The Logic of Bargaining

Dongmo Zhang

Intelligent Systems Laboratory
University of Western Sydney
Locked Bag 1797, Penrith South DC
NSW 1797, Australia
dongmo@scm.uws.edu.au

Abstract. This paper reexamines the game-theoretic bargaining theory from logic and Artificial Intelligence perspectives. We present an axiomatic characterization of the logical solutions to bargaining problems. A bargaining situation is described in propositional logic with numerical representation of bargainers' preferences. A solution to the n-person bargaining problems is proposed based on the maxmin rule over the degrees of bargainers' satisfaction. The solution is uniquely characterized by four axioms *collective rationality*, *scale invariance*, *symmetry* and *mutually comparable monotonicity* in conjunction with three other fundamental assumptions *individual rationality*, *consistency* and *comprehensiveness*. The Pareto efficient solutions are characterized by the axioms *scale invariance*, *Pareto optimality* and *restricted mutually comparable monotonicity* along with the basic assumptions. The relationships of these axioms and assumptions and their links to belief revision postulates and game theory axioms are discussed. The framework would help us to identify the logical reasoning behind bargaining processes and would initiate a new methodology of bargaining analysis.

Keywords. Bargaining theory, belief revision, game theory

1 Introduction

As one of the most fundamental models in modern economic theory, the Nash bargaining solution (Nash 1950 [1]) has been developed through the investigations in game theory in the past five decades into a high sophisticated theory with varieties of models and extensions, and has been extensively applied to economics, sociology, politics and management science [2,3,4,5,6,7]. Computer Scientists and Artificial Intelligence (AI) researchers have found it useful in modeling interactions among distributed computer systems and autonomous software agents since early 90s [8,9,10,11]. Many applications have been developed for the design and evaluation of high-level interaction protocols among autonomous agents for task assignment, resource allocation, conflict resolution, electronic trading and web services, which forms the first force of the research on *automated negotiation* [12,13,14,10,15].

Traditionally, a bargaining situation is abstracted as a numerical game. The game-theoretic theory of bargaining provides a quantitative method to facilitate bargaining analysis. Its highly abstract model allows us to directly apply the approach to computing-related applications. Most existing work on automated negotiation is actually built up

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on game-theoretic models thanks to its well-established framework and simplicity in implementation. However, with the advance of intelligent software agents, it has been found that the purely quantitative approach is insufficient and sometimes is inappropriate to computing-related applications [16,17,18]. To demonstrate this point, let us consider a typical example in bargaining theory.

Example 1 *(Wage negotiation) A labor union bargains with the management of a firm for a wage raise. The management claims that if the raise is granted, it will have to lay off employees to regain the balance. The union threatens that if it is not granted, a strike will be called*[1].

The modeling of the scenario in game theory requires quantifying each party's situation with utility functions. The firm's utility can be simply defined as the excess of its revenue over labor costs in conjunction with the possible loss due to a strike. The union's utility can be extremely complicated, as "it is an old question in labor economics", which needs a comprehensive consideration of the levels of payment, chance of layoff, potential alternative offers and living costs [19,20,5]. Acquiring these data can never be an easy job for a software agent.

Bargaining is intelligent rivalry between agents, logical reasoning must play an essential role in most bargaining activities. A bargaining theory should be able to identify the logical reasoning behind bargaining. However, the language of utility has limited expressive power. The process of quantifying a bargaining situation loses information about bargaining reasoning. A number of factors, such as negotiation demands, conflicting claims, and mutual threats, which in fact determine the outcome of a negotiation, cannot be explicitly specified in game-theoretic models and hence are mostly ignored.

To gain an insight into bargaining reasoning, let us restate the wage negotiation problem in logical language. On the one hand, the union demands a wage raise (*raise*) and is aware of the management's claim that if the raise is granted, a layoff may be carried out ($raise \rightarrow layoff$). It is believed that a layoff will result in a number of union members losing their jobs ($layoff \rightarrow \neg jobs$), which is a situation the union has to avoid (*jobs* are demanded). On the other hand, the management realizes that the failure to agree on the wage raise would lead to a strike ($\neg raise \rightarrow strike$), which will surely paralyze the normal production of the firm ($strike \rightarrow \neg production$). The management must prevent such a situation from happening (*production* is demanded).

With the new description, it is easy to identify the conflicts between the negotiation parties. Besides the direct conflicts: *raise* and $\neg raise$, there are two indirect conflicts can be identified:

1. $\{raise, raise \rightarrow layoff, layoff \rightarrow \neg jobs, jobs\} \vdash \bot$;
2. $\{\neg raise, \neg raise \rightarrow strike, strike \rightarrow \neg production, production\} \vdash \bot$.

Resolving the conflicts is actually a logical issue even though the classical propositional logic may not be sufficient to supply a solution (see a formal solution to the problem in Section 5).

