to be a sum of products of values of the zeta function.

This joint investigation started in July 2004 when Robin Pemantle asked the following question to Herbert Wilf and Doron Zeilberger:

> I have a sum that, when I evaluate numerically, looks suspiciously like it comes out to exactly 1. Is there a way I can automatically decide this? The sum may be written in many ways, but one is[1]:
>
> $$S := \sum_{j,k=1}^{\infty} \frac{H_j(H_{k+1}-1)}{jk(k+1)(j+k)}. \qquad (1)$$
>
> Of course you can expand out the H’s and get a quadruple sum. There are zillions of ways to play with it, summing by parts, but I have never managed to get rid of all the summations.

D. Zeilberger replied to R. Pemantle and H. Wilf as follows:

> I am willing to bet that Carsten Schneider’s Sigma package for handling sums with harmonic numbers (among others) can do it in a jiffy. [...] Carsten: please do it, and Cc- the answer to me.

* Supported by the SFB-grant F1305 and the grant P16613-N12 of the Austrian FWF.

[1] $H_j = \sum_{i=1}^{j} \frac{1}{i}$ denotes the harmonic numbers.

After receiving these emails I was eager to win the bet, i.e., in solving the posed problem. As it turned out, this task was not as simple as it was supposed to be[2].

Finally, after one week of various attempts I managed to show that the sum (1) is not 1, but evaluates to

$$S = -4\zeta(2) - 2\zeta(3) + 4\zeta(2)\zeta(3) + 2\zeta(5) \simeq 0.99922283776383000876 \qquad (2)$$

where $\zeta(r) = \sum_{i=1}^{\infty} \frac{1}{i^r}$ denotes the Riemann zeta function.

Within these computations, see [1] for all the details, the crucial step consists of deriving the identity

$$S = \lim_{a,b\to\infty} (A(a,b) + B(a,b) + C(a,b)) \qquad (3)$$

where[3]

$$\begin{aligned} A(a,b) :=& \frac{1}{2(b+1)^2}\Big(6H_b + 4bH_b + 4H_b^2 + 3bH_b^2 + H_b^3 + bH_b^3 - 6bH_a^{(2)} \\ &+ 2H_bH_a^{(2)} + 2bH_bH_a^{(2)} - 2H_b^{(2)} - 7bH_b^{(2)} + H_bH_b^{(2)} + bH_bH_b^{(2)}\Big), \\ B(a,b) :=& -\frac{2b^2}{(b+1)^2}\Big(H_a^{(2)} + H_b^{(2)}\Big) \end{aligned}$$

and

$$C(a,b) := (H_a^{(2)} - 1)\sum_{i=1}^{b} \frac{H_i}{i^2} - \sum_{i=1}^{b} \frac{H_i^2}{i^3} + \frac{1}{2}\sum_{i=1}^{b} \frac{H_i^3}{i^2} + \frac{1}{2}\sum_{i=1}^{b} \frac{H_i H_i^{(2)}}{i^2}.$$

Note that (3) has been found with the summation package Sigma [2] implemented in the computer algebra system Mathematica. More precisely, using the algorithms of Sigma I could apply the summation principles from [3] (telescoping, creative telescoping and solving recurrences) to (1) in order to derive (3). Moreover, I want to emphasize that Sigma not only finds (3), but also provides proof certificates that enable the user to verify (3) in easy steps; see [1].

Given this result, simple limit considerations show that

$$\lim_{a,b\to\infty} A(a,b) = 0 \qquad \text{and} \qquad \lim_{a,b\to\infty} B(a,b) = -4\zeta(2).$$

Finally, using zeta-relations from [4,5,6] I was capable of deriving the evaluation

$$\lim_{a,b\to\infty} C(a,b) = -2\zeta(3) + 4\zeta(2)\zeta(3) + 2\zeta(5);$$

[2] First, I looked up the expression "in a jiffy" in a dictionary and found out that I should solve the problem in a "moment" or "instant". (Actually, this was the only task that I managed to do in a jiffy.)

[3] $H_j^{(2)} = \sum_{i=1}^{j} \frac{1}{i^2}$ denotes the generalized harmonic numbers.

see [1]. This result produces (2).

To this end, when I reported on my results, I obtained the following concluding remark from D. Zeilberger:

> Wow, you (and your computer!) are wizhes! I suggest that Carsten and Robin write a short [...] paper, that will serve, among other things, as a cautionary tale not to confuse .99999 with 1, and also the sad fact, that, at least for now, Carsten had to cheat and use some human-previously-proved identities, and hence the proof is not fully rigorous (from my point of view, since it uses human mathematics).
> Anyway, even though the bet was one sided, I still feel that Robin and/or Herb owe me a free lunch (and they owe Carsten, and his computer, a free dinner).

References

1. Pemantle, R., Schneider, C.: When is 0.999... equal to 1. SFB-Report 2004-30, J. Kepler University, Linz (2004)
2. Schneider, C.: The summation package Sigma: Underlying principles and a rhombus tiling application. Discrete Math. Theor. Comput. Sci. **6** (2004) 365–386
3. Petkovšek, M., Wilf, H.S., Zeilberger, D.: $A = B$. A. K. Peters, Wellesley, MA (1996)
4. Borwein, D., Borwein, J., Girgensohn, R.: Explicit evaluation of Euler sums. In: Proceedings of the Edinburgh Mathematical Society. Volume 38. (1995) 277–294
5. Borwein, J., Girgensohn, R.: Evaluation of triple Euler sums. Electron. J. Combin. **3** (1996) 1–27
6. Flajolet, P., Salvy, B.: Euler sums and contour integral representations. Experimental Math. **7** (1998) 15–35