Vertex reconstruction in Cayley graphs

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Abstract

In this report paper we collect recent results on the vertex reconstruction in Cayley graphs $Cay(G, S)$. The problem is stated as the problem of reconstructing a vertex from the minimum number of its r-neighbors that are vertices at distance at most r from the unknown vertex. The combinatorial properties of Cayley graphs on the symmetric group S_n and the signed permutation group B_n with respect to this problem are presented. The sets S of generators are specified by applications in coding theory, computer science, molecular biology and physics.

Key words: reconstruction problems; Cayley graphs; symmetric group; signed permutation group; sorting by reversals; pancake problem;

1 Introduction

The basic question asked in classical combinatorial reconstruction problems is whether certain information about the isomorphism types of the subobjects of an unknown object allows to reconstruct this object up to isomorphism. Graph reconstruction problems based on the decks of a graph are important instances of this situation [1–3]. Some reconstruction problems arise naturally, for instance in the representation theory of symmetric and Lie groups when a partition of an integer is reconstructible from certain of its subpartitions [4–6]. This problem is connected to the efficient reconstruction of a sequence from its sub– and supersequences which has been considered in coding theory [7]. The efficient reconstruction of arbitrary sequences has been investigated in [8,9] for combinatorial channels with errors of interest in coding theory such as substitutions, transpositions, deletions and insertions of symbols. Sequences

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are considered as a vertex set V of a graph $\Gamma = (V, E)$, where $(x, y) \in E$ if there exist single errors of the type under consideration which transform x into y and y into x. Then the corresponding efficient reconstruction problem can be treated as the following graph theory problem. Given a graph $\Gamma = (V, E)$ and an integer r, is there the minimum number N of vertices in the metric balls $B_r(x)$ such that an arbitrary vertex $x \in V$ can be identified from any $N + 1$ distinct vertices in $B_r(x)$? It is reduced to finding the value

$$N(\Gamma, r) = \max_{x,y \in V(\Gamma),\ x \neq y} |B_r(x) \cap B_r(y)|, \tag{1}$$

since $N(\Gamma, r) + 1$ is the minimum number of distinct vertices in the metric ball $B_r(x)$ of an unknown vertex $x \in V$ which are sufficient to reconstruct this vertex x under the condition that at most r single errors were happened. The vertices from the set $B_r(x)$ are r-neighbors of the vertex x.

Reconstruction problems mentioned above deal with some data (graphs, partitions, sequences) which are distorted by some operations on the data. These investigations were continued for reconstructing permutations and signed permutations from their erroneous patterns which are distorted by transpositions or reversals [10–13], where reversals are the operations reversing the order of a substring of a permutation. From the graph–theoretical point of view this is the problem of reconstructing vertices from the minimum number of their r-neighbors in Cayley graphs when the symmetric group S_n and the signed permutation group B_n are considered as a group, and the sets of generators are specified by transpositions and reversals. Since vertex sets of Cayley graphs are presented by the group elements, at that time one can say that we have the problem of reconstructing group elements which considered in [14].

These graphs, groups and generators are of exclusive interest in molecular biology, computer science and physics. Why permutations are considered with respect to these operations? In molecular biology, permutations and signed permutations are used to represent sequences of genes in chromosomes and genomes as well. Some of the operations on permutations called *genome rearrangements* represent evolutionary events. In the 1980's it was shown that the difference in genomes may be explained by a small number of reversals. The problem of determining the smallest number of reversals transforming a given permutation into the identity permutation is called *sorting by reversals* [15].

Permutations are also used in the representation of interconnection networks which are modeled by Cayley graphs generated by transpositions and reversals [16]. The vertices in such Cayley graphs correspond to processing elements, memory modules, or just switches, and the edges correspond to communication lines. The main advantage in using Cayley graphs as models for interconnection networks is their vertex–transitivity which makes it possible to implement the same routing and communication schemes at each node (vertex) of the network they model. Furthermore, some of them have other advantages such as edge–transitivity (line symmetry), hierarchical structure (allowing recursive construction), high fault tolerance and so on [17–19].

The Cayley graph on the symmetric group S_n generated by all transpositions swapping any two neighbors elements of a permutations is an important instance in combinatorics of Coxeter groups (see [20]). This set of generators is also known as the set of the $(n-1)$ Coxeter generators of S_n. The combinatorial properties of this graph are fundamental to physics and Lie theory. In computer science this graph is called the bubble–sort Cayley graph.

The Cayley graphs on permutations and signed permutations generated by the prefix–reversals of any substring $[1,i],\ i \leq n$, are known with respect to the open combinatorial *pancake problem* of sorting the pancakes in increasing order by diameter [21]. The only possible action is to lift the top of the stack, flip it over and place it back on the top. A stack of n pancakes is represented by a permutation on n elements and the problem is to find the minimum number of flips (prefix–reversals) transforming a permutation into the identity permutation. In molecular biology this problem is called *sorting by prefix–reversal*. It has also practical applications in parallel processing [17].

The main task of this report paper is to observe the main combinatorial properties of Cayley graphs on S_n and B_n with respect to the problem of reconstructing a vertex from the minimum number of its r-neighbors.

The paper is organized as follows. Firstly, we give the main definitions, notations and general results. Then we describe the main results for Cayley graphs on the symmetric group S_n generated by: 1) all transpositions (this graph is called the *transposition network* in computer science); 2) transpositions of two neighbors elements (the *bubble–sort Cayley graph*); 3) prefix–transpositions swapping the first element of a permutation with any other element of a permutation (the *star Cayley graph*); 4) all reversals; and 5) the prefix–reversals on substring $[1,i],\ i \leq n$, of a permutation (the *unburnt pancake Cayley graph*).

Then we observe Cayley graphs on the signed permutation group B_n. The main difference in this case is that the transpositions and reversals on signed permutations are considered with flipping signs of swapping and reversing elements. We call such operations as the *sign–change transpositions* and *sign–change reversals*, and present results for Cayley graphs generated by: 1) all sign–change transpositions; 2) the sign–change transpositions of two neighbors elements; 3) the sign–change prefix–transpositions swapping the first element of a permutation with any other element of a permutation; 4) all sign–change reversals; and 5) the sign–change prefix–reversals. In the last case the corresponding Cayley graph is called the *burnt pancake Cayley graph* [22,23].

2 Definitions, notations, general results

Let G be a finite group and let $S \subset G$ be a set of generators such that the identity element e of G does not belong to S, i.e. $e \notin S$, and $S = S^{-1}$, where $S^{-1} = \{s^{-1} : s \in S\}$. In the *Cayley graph* $\Gamma = Cay(G, S) = (V, E)$ the vertices correspond to the elements of the group, i.e. $V = G$, and the edges correspond to the action of the generators, i.e. $E = \{(g, gs) : g \in G, s \in S\}$. The condition $e \notin S$ is imposed so that there are no loops in Γ. Also, S is required to be a generating set of G so that Γ is connected. The basic facts about Cayley graphs are recorded in the following

Lemma 1 *Let S be a set of generators for a group G. The Cayley graph $Cay(G, S)$ has the following properties:*
(i) it is a connected regular graph of degree equal to the cardinality of S;
(ii) it is a vertex–transitive graph.