This paper aims to develop a logical framework of bargaining reasoning. A bargaining situation is modeled in terms of bargainers' belief states that specify the bargainers'

[1] The description of the scenario is based on the similar examples in [16] and [7].

negotiation items, including their beliefs, demands and threats, as well as their preferences. The negotiation items will be represented in propositional logic. The preferences of bargainers are quantified in terms of the logical structure of bargainers' belief states. We propose a logical solution to the n-person bargaining situations based on bargainers' satisfaction from the negotiation outcome. We characterize the solution with logically represented axioms and assumptions. These axioms and assumptions draw on two difference resources: *belief revision* and *game theory*. Some of them are the multiagent version of the AGM postulates [21,22], which specify the purely logical properties of bargaining solutions in conflict resolving. Some are the analogue of game-theoretic axioms [3,1], which balance the satisfaction of negotiating parties. The others are the combination of both. These axioms specify two basic elements of bargaining reasoning: conflict resolving and gain balancing.

We shall restrict ourselves in this paper to the axiomatic model of bargaining by following Nash's tradition: *the cooperative model of bargaining* [1,4,7]. In stead of specifying concrete negotiation procedures, we focus on the generic properties of bargaining solutions. This differentiates our work from the argumentation-based framework of negotiation whereby negotiation is modeled as a procedure of argumentation that brings about agreement in noncooperative situations [16,14]. Different from the game-theoretic cooperative models, our framework is built up on a different level of abstraction. We explicitly express the physical negotiation items. Rather than given, the possible agreements are generated from our representation of bargaining situations. A solution gives the actual contract of agreements. However, the low abstraction does not complicate analysis and will not restrict our model to certain application domains. In fact, the logical abstraction allows us to address even more general bargaining problems, such as non-convex domains and multi-issue bargaining. We will discuss the relationships and the differences of our work with respect to the existing work in Section 6.

This paper is organized as follows. In Section 2 we introduce a few logical and game-theoretic concepts and conventions that will be used throughout the paper. Section 3 describes the logical representation of bargaining situations. Section 4 presents our solution to the logically represented bargaining problems and its logical characterization. Section 5 examines our solution by using the above wage negotiation example. Section 6 briefly summarizes the related work. Section 7 will conclude the paper with a discussion of the issues related to the methodology that is developed in this paper.

2 Formal Preliminaries

Throughout this paper, a finite propositional language $\mathcal{L}$ is assumed. The language consists of a non-empty finite set of propositional variables, the standard propositional connectives $\{\neg, \vee, \wedge, \rightarrow, \equiv\}$ and two logical constants $\top$ **(true)** and $\bot$ **(false)**. Propositional sentences are denoted by φ, ψ, χ, $\cdots$. As usual, the symbol $\vdash$ denotes derivability and Cn the corresponding logical closure operation, *i.e.*, $Cn(X) = \{\varphi \in \mathcal{L} : X \vdash \varphi\}$. We call a set, K, of sentences to be a *belief set* if it is logically closed and consistent, *i.e.*, $K = Cn(K)$ and $K \not\vdash \bot$.

We also assume a few standard game-theory concepts and conventions. We shall use $\Re^n$ to represent the n-dimensional Euclidean space. For any $x \in \Re^n$, by default, x_i will indicate the i^{th} component of x. Such a convention will also apply to any vector, for example, a vector of sets of logical sentences.

For any $x, y \in \Re^n$, $x \geq y$ means $x_i \geq y_i$ for all i. $x \geqslant y$ means $x_i \geq y_i$ for all i but $x \neq y$. $x > y$ means $x_i > y_i$ for all i.

A *positive affine transformation* $\tau = (\tau_1, \cdots, \tau_n)$ over $\Re^n$ means that $\tau_i(x_i) = a_i x_i + b_i$ for some real numbers a_i and b_i with $a_i > 0$.

We say $S \subseteq \Re^n$ to be *d-comprehensive* if $x \in S$ and $d \leqslant y \leqslant x$ implies $y \in S$.

The *convex hull* of S, represented by $conh(S)$, is the smallest convex set containing the set S. The *comprehensive hull* of S, denoted by $comh(S)$, is the smallest **0**-comprehensive set containing S, where $\mathbf{0} = (0, \cdots, 0) \in \Re^n$.

We define the *weak Pareto frontier* of S as: $WP(S) = \{x \in S : y > x$ implies $y \notin S\}$ and the *strong Pareto frontier* of S as: $P(S) = \{x \in S : y \geqslant x$ implies $y \notin S\}$.

A bargaining game in game theory is a pair (S, d), where $S \subseteq \Re^n$ represents the feasible set that can be derived from possible agreements and $d \in S$ stands for the disagreement point. In traditional bargaining theory, we assume that S is convex and compact. A bargaining solution f is a function that assigns to each bargaining game a unique point of S, i.e., $f(S, d) \in S$.

A bargaining solution N is the *Nash solution* if $N(S, d)$ is the maximizer of the product $\prod(x_i - d_i)$ over S for any bargaining game (S, d).

A barging solution KS is the *Kalai-Smorodinsky solution* (KS-solution) if $KS(S, d)$ is the maximal point of S on the segment connecting d to $a(S, d)$, where $a_i(S, d) = \max\{x_i : x \in S \; \& \; x \geq d\}$ for all i.

Nash in [1] shows that a bargaining solution $f = N$ if and only if it satisfies the following axioms:

- *Pareto-Optimality*: $f(S, d) \in P(S)$.
- *Symmetry*: If (S, d) is a symmetric game, then $f_i(S, d) = f_j(S, d)$ for all i, j.
- *Scale Invariance*: For any positive affine transformation $\tau = (\tau_1, \cdots, \tau_n)$, $\tau(f(S, d)) = f(\tau(S), \tau(d))$
- *Independence of Irrelevant Alternatives*: If $S' \subseteq S$ and $f(S, d) \in S'$, then $f(S', d) = f(S, d)$.

Kalai and Smorodinsky in [3] shows that a solution $f = KS$ for 2-person bargaining games if and only if it satisfies Pareto-Optimality, Symmetry, Scale Invariance as well as the following Restricted Monotonicity:

- *Restricted Monotonicity*: If $S' \subseteq S$ and $a(S', d) = a(S, d)$, then $f(S', d) \leq f(S, d)$.

In spite of the large number of other solutions that have been proposed in the literature, these two solutions are most outstanding (see [7] for a comprehensive survey).

We assume that the reader is familiar with the basic concepts of game theory, especially the cooperative theory of bargaining, and belief revision. For an introductory survey of each area, see Thomson's article [7] and Gärdenfors's article [23], respectively.

3 Logical Representation of bargaining situations

Different from the game-theoretic model of bargaining, we shall represent a bargaining situation in terms of bargainers' belief states, in which the negotiation items are explicitly described in logical statements while the preferences of the bargainers over their bargaining items are quantified in terms of the logical structure of bargainers' belief states.

We assume that all negotiation items of each bargainer, including her demands, beliefs and threats, are represented in a certain finite propositional language $\mathcal{L}$. We mix all the items together but intuitively, a demand is a term the negotiator requests the other parties to accept (to be included in the final contract); a belief is an item she wants to retain (to be consistent with the final contract); a threat is a statement the bargainer intends the other parties to believe. In Example 1, for instance, *raise* and ¬*raise* are the respective demands of the union and the management. The statements *layoff*→ ¬*jobs* and *strike*→ ¬*production* are the respective beliefs. The statements *raise*→*layoff* and ¬*raise*→*strike* are threats made by each side, respectively[2].

3.1 Entrenchment measure

In belief revision, a belief state of an agent is modeled by a *belief set*, indicating what the agent believes, and an *epistemic entrenchment ordering*, specifying how the agent believes its beliefs [21]. An epistemic entrenchment ordering, introduced by Gärdenfors and Makinson in [24], is a qualitative measurement of the degrees in which an agent entrenches her beliefs when a contraction or a revision operation is carried out. Zhang in [25] extends the concept by quantifying the strengths of entrenchment. More precisely, he assign to each statement in the underlying language a real number, interpreted as the strength in which the agent entrenches her negotiation item that the statement stands for. We will use the same notion to specify the belief states of a negotiating agent.

Definition 1 An *entrenchment measure* ρ is a function that assigns to each sentence in the language $\mathcal{L}$ a real number and satisfies the following condition:

(LR) If $\varphi_1, \cdots, \varphi_m \vdash \varphi$, $\min\{\rho(\varphi_1), \cdots, \rho(\varphi_m)\} \le \rho(\varphi)$. *(Logical rationality)*

As the name suggests, there is an inherent connection between entrenchment measure and epistemic entrenchment ordering. In [24], an epistemic entrenchment ordering is specified by five assumptions, named as (EE1)-(EE5). (EE4) and (EE5) describe the properties of the ordering over a particular belief set, therefore they are not applicable to entrenchment measure. The following observation shows that an entrenchment measure satisfies (EE1)-(EE3).

Observation 1 *An entrenchment measure ρ satisfies the following properties:*

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[2] In many cases, it is not clear-cut what is which. For instance, *jobs* and *production* can be viewed either as beliefs or as demands. The most important factor is how the negotiators entrench the items. In the sequel we will not explicitly distinguish them from each other but simply refer them to as negotiation items.