Denote by $d(x, y)$ the path distance between vertices x and y in a graph and by $d(\Gamma) = max_{x,y \in V(\Gamma)} d(x, y)$ the diameter of Γ. Another words, the diameter of a Cayley graph is the maximum over $g \in G$ of the length of the shortest expression of g as a product of the generators. For the vertex $x \in V(\Gamma)$ let $S_r(x) = \{y \in V(\Gamma), d(x, y) = r\}$ and $B_r(x) = \{y \in V(\Gamma), d(x, y) \leq r\}$ be the sphere and ball of radius r centered at the vertex x respectively. Then all vertices $y \in B_r(x)$ are *r-neighbors* of the vertex x. As it was mentioned in Introduction, reconstructing an unknown vertex x from its r-neighbors is reduced to finding the value (1) which was initially studied in coding theory for the Hamming and Johnson graphs. These graphs are distance–regular graphs as well as Cayley graphs. Let us recall that a simple connected graph Γ is called *distance–regular* if there are integers $b_i, c_i, i \geq 0$, such that for any two vertices x and y at distance $i = d(x, y)$, there are precisely c_i neighbors of y in $S_{i-1}(x)$ and b_i neighbors of y in $S_{i+1}(x)$. In particular, Γ is *regular* of valency $k = b_0$, or *k-regular.* A distance-regular graph of diameter 2 with v vertices, not complete or null, is *a strongly regular graph* with parameters (v, k, λ, μ) in which the number of common neighbors of x and y is k, λ or μ according as x and y are equal, adjacent or non-adjacent respectively.

The Hamming graph $L_n(q)$ is defined on the Hamming space F_q^n consisting q^n vectors of length n over the alphabet $\{0, 1, ..., q-1\}$, $q \geq 2$, with Hamming distance given by the number of distinct coordinates of vectors x and y. The vertex set of $L_n(q)$ is presented by vectors of F_q^n and two vectors in $L_n(q)$ are connected by an edge if and only if they differ in one coordinate. The Hamming graph is the Cayley graph on $G = F_q^n$ with the set of generators $S = \{xe_i : x \in (F_q)^\times, 1 \leq i \leq n\}$, where the $e_i = (0, ..., 0, 1, 0, ...0)$ are the standard basis vectors of F_q^n.

Theorem 1 *[8], [9] For any n, q and r,*

$$N(L_n(q), r) = q \sum_{i=0}^{r-1} \binom{n-1}{i} (q-1)^i \tag{2}$$

In the particular case, when $n = 2$, the Hamming graph $L_2(q)$ is called *the lattice graph*. $L_2(q)$ is strongly regular with parameters $v = q^2$, $k = 2(q-1)$, $\lambda = q - 2$, $\mu = 2$, and from (2) we get $N(L_2(q), 1) = q$ and $N(L_2(q), 2) = q^2$.

The Johnson graph J_e^n is defined on the Johnson space J_e^n which consists of $\binom{n}{e}$ binary vectors of length $n \geq 2$, $1 \leq e \leq n-1$. The Johnson distance equals half of the Hamming distance. The vertex set of J_e^n is presented by vectors of the Johnson space and two vectors are connected by an edge if and only if they obtain one from other by transposition of two symbols. Thus the Johnson graph is the Cayley graph on $G = J_e^n$ with the set of generators presented by all possible transpositions of two any symbols of a binary vector.

Theorem 2 *[8], [9] For any n, e and r,*

$$N(J_e^n, r) = n \sum_{i=0}^{r-1} \binom{e-1}{i} \binom{n-e-1}{i} \frac{1}{i+1} \tag{3}$$

In the particular case, when $e = 2$, the Johnson graph J_2^n is called *the triangular graph* $T(n), n \geq 4$. It has as vertices the 2-element subsets of an n-set. Two vertices are adjacent if they are not disjoint. $T(n)$ is strongly regular with parameters $v = \frac{n(n-1)}{2}$, $k = 2(n-2)$, $\lambda = n-2$, $\mu = 4$, and from (3) we obtain $N(T(n), 1) = n$ and $N(T(n), 2) = \frac{n(n-1)}{2}$.

The presented two results were the first analytic formulas which were obtained in the vertex reconstruction we are interested in. What are general results for simple graphs and Cayley graphs? We give a few observations from [14] for any connected simple graphs $\Gamma = (V, E)$. Let us define numbers $c_i(x, y), b_i(x, y)$ and $a_i(x, y)$ for any two vertices $x \in V$ and $y \in S_i(x)$ by analogy with the corresponding numbers of distance–regular graphs such that $c_i(x, y) = |\{z \in S_{i-1}(x) : d(z, y) = 1\}|$, $b_i(x, y) = |\{z \in S_{i+1}(x) : d(z, y) = 1\}|$,$a_i(x, y) = |\{z \in S_i(x) : d(z, y) = 1\}|$. From this, $a_1(x, y)$ is the number of triangles with the edge (x, y) and $c_2(x, y)$ is the number of common neighbors of $x \in V$ and $y \in S_2(x)$. Let $\lambda = \lambda(\Gamma) = \max_{x \in V,\ y \in S_1(x)} a_1(x, y)$ and $\mu = \mu(\Gamma) = \max_{x \in V,\ y \in S_2(x)} c_2(x, y)$. Since $|B_r(x) \cap B_r(y)| > 0$ for $x \neq y$ if and only if $1 \leq d(x, y) \leq 2r = d(\Gamma)$, we have

$$N(\Gamma, r) = \max_{1 \leq s \leq 2r} N_s(\Gamma, r) \tag{4}$$

where $N_s(\Gamma, r) = \max_{x,y \in V;\ d(x,y)=s} |B_r(x) \cap B_r(y)|$. In particular, $N_1(\Gamma, 1) = \lambda + 2$ and $N_2(\Gamma, 1) = \mu$ so that

$$N(\Gamma, 1) = \max(\lambda + 2, \mu). \tag{5}$$

One can easily check that by using this formula for the lattice graph $L_2(q)$ and the triangular graph $T(n)$ we again obtain the same results as we have got from formulas (1) and (2). Indeed, since $\lambda = n - 2$ and $\mu = 4$ for $T(n), n \geq 4$, then from (5) we have $N(T(n), 1) = n$ for this graph. By the same reason we have $N(L_2(q), 1) = q$ since $\lambda = q - 2$ and $\mu = 2$ in this case.

Now let us assume that Γ is a Cayley graph $Cay(G, S)$ with identity element $e \notin S$. Let us put $S^0 = \{e\}$ and set $S^i = SS^{i-1}$. Moreover, by transitivity it is sufficient to consider only the spheres and balls with center e so that $S_i = S_i(e)$.

Lemma 2 *[14] For any Cayley graph Γ of a group G and for $i > 0$ we have $S_i = S^i \setminus (S^{i-1} \cup S^{i-2} \cup ... \cup S^0)$. In particular, μ is the maximum number of representations of an element in $S^2 \setminus (S \cup e)$ as a product of two elements of S and λ is the maximum number of representations of an element in S as a product of two elements of S, i,e.*

$$\lambda(\Gamma) = \max_{s \in S} \mid \{(s_i s_j) \in S^2 \ : \ s = s_i s_j\} \mid, \tag{6}$$

$$\mu(\Gamma) = \max_{s \in S^2 \setminus (S \cup e)} \mid \{(s_i s_j) \in S^2 \ : \ s = s_i s_j\} \mid. \tag{7}$$

This Lemma allows to find $N(\Gamma, 1)$ from (5) for a general Cayley graph. The concrete results in finding and estimating the value $N(\Gamma, r)$ for Cayley graphs on the given groups are observed in the next sections.

Remark 1. Let us note here that one more problem which will be discussed below arises on Cayley graphs. This is the problem of establishing the diameter of a Cayley graph. General upper and lower bounds are very difficult to obtain. It is known [24] that every non–abelian finite simple group has a set of at most seven generators for which the diameter of the Cayley graph $Cay(G, S)$ is at most $c \log_2(|G|)$ where c is a constant. This property does not hold for Cayley graphs of abelian groups [25]. It was also proved [26] that the diameter of every Cayley graph of the symmetric group S_n or the alternating group A_n is at most $exp((n \ln n)^{(1/2)}(1 + o(1)))$. Computing the diameter of an arbitrary Cayley graph over a set of generators is NP–hard [27]. We will see below that in some cases it is possible to get an exact formula of the diameter of the corresponding Cayley graph. But in another cases the diameter is unknown and there are only bounds.

3 Cayley graphs on S_n generated by transpositions and reversals

In this section we consider Cayley graphs on the symmetric group S_n whose elements are permutations π written in one–line notation as $[\pi_1, \pi_2, \ldots, \pi_n]$ where $\pi_i = \pi(i)$ for every $i \in \{1, \ldots, n\}$. The presented graphs are generated by the transpositions $t_{i,j}, 1 \le i < j \le n$, where $t_{i,j}$ interchanges positions i and j when acting to the right, i.e., $[\ldots, \pi_i, \ldots, \pi_j, \ldots]t_{i,j} = [\ldots, \pi_j, \ldots, \pi_i, \ldots]$, and by the reversals $r_{i,j}, 1 \le i < j \le n$, which are the operations of reversing segments $[i, j]$ of a permutation, i.e., $[\ldots, \pi_i, \pi_{i+1}, \ldots, \pi_{j-1}, \pi_j, \ldots]r_{i,j} = [\ldots, \pi_j, \pi_{j-1}, \ldots, \pi_{i+1}, \pi_i, \ldots]$.

3.1 The transposition Cayley graphs $\mathrm{S}_n(T)$, $\mathrm{S}_n(t)$ *and* $\mathrm{S}_n(st)$

The *transposition Cayley graph* $\mathrm{S}_n(T)$ is defined on the symmetric group S_n and generated by the transpositions from the set $T = \{t_{i,j} \in \mathrm{S}_n,\ 1 \le i < j \le n\},\ |T| = \binom{n}{2}$. The distance in this graph is defined as the minimal number of transpositions transforming one permutation into another. The diameter is at most $(n-1)q$ since this number of transpositions suffice to transform any permutation of n elements into another. On the other hand, transforming the identity permutation I to an n-cycle requires $(n-1)$ transpositions. The graph is bipartite since the endpoints of every edge consist of an even and odd permutation. Thus, all these properties as well as other basic facts are collected in the following

Lemma 3 *[12] The transposition Cayley graph* $\mathrm{S}_n(T),\ n \ge 3$,
(i) is a connected bipartite $\binom{n}{2}$*-regular graph of order* $n!$ *and diameter* $(n-1)$*;*
(ii) it does not contain subgraphs isomorphic to $K_{2,4}$;
(iii) each its vertex belongs to $\binom{n}{3}$ *subgraphs isomorphic to* $K_{3,3}$,

where $K_{p,q}$ is a complete bipartite graph with p and q vertices in the two parts.

Theorem 3 *[12] [14] For any* $n \ge 3, \quad N(\mathrm{S}_n(T), 1) = 3.$

This means that any unknown permutation is uniquely reconstructible from 4 its distinct 1–neighbors. Proofs of these statements are based on considering a permutation $\pi \in \mathrm{S}_n$ in the cycle notation with the *cycle type* $\mathrm{ct}(\pi) = 1^{h_1}2^{h_2}..\,n^{h_n}$, where h_i is the number of cycles of length i and $\sum_i^n ih_i = n$. A permutation can be also presented as the product of a minimal number of transpositions which is equivalent to the number of distinct paths between two permutations in the transposition Cayley graph $\mathrm{S}_n(T)$. This number is based on Ore's theorem on the number of trees with n labeled vertices and presented by the following

Theorem 4 *[28] Let $\pi \in \mathrm{S}_n$ has cycle type* $\mathrm{ct}(\pi) = 1^{h_1}2^{h_2}...n^{h_n}$*, consisting of* $\sum_{j=1}^{n} h_j = n - i$, $1 \leq i \leq n-1$*, cycles, then the number of distinct ways to express π as a product of i transpositions is equal to*

$$i! \prod_{j=1}^{n} \left(\frac{j^{j-2}}{(j-1)!} \right)^{h_j}.$$

According to the above Theorem, the following Lemma gives us formulas on the numbers $c_i(\pi) = c_i(\pi, I)$, $b_i(\pi) = b_i(\pi, I)$, $a_i(\pi) = a_i(\pi, I), 1 \leq i \leq n-1$.

Lemma 4 *[14] In the transposition Cayley graph $\mathrm{S}_n(T)$ the sets $S_i = S_i(I), 1 \leq i \leq n-1$, are the permutations consisting of $(n-i)$ disjoint cycles. For any $\pi \in S_i$ with cycle type* $\mathrm{ct}(\pi) = 1^{h_1}2^{h_2}\dots n^{h_n}$,

$$c_i(\pi) = \frac{1}{2}\left(\sum_{j=1}^{n} j^2 h_j - n\right), \qquad b_i(\pi) = \frac{1}{2}\left(n^2 - \sum_{j=1}^{n} j^2 h_j\right), \qquad a_i(\pi) = 0.$$

In particular, since $a_i(\pi) = 0$ for any $1 \leq i \leq n-1$, then from this Lemma and by the definition of λ we have $\lambda(\mathrm{S}_n(T)) = 0$. Moreover, the well-known fact is that two permutations are conjugate by an element of S_n if and only if they have the same cycle type. It is shown in [14] that for any $\pi \in S_i, 1 \leq i \leq n-1$, such that $(1^{h_1}2^{h_2}\dots n^{h_n})^G$ is the conjugacy class of π, the set S_i, $1 \leq i \leq n-1$, is the disjoint union

$$S_i = \bigcup_{h_1+h_2+\dots+h_n=n-i} (1^{h_1}\, 2^{h_2}\, \dots\, n^{h_n})^G, \tag{8}$$

where

$$|(1^{h_1}2^{h_2}\dots n^{h_n})^G| = \frac{n!}{1^{h_1}h_1!2^{h_2}h_2!\cdots n^{h_n}h_n!}. \tag{9}$$

Hence, from (8) we have $S_2 = (1^{n-3}\,3^1)^G \cup (1^{n-4}\,2^2)^G$ and then by Lemma 4 $c_2(\pi) = 3$ if $ct(\pi) = 1^{n-3}\,3^1$, and $c_2(\pi) = 2$ if $ct(\pi) = 1^{n-4}\,2^2$. From these and by the definition of μ, we have $\mu(\mathrm{S}_n(T)) = 3$, and therefore, by (5) we get Theorem 3. The number $\binom{n}{3}$ of subgraphs isomorphic to $K_{3,3}$ and having I as one of the vertices is obtained from (9) for any $\pi \in (1^{n-3}\,3^1)^G$.

So, any unknown permutation is uniquely reconstructible from 4 its distinct 1–neighbors. The reconstruction of a permutation in the case of at most two transpositions requires many more its distinct 2–neighbors.

Theorem 5 *[12] [14] For* $n \geq 3$,

$$N(\mathrm{S}_n(T), 2) = N_2(\mathrm{S}_n(T), 2) = \frac{3}{2}(n-2)(n+1) \tag{10}$$

which follows from the fact that the normalizer of T is $G = \mathrm{S}_n$ itself and from

Lemma 5 *[14] For any* $\pi \in S_i$, $1 \leq i \leq n-1$, *the number of all vertices in* $(1^{h_1} 2^{h_2} \ldots n^{h_n})^G$, *at a given distance from* π, *depends only on the conjugacy class to whom* π *belongs.*

So, to prove Theorem 5 it is enough to consider the numbers of vertices in all subsets of $B_2(I)$ at the minimal distance at most 2 from a given vertex $\pi \in S_i$, $1 \leq i \leq 4$, where as follows from (8), $S_1 = (1^{n-2}\, 2^1)^G$, $S_2 = (1^{n-3}\, 3^1)^G \cup (1^{n-4}\, 2^2)^G$, $S_3 = (1^{n-4}\, 4^1)^G \bigcup (1^{n-5}\, 2^1\, 3^1)^G \bigcup (1^{n-6}\, 2^3)^G$, and $S_4 = (1^{n-5}\, 5^1)^G \bigcup (1^{n-6}\, 2^1\, 4^1)^G \bigcup (1^{n-6}\, 3^2)^G \bigcup (1^{n-7}\, 2^2\, 3^1)^G \bigcup (1^{n-8}\, 2^4)^G$. It is shown that $N_4(\mathrm{S}_n(T), 2) = 20$ for $n \geq 5$, $N_3(\mathrm{S}_n(T), 2) = 12$ for $n \geq 4$, $N_2(\mathrm{S}_n(T), 2) = \frac{3}{2}(n-2)(n+1)$ and $N_1(\mathrm{S}_n(T), 2) = (n-1)n$ for all $n \geq 3$. From these and by (4) one can conclude (10).

Now let us consider the *bubble–sort Cayley graph* $\mathrm{S}_n(t)$ on the symmetric group S_n generated by the bubble–sort transpositions from the set $t = \{t_{i,i+1} \in \mathrm{S}_n,\ 1 \leq i < n\},\ |t| = n-1$. The distance in this graph is defined as the minimal number of the bubble–sort transpositions needed to transform one permutation into another. The diameter of this graph is $\binom{n}{2}$ since this number of the bubble–sort transpositions is required to transform the identity permutation to its inverse permutation and it also suffices to transform any permutation of n elements into another.

Lemma 6 *[12] The bubble–sort Cayley graph* $\mathrm{S}_n(t), n \geq 3$,
(i) is a connected bipartite $(n-1)$*-regular graph of order* $n!$ *and diameter* $\binom{n}{2}$;
(ii) it does not contain subgraphs isomorphic to $K_{2,3}$;
(iii) each its vertex belongs to $\binom{n-2}{2}$, $n \geq 4$, *subgraphs isomorphic to* $K_{2,2}$.

This graph does not contain triangles since it is a bipartite, i.e. $\lambda(\mathrm{S}_n(t)) = 0$, and pentagons as well. Now let us consider a permutation $\pi \in S_2 = S_2(I)$ as a product of two bubble–sort transpositions $t_{i,i+1}$ and $t_{j,j+1}$ acting to the identity permutation, where $1 \leq i < j < n$. There are exactly $\binom{n-2}{2}$ distinct ways to express a permutation $\pi \in (1^{n-4}\, 2^2)^G$ as the following representation $\pi = t_{i,i+1} t_{j,j+1} = t_{j,j+1} t_{i,i+1} = \pi^{-1}$, when $j \neq i+1$. Hence $c_2(\pi) = 2$ by Lemma 2, and by the definition of μ, we have $\mu(\mathrm{S}_n(t)) = 2$. It can be also verified that for $r = 2$ we have $N_4(\mathrm{S}_n(t), 2) = 4$ for $n \geq 5$, $N_3(\mathrm{S}_n(t), 2) = 2$ for $n \geq 4$, $N_2(\mathrm{S}_n(t), 2) = N_1(\mathrm{S}_n(t), 2) = 2(n-1)$ for $n \geq 3$. From all these and by (4) and (5) we get

Theorem 6 *[12] For any $n \geq 3$,*

$$N(\mathrm{S}_n(t), 1) = 2, \quad N(\mathrm{S}_n(t), 2) = N_2(\mathrm{S}_n(t), 2) = N_1(\mathrm{S}_n(t), 2) = 2(n-1).$$

Almost the same results appear for the *star Cayley graph* $\mathrm{S}_n(st)$ generated by the prefix–transpositions from the set $st = \{(1,i) \in \mathrm{S}_n,\ 1 < i \leq n\},\ |st| = n-1$. The distance in the graph $\mathrm{S}_n(st)$ is defined as the minimal number of the prefix– transpositions transforming one permutation into another.

Lemma 7 *[17] The star Cayley graph $\mathrm{S}_n(st), n \geq 3$, is a connected bipartite $(n-1)$-regular graph of order $n!$ with diameter $d(\mathrm{S}_n(st)) = \lfloor \frac{3(n-1)}{2} \rfloor$.*

It is also known that there are no cycles of lengths of 3,4,5,7 in the graph $\mathrm{S}_n(st)$, hence $\lambda(\mathrm{S}_n(st)) = 0$ and $\mu(\mathrm{S}_n(st)) = 1$. Moreover, it is easy to verify that if $r = 2$ then $N_4(\mathrm{S}_n(st), 2) = 4$ for $n \geq 5$, $N_3(\mathrm{S}_n(st), 2) = 4$ for $n \geq 4$, $N_2(\mathrm{S}_n(st), 2) = n$ for $n \geq 5$ and $N_1(\mathrm{S}_n(st), 2) = 2(n-1)$ for $n \geq 4$. From all these properties and by (4) and (5) we get

Theorem 7 *[12] For any $n \geq 4$,*

$$N(\mathrm{S}_n(st), 1) = 2, \quad N(\mathrm{S}_n(st), 2) = N_1(\mathrm{S}_n(st), 2) = 2(n-1).$$

Thus, in the bubble–sort and star Cayley graphs any unknown permutation is uniquely reconstructible from 3 its distinct 1–neighbors. The reconstruction of a permutation in the case of at most two bubble–sort transpositions or prefix–transpositions requires $(2n-1)$ its distinct permutations which are 2–neighbors of the unknown permutation.

Remark 2. The presented transposition Cayley graphs have been extensively studied in computer science. As it was observed in [17] many interconnection network topologies have a natural algebraic representation as coset graphs, of which Cayley graphs are a special case. This follows from the symmetric nature of network topologies and a fundamental result saying that all vertex–transitive graphs can be represented as coset graphs [29]. Other symmetry properties of Cayley graphs are discussed in [19]. For instance, it is shown there that the transposition Cayley graph $\mathrm{S}_n(T)$ is edge–transitive but not distance–regular and not distance–transitive; the bubble–sort Cayley graph $\mathrm{S}_n(t)$ is not edge–transitive, not distance–regular and not distance–transitive; the star Cayley graph $\mathrm{S}_n(st)$ is also not distance–regular and not distance–transitive for $n \geq 4$.

Remark 3. Among the considered in this section graphs the most investigated in the theory of interconnection networks is the star Cayley graph since many parallel algorithms can be efficiently mapped on this graph.

The *reversal Cayley graph* $\mathrm{S}_n(R)$ is defined on the symmetric group S_n and generated by the reversals from the set $R = \{r_{i,j} \in \mathrm{S}_n, 1 \le i < j \le n\}$, $|R| = \binom{n}{2}$. The distance in this graph is defined as the minimal number of reversals transforming one permutation into another. In [15] it was proved that the diameter of $\mathrm{S}_n(R)$ is $(n-1)$.

Lemma 8 *[10] The reversal Cayley graph* $\mathrm{S}_n(R)$, $n \ge 3$,
(i) is a connected $\binom{n}{2}$*-regular graph of order* $n!$ *and diameter* $n-1$*;*
(ii) it does not contain triangles nor subgraphs isomorphic to $K_{2,4}$*;*
(iii) each its vertex belongs to $(n-2)$ *subgraphs isomorphic to* $K_{3,3}$ *and to* $\frac{1}{12}(n-3)(n-1)(n^2+2n+4), n \ge 4$, *subgraphs isomorphic to* $K_{2,2}$ *that are not subgraphs of* $K_{3,3}$.

The verification of these facts is based on the careful combinatorial analysis of the spheres $S_1 = S_1(I)$ and $S_2 = S_2(I)$. It is shown that $a_1(\pi) = a_1(\pi, I) = 0$ for any $\pi \in S_1$, hence there are no triangles in $\mathrm{S}_n(R)$. It is also shown that $c_2(\pi) = c_2(\pi, I) = 3$ if and only if $\pi = \pi_k$ or $\pi = \pi_k^{-1}$ for any $k = 1, \ldots, n-2$ where $\pi_k = [1, ..., k-1, k+1, k+2, k, k+3, ..., n] \in S_2$. It is true that

$$\pi_k = r_{k,k+1}\ r_{k+1,k+2} = r_{k,k+2}\ r_{k,k+1} = r_{k+1,k+2}\ r_{k,k+2} \quad \text{and} \tag{11}$$

$$\pi_k^{-1} = r_{k,k+1}\ r_{k,k+2} = r_{k,k+2}\ r_{k+1,k+2} = r_{k+1,k+2}\ r_{k,k+1}. \tag{12}$$

As one can see, the reversals on intervals of two and three elements of a permutation are used in (11) and (12) that correspond to considering the transpositions $t_{i,i+1},\ 1 \le i \le n-1$, and $t_{i,i+2},\ 1 \le i \le n-2$. Moreover, there is no a permutation $\pi \in S_2$ such that $c_2(\pi) = 4$, hence there are no subgraphs isomorphic to $K_{2,4}$ and there exist exactly $(n-2)$ subgraphs isomorphic to $K_{3,3}$ having I, π_k, π_k^{-1} in the first part and $r_{k,k+1}, r_{k,k+2}, r_{k+1,k+2}$ in the second part for any $k = 1, ..., n-2$. By vertex–transitivity, this holds for any vertex of $\mathrm{S}_n(R)$. To check the last statement of Lemma 8 it is enough to calculate the total number of permutations $\pi \in S_2$ such that $c_2(\pi) = 2$. It is shown that $c_2(\pi) = 2$ if and only if π has one of the following representations

$$r_{k,k+2}\ r_{k+1,k+3} = r_{k+1,k+3}\ r_{k,k+2}, \qquad k \le n-3, \tag{13}$$

$$r_{k,l}\ r_{k,j} = r_{k+l-j,l}\ r_{k,l}, \qquad k+1 \le j \le l-1,\ l > k+2, \tag{14}$$

$$r_{k,l}\ r_{k,j} = r_{k,j}\ r_{k+j-l,j}, \qquad l+1 \le j \le n,\ j > k+2, \tag{15}$$

$$r_{k,l}\ r_{i,j} = r_{i,j}\ r_{k,l}, \qquad k < l < i < j, \tag{16}$$

$$r_{k,l}\ r_{i,j} = r_{l-j+k,l-i+k}\ r_{k,l}, \qquad k < i < j < l, \tag{17}$$

where $1 \leq k < l \leq n$. So, there are exactly $(n-3)$ permutations having representations (13); the number of representations (14) and (15) equals $2\sum_{k=1}^{n-3}\sum_{i=3}^{n-k}(i-1) = \frac{1}{3}(n-3)(n^2-4)$ and the number of representations (16)-(17) equals $2\binom{n}{4}$. The summation of all these numbers gives the required number in Lemma. As the result, $\lambda(\mathrm{S}_n(R)) = 0,\ \mu(\mathrm{S}_n(R)) = 3$ and by (5) we get

Theorem 8 *[10] For any* $n \geq 3, \quad N(\mathrm{S}_n(R), 1) = 3.$

Thus, any unknown permutation is uniquely reconstructible from 4 its distinct 1-neighbors. It is also shown that a permutation is reconstructible from 3 its 1-neighbors with probability $p_3 \to 1$ as $n \to \infty$ and it is reconstructible from 2 its 1-neighbors with probability $p_2 \sim \frac{1}{3}$ as $n \to \infty$ under the conditions that these permutations are uniformly distributed.

Theorem 9 *[10] For any* $n \geq 3$,

$$N(\mathrm{S}_n(R), 2) \geq \frac{3}{2}(n-2)(n+1). \tag{18}$$

This result is obtained by showing that $|\cup_{i=1}^{3} B_1(r_{k_i,l_i})| \geq \frac{3}{2}(n-2)(n+1)$ for any $r_{k_i,l_i} \in S_1, i = 1,2,3$. Indeed, the metric balls $B_1(r_{k_i,l_i}), i = 1,2,3$, belong to $B_2(I) \cap B_2(\pi),\ \pi \in S_2$ such that $c_2(\pi) = 3$, and have three joint vertices I, π, π^{-1} where $\pi = \pi_k,\ k = 1, \ldots, n-2$. Each of the metric balls has size $\binom{n}{2}+1$. So the required statement is getting by $3(\frac{n(n-1)}{2}+1)-6 = \frac{3}{2}(n-2)(n+1)$. For example, for $\pi = [1342]$ or $\pi^{-1} = [1423]$, and for $r_{1,3} = [1432], r_{2,3} = [1324], r_{3,4} = [1243]$ one can check that $|B_1(r_{1,3}) \cup B_1(r_{2,3}) \cup B_1(r_{3,4})| = 15$.

Let us mentioned here, that inequality (18) is attained for the permutations having representations (11) and (12) where reversals can be considered as transpositions like in the above example (compare also with equality (10)).

The *pancake Cayley graph* $\mathrm{S}_n(PR)$, also called the *prefix–reversal graph*, is defined on the symmetric group S_n and generated by the prefix–reversals from the set $PR = \{r_{1,j} \in \mathrm{S}_n, 1 < i \leq n\},\ |PR| = n-1$. Sometimes this graph is also called the unburnt pancake Cayley graph. As one can see there is a similarity between the star Cayley graph $\mathrm{S}_n(st)$ and the pancake Cayley graph $\mathrm{S}_n(PR)$. In particular, $S_2(st) = S_2(PR) = K_2$ and $S_3(st) = S_3(PR) = C_6$. The distance in this graph is defined as the minimal number of the prefix–reversals transforming one permutation into another. This distance is also called the *prefix–reversal distance* and the diameter of $\mathrm{S}_n(PR)$ is called the *prefix–reversal diameter*.

The question about the prefix–reversal diameter is open. The problem is known as the *pancake flipping problem*. Currently, exact values of the prefix–reversal diameter are known for $n \leq 13$, for instance, it is 15 for $n = 13$. Bounds are given in [30] where it is shown that the diameter of $\mathrm{S}_n(PR)$ is at most $(5n+5)/3$ for all n, and at least $17n/16$ for infinitely many n. A lower bound was improved in [23] such that the prefix–reversal diameter is at least $15n/14$. It was also shown there that sorting by the prefix–reversals (finding a sequence of the prefix–reversals sorting a permutation to the identity permutation) is an NP–hard problem. Some combinatorial properties of $\mathrm{S}_n(PR)$ are collected in the following

Lemma 9 *The pancake Cayley graph* $\mathrm{S}_n(PR)$, $n \geq 3$, *is a connected* $(n-1)$-*regular graph of order* $n!$ *without cycles of lengths of 3,4,5.*

From this Lemma $\lambda(\mathrm{S}_n(PR)) = 0$ and $\mu(\mathrm{S}_n(PR)) = 1$ since there are no triangles and quadrangles as well. It is not difficult to observe that $N_2(\mathrm{S}_n(PR)) = n$ for $n \geq 4$ and $N_1(\mathrm{S}_n(PR)) = 2(n-1)$ for $n \geq 4$. So, by (4) and (5) we get

Theorem 10 *For any* $n \geq 4$,

$$N(\mathrm{S}_n(PR), 1) = 2, \quad N(\mathrm{S}_n(PR), 2) = N_1(\mathrm{S}_n(PR), 2) = 2(n-1).$$

Comparing this statements with Theorem 7 one can see that there is one and the same result for the star and pancake Cayley graphs. Indeed, any unknown permutation is uniquely reconstructible from 3 distinct its 1-neighbors and from $(2n-1)$ distinct its 2-neighbors in the both cases.

Remark 4. The pancake Cayley graph corresponds to the n-dimensional pancake network in computer science such that this network has processors labeled with each of the $n!$ distinct permutations of length n. Two processors are connected when the label of one is obtained from the other by some prefix–reversal. Each permutation is considered as a stack of different size pancakes. The diameter of this network corresponds to the worst communication delay for transmitting information in a system. Symmetries of this network were considered in [19]. It was shown that the pancake Cayley graph $\mathrm{S}_n(PR)$ is not edge–transitive, not distance–regular and not distance–transitive.

Remark 5. Sorting by fixed–length reversals (k-reversals) was considered in [31]. In particular, it was shown that $O(n^{3/2})$ k-reversals suffice to transform any permutation to the identity permutation when $k \approx \sqrt{n}$. The number of connected components on the Cayley graphs generated by the fixed–length reversals was also discussed thereq. For instance, the trivial case is n-reversals and there are $n!/2$ connected components for S_n, $k = n > 2$.

4 Cayley graphs on B_n generated by transpositions and reversals

In this section we observe Cayley graphs on the signed permutation group B_n which is also known as the hyperoctahedral group [20]. The elements of B_n are signed permutations, i.e., permutations with a sign attached to every entry. We use the compact one–line notation for a signed permutation $\pi = [\pi_1, \overline{\pi}_2, \ldots, \overline{\pi}_i, \ldots, \pi_n]$, where a bar is written over an element with a negative sign. The *sign–change transpositions* $t^{\sigma}_{ij}, 1 \leq i < j \leq n$, switches two elements i and j and their signs, i.e., $[\ldots, \pi_i, \ldots, \overline{\pi}_j, \ldots]t^{\sigma}_{ij} = [\ldots, \pi_j, \ldots, \overline{\pi}_i, \ldots]$, and the *sign–change "transpositions"* $t^{\sigma}_{ii}, 1 \leq i \leq n$, changes the sign of the i-th element, i.e., $[\ldots, \pi_i, \ldots]t^{\sigma}_{ii} = [\ldots, \overline{\pi}_i, \ldots]$. The sign–change reversals $r^{\sigma}_{i,j}$ flip the signs of elements on the segments $[i, j]$, $1 \leq i \leq j \leq n$, i.e., $[\ldots, \pi_i, \overline{\pi}_{i+1}, \ldots, \pi_{j-1}, \pi_j, \ldots]r^{\sigma}_{i,j} = [\ldots, \overline{\pi}_j, \overline{\pi}_{j-1}, \ldots, \pi_{i+1}, \overline{\pi}_i, \ldots]$.

4.1 The transposition Cayley graphs $\mathrm{B}_n(T^{\sigma})$, $\mathrm{B}_n(t^{\sigma})$ *and* $\mathrm{B}_n(st^{\sigma})$

The *transposition Cayley graph* $\mathrm{B}_n(T^{\sigma})$ on the signed permutation group B_n is generated by the sign–change transpositions from the set $T^{\sigma} = \{t^{\sigma}_{ii} \in \mathrm{B}_n,\ 1 \leq i \leq n\} \bigcup \{t^{\sigma}_{ij} \in \mathrm{B}_n,\ 1 \leq i < j \leq n\},\ |T^{\sigma}| = \binom{n+1}{2}$. The distance in this graph is defined as the minimal number of the sign–change transpositions transforming one permutation into another. The order of this graph corresponds to the order of B_n that is $2^n n!$. The basic facts about $\mathrm{B}_n(T^{\sigma})$ are collected in

Lemma 10 *The transposition Cayley graph* $\mathrm{B}_n(T^{\sigma}),\ n \geq 2$,
(i) is a connected bipartite $\binom{n+1}{2}$*-regular graph of order* $2^n n!$;
(ii) it does not contain subgraphs isomorphic to $K_{2,3}$;
(iii) each its vertex belongs to $\frac{1}{2}(n^3 + 9n^2 - 58n + 90), n \geq 3$, *subgraphs isomorphic to* $K_{2,2}$.

All these facts are based on the properties of the signed permutations belonging to the sphere $S_2 = S_2(I)$. There is no a signed permutation $\pi \in S_2$ for which $c_2(\pi) = 3$ and this means that there are no subgraphs isomorphic to $K_{2,3}$ in $B_n(T^{\sigma})$. From the other side, $c_2(\pi) = 2$ for a signed permutation $\pi \in S_2$ if and only if π has one of the following representations

$$t^{\sigma}_{i,i}\ t^{\sigma}_{j,j} = t^{\sigma}_{j,j}\ t^{\sigma}_{i,i}, \qquad 1 \leq i < j \leq n, \tag{19}$$

$$t^{\sigma}_{k,k}\ t^{\sigma}_{i,j} = t^{\sigma}_{i,j}\ t^{\sigma}_{l,l}, \qquad 1 \leq i < j \leq n,\ k = i, l = j \text{ or } k = j, l = i, \tag{20}$$

$$t^{\sigma}_{k,k}\ t^{\sigma}_{i,j} = t^{\sigma}_{i,j}\ t^{\sigma}_{k,k}, \qquad 1 \leq i < j \leq n,\ k \neq i \text{ and } k \neq j,\ 1 \leq k \leq n, \tag{21}$$

$$t^{\sigma}_{i,j}\ t^{\sigma}_{k,l} = t^{\sigma}_{i,j}\ t^{\sigma}_{k,l}, \qquad \text{for some } i, j, k, l. \tag{22}$$

So, there are exactly $3\binom{n}{2}$ signed permutations having representations (19) and (20); the number of representations (21) equals $n\binom{n-1}{2}$ and there are $3\binom{n-2}{2}+6\binom{n-3}{2}$ signed permutations for which (22) holds. The required number in Lemma 10 is obtained by the summation of all these numbers. So, by this Lemma, $\lambda(\mathrm{B}_n(T^\sigma))=0,\ \mu(\mathrm{B}_n(T^\sigma))=2$ and by (5) we have

Theorem 11 *For any* $n\geq 2, \quad N(\mathrm{B}_n(T^\sigma),1)=2.$

The question about the reconstruction of a signed permutation from its distinct 2–neighbors is unanswered. The conjecture is $N(\mathrm{B}_n(T^\sigma),2)=n(n+1)$ for any $n\geq 2$.

The *bubble–sort Cayley graph* $\mathrm{B}_n(t^\sigma)$ on the group B_n is generated by the sign–change bubble–sort transpositions from the set $t^\sigma=\{t^\sigma_{i,i}\in \mathrm{B}_n,\ 1\leq i\leq n\}\bigcup\{t^\sigma_{i,i+1}\in \mathrm{B}_n,\ 1\leq i<n\},\ |t^\sigma|=2n-1$. The main properties of this graph are presented by

Lemma 11 *The bubble–sort Cayley graph* $\mathrm{B}_n(t^\sigma), n\geq 2,$
(i) is a connected bipartite $(2n-1)$*-regular graph of order* $2^n n!$;
(ii) it does not contain subgraphs isomorphic to $K_{2,3}$;
(iii) each its vertex belongs to $(2n^2-4n+3),\ n\geq 3,$ *subgraphs isomorphic to* $K_{2,2}$.

From this Lemma, $\lambda(\mathrm{B}_n(t^\sigma))=0$ since $\mathrm{B}_n(t^\sigma)$ is a bipartite. There also does not exist a signed permutation $\pi\in S_2$ such that $c_2(\pi)=3$, hence there are no subgraphs isomorphic to $K_{2,3}$. The number of subgraphs isomorphic to $K_{2,2}$ having I as one of the vertices can be calculated by the formulas (19)-(22) taking into account that $j=i+1$ in (20) and (21), and $j=i+1,\ l=k+1,\ k\neq i+1$ in (22). From this, there are $\binom{n}{2}$ and $\binom{n-2}{2}$ signed permutations represented by (19) and (22); the numbers of representations (20) and (21) equal $2(n-1)$ and $(n-2)(n-1)$, respectively. These gives the total number. So, $\mu(\mathrm{B}_n(t^\sigma))=2$ and by (5) the following theorem takes place

Theorem 12 *For any* $n\geq 2, \quad N(\mathrm{B}_n(t^\sigma),1)=2.$

The similar results are obtained for the *star Cayley graph* $\mathrm{B}_n(st^\sigma)$ on the group B_n generated by the sign–change prefix–transpositions from the set $st^\sigma=\{t^\sigma_{i,i}\in \mathrm{B}_n,\ 1\leq i\leq n\}\bigcup\{t^\sigma_{1,i}\in \mathrm{B}_n,\ 1<i\leq n\},\ |t^\sigma|=2n-1$.

Lemma 12 *The star Cayley graph* $\mathrm{B}_n(st^\sigma), n\geq 2,$
(i) is a connected bipartite $(2n-1)$*-regular graph of order* $2^n n!$;
(ii) it does not contain subgraphs isomorphic to $K_{2,3}$;
(iii) each its vertex belongs to $\frac{3}{2}n(n-1),\ n\geq 3,$ *subgraphs isomorphic to* $K_{2,2}$.

Again, we have $\lambda(\mathrm{B}_n(st^\sigma)) = 0$ and $\mu(\mathrm{B}_n(st^\sigma)) = 2$ by the same reasons as in the previous case. The number of signed permutations $\pi \in S_2$ for which $c_2(\pi) = 2$ is calculated by the formulas (19)-(21) when $i = 1, k = 1, l = j$ or $i = 1, l = 1, k = j$ in (20), and $i = 1, k \neq 1$, in (21); the formula (22) doesn't hold in this case. So, the total number is $\binom{n}{2} + 2(n-1) + (n-2)(n-1) = \frac{3}{2}n(n-1)$. From this Lemma, and by (5) we have

Theorem 13 *For any* $n \geq 2, \quad N(\mathrm{B}_n(st^\sigma), 1) = 2.$

Thus, for all considered Cayley graphs in this section, any unknown signed permutation is uniquely reconstructible from 3 its distinct 1-neighbors. There are no results on reconstructing a signed permutation from its distinct 2–neighbors in the graphs $\mathrm{B}_n(t^\sigma)$ and $\mathrm{B}_n(st^\sigma)$.

Remark 6. The Cayley graphs on the signed permutation group generated by the sign–change transpositions $t^\sigma_{i,j}$, when $i \neq j$, are considered in [12]. It is shown there that the connected components arise for these graphs. The number of these connected components depends on the set of generators. For example, if all transpositions $t^\sigma_{i,j}, 1 \leq i < j \leq n$, are the generators, then there are 2 connected bipartite $\binom{n}{2}$–regular components of order $2^{n-1}n!$. Each of these connected components represents a subgroup of B_n and by symmetry these subgroups are isomorphic. The even–signed permutation group D_n [20] which is the normal subgroup of B_n of index 2 whose elements are signed permutations with even numbers of negative elements is one of these subgroups for this Cayley graph. In the case, when the sign–change bubble–sort transpositions $t^\sigma_{i,i+1}, 1 \leq i < n$, are the generators, then there are 2^n connected bipartite $\binom{n}{2}$–regular components of order $n!$. Each of these connected components represents a subgroup of B_n isomorphic to S_n and these components are isomorphic to the Cayley graph $\mathrm{S}_n(t)$ (see section 3.1). The similar situation appears when the prefix–transpositions $t^\sigma_{1,i}, 1 \leq i < n$, are considered as the generators. In this case, there are 2^n connected bipartite $(n-1)$–regular components of order $n!$ each of which is isomorphic to the Cayley graph $\mathrm{S}_n(st)$ (see section 3.1). It is required 3 distinct 1–neighbors to reconstruct any unknown signed permutation in all these cases.

4.2 The reversal Cayley graphs $\mathrm{B}_n(R^\sigma)$ *and* $\mathrm{B}_n(PR^\sigma)$

The *reversal Cayley graph* $\mathrm{B}_n(R^\sigma)$ is defined on the signed permutation group B_n and generated by the sign–change reversals from the set $R^\sigma = \{r^\sigma_{i,j} \in \mathrm{B}_n, 1 \leq i \leq j \leq n\}$, $|R^\sigma| = \binom{n+1}{2}$. The distance in the graph $\mathrm{B}_n(R^\sigma)$ is defined as the minimal number of sign–change reversals transforming one signed permutation into another. It is shown in [32] that the diameter of $\mathrm{B}_n(R^\sigma)$ is $(n+1)$ and the permutations $[+n, +(n-1), \ldots, +1]$, when n is even, and

$[+2, +1, +3, +n, +(n-1), \ldots, +4]$, when $n > 3$ is odd, are at this maximum distance from the identity permutation.

Lemma 13 *[13] The reversal Cayley graph $B_n(R^\sigma)$, $n \geq 2$,*
(i) is a connected $\binom{n+1}{2}$-regular graph of order $2^n n!$ and diameter $(n+1)$;
(ii) it does not contain triangles and subgraphs isomorphic to $K_{2,3}$;
(iii) each its vertex belongs to $\frac{1}{12}(n-1)n(n+1)(n+4)$ subgraphs isomorphic to $K_{2,2}$.

It is also shown that $c_2(\pi) = 2$ for any signed permutation $\pi \in S_2$ if and only if π has one of the following representations

$$r^\sigma_{k,l}\, r^\sigma_{k,j} = r^\sigma_{k+l-j,l}\, r^\sigma_{k,l}, \qquad k \leq j \leq l-1, \tag{23}$$

$$r^\sigma_{k,l}\, r^\sigma_{k,j} = r^\sigma_{k,j}\, r^\sigma_{k+j-l,j}, \qquad l+1 \leq j \leq n, \tag{24}$$

$$r^\sigma_{k,l}\, r^\sigma_{i,j} = r^\sigma_{i,j}\, r^\sigma_{k,l}, \qquad k \leq l < i \leq j, \tag{25}$$

$$r^\sigma_{k,l}\, r^\sigma_{i,j} = r^\sigma_{l-j+k,l-i+k}\, r_{k,l}, \qquad k < i \leq j < l, \tag{26}$$

where $1 \leq k \leq l \leq n$. The number of representations (23) and (24) equals $2\sum_{k=1}^{n-1}\sum_{i=1}^{n-k} i = \frac{1}{3}n(n-1)(n+1)$; the number of representations (25) and (26) equals $\binom{n}{2}$, when $k = l < i = j$, equals $\binom{n}{3}$ in each of three cases $k = l < i < j$, $k < l < i = j$, $k < i = j < l$, and equals $2\binom{n}{4}$ in the cases when all i, j, k, l differ. The required number is obtained by the summation of all these numbers. So, from this Lemma, $\lambda(B_n(R^\sigma)) = 0$, $\mu(B_n(R^\sigma)) = 2$ and by (5) we get

Theorem 14 *[13] For any $n \geq 2$, $N(B_n(R^\sigma), 1) = 2$.*

Thus, any unknown signed permutation is uniquely reconstructible from 3 its distinct 1-neighbors. It is also shown that a signed permutation is reconstructible from 2 its distinct 1-neighbors with probability $p_2 \sim \frac{1}{3}$ as $n \to \infty$ under the conditions that these signed permutations are uniformly distributed. In the case, when $r = 2$ the following theorem holds

Theorem 15 *[13] For any $n \geq 2$,*

$$N(B_n(R^\sigma), 2) \geq n(n+1). \tag{27}$$

This result is obtained by the same method as for the reversal Cayley graph $\mathrm{S}_n(R)$. Let us mentioned here, that inequality (27) is attained for the permutations π with $c_2(\pi) = 2$ having the representations (23)–(26) where the sign–change reversals correspond to the sign–change transpositions.

The *burnt pancake Cayley graph* $\mathrm{B}_n(PR^\sigma)$, also called *burnt prefix–reversal graph*, is defined on the signed permutation group B_n and generated by the sign–change prefix–reversals from the set $PR^\sigma = \{r^\sigma_{1,i} \in \mathrm{B}_n, 1 \leq i \leq n\}$, $|PR^\sigma| = 2n-1$. The distance in this graph is defined as the minimal number of the sign–change prefix–reversals transforming one signed permutation into another. This distance is also called the *burnt prefix–reversal distance* and the diameter of $\mathrm{B}_n(PR^\sigma)$ is called the *burnt prefix–reversal diameter*. The problem of finding the burnt prefix–reversal diameter is known as the *burnt pancake flipping problem*. It was shown in [22] that the burnt prefix–reversal diameter is at most $3n/2$ and at least $2n-2$ where the upper bound holds for $n \geq 10$. It is conjectured that the diameter is achieved by the negative identity permutation $-I = [-1, -2, \ldots, -n]$.

Lemma 14 *The burnt pancake Cayley graph* $B_n(PR^\sigma)$, $n \geq 2$,
(i) is a connected $(2n-1)$*-regular graph of order* $2^n n!$;
(ii) it does not contain triangles and subgraphs isomorphic to $K_{2,3}$;
(iii) each its vertex belongs to $\frac{3}{2}n(n-1)$ *subgraphs isomorphic to* $K_{2,2}$.

There are no subgraphs isomorphic to $K_{2,3}$ since there is no a signed permutation $\pi \in S_2$ for which $c_2(\pi) = 3$. The number of subgraphs isomorphic to $K_{2,2}$ having I as one of the vertices can be calculated by the formulas (23)-(25) taking into account that $k = 1$ in (23) and (24), and $i = 1,\ k = 1,\ 1 \leq l < j \leq n$ in (25). The formula (26) coincide with (23) when $k = 1$. So, the number of representations (23)–(25) equals $3\binom{n}{2}$ and this gives the required answer. From this Lemma, $\lambda(\mathrm{B}_n(PR^\sigma)) = 0$ and $\mu(\mathrm{B}_n(PR^\sigma)) = 2$, this means that any sign–change permutation is uniquely reconstructible from 3 its distinct 1–neighbors in this case.

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