(EE1) *If* $\rho(\varphi) \leq \rho(\psi)$ *and* $\rho(\psi) \leq \rho(\chi)$*, then* $\rho(\varphi) \leq \rho(\chi)$.
(EE2) *If* $\varphi \vdash \psi$*, then* $\rho(\varphi) \leq \rho(\psi)$.
(EE3) *For any* φ *and* ψ*,* $\rho(\varphi) \leq \rho(\varphi \wedge \psi)$ *or* $\rho(\psi) \leq \rho(\varphi \wedge \psi)$.

Proof: Straightforward from Definition 1. □

It is easy to see that the logical constraint **LR**, introduced by Zhang and Foo in [26], is actually the combination of (EE2) and (EE3). It implies that logically equivalent statements have equal values of entrenchment measure. In other words, the values of entrenchment measure are syntactically independent. Since a finite propositional language has only finite number of non-equivalent sentences, the value range of an entrenchment measure is always finite.

We remark that an entrenchment measure is not a payoff or utility function. For instance, suppose that p_1 represents the demand of a seller "*the price of the good is no less than \$100*" and p_2 denotes "*the price of the good is no less than \$90*". Obviously the seller could get higher payoff from p_1 than p_2. However, since p_1 implies p_2, she will entrench p_2 more than p_1, *i.e.*, $\rho(p_2) > \rho(p_1)$, because, if she fails to keep p_1, she can still bargain for p_2 but the loss of p_2 means the loss of both.

In Appendix, we present a procedure that generates an entrenchment measure from a partial assignment over a non-empty subset of $\mathcal{L}$ provided the partial assignment satisfies **LR**. With this procedure, an agent does not have to provide its entrenchment measure for the whole language. She only needs to represent the preferences over her own negotiation items. This is importance because beliefs are private information of each bargainer. We assume that an arbitrator or an analyser is responsible to perform the procedure.

With the notion of entrenchment measure, we can introduce a similar concept in the belief revision literature known as "cut" in (Grove [27]) or "EE-cut" in (Rott [28]):

$$Cut(\rho, \eta) = \{\varphi \in \mathcal{L} : \rho(\varphi) > \eta\} \tag{1}$$

where η is a real number, referred to as the *cut-off* of the cut. Intuitively, $Cut(\rho, \eta)$ consists of only the sentences with the degrees of entrenchment higher than η.

The following facts are the direct consequences of **LR**, so we omit their proofs.

Lemma 1 *Let* ρ *be an entrenchment measure. Then*

1. $\rho(\bot) \leq \rho(\varphi) \leq \rho(\top)$ *for any* $\varphi \in \mathcal{L}$.
2. *For any* $\eta \in \Re$, $Cut(\rho, \eta)$ *is logically closed.*
3. *For any* $\eta > \rho(\bot)$, $Cut(\rho, \eta)$ *is consistent.*

We now extend the concept of entrenchment measure from single sentences to sets of sentences. To avoid too many notations, we overload the function ρ with the parameter of a set of formulas: for any *consistent* set, X, of sentences, we define

$$\rho(X) = \max\{\rho(\varphi) : \varphi \in \mathcal{L} \setminus X\}. \tag{2}$$

The definition is well founded because the language $\mathcal{L}$ is finite and at least $\perp \notin X$.[3]

The following concept is an analogue of comprehensiveness in game theory (see Section 2). We shall refer it to as the *logical comprehensiveness*.

Definition 2 A set, X, of sentences is *comprehensive* w.r.t. ρ if $\varphi \in X$ and $\rho(\varphi) \leq \rho(\psi)$ imply $\psi \in X$.

Interestingly, the following lemma shows that the concept of logical comprehensiveness coincides with the notion of "cut" in belief revision.

Lemma 2 *A set of sentences X is comprehensive w.r.t. ρ if and only if $X = Cut(\rho, \rho(X))$.*

Proof: The 'if' part is obvious. For the 'only if' part, let $\eta = \rho(X) = \max\{\rho(\varphi) : \varphi \notin X\}$. It follows that there exists $\psi \notin X$ such that $\rho(\psi) = \eta$. Meanwhile, for any $\varphi \in \mathcal{L}$, $\rho(\varphi) > \eta$ implies $\varphi \in X$. Thus $Cut(\rho, \eta) \subseteq X$. On the other hand, if there is a $\varphi \in X$ such that $\rho(\varphi) \leq \eta = \rho(\psi)$, by the comprehensiveness of X, we have $\psi \in X$, which contradicts the fact $\psi \notin X$. Therefore $X = Cut(\rho, \eta)$, *i.e.*, $X = Cut(\rho, \rho(X))$. □

3.2 Belief state

As we have mentioned in the previous subsection, a belief state in the AGM theory consists of a belief set and an epistemic entrenchment ordering. With the concept of entrenchment measure we can update the definition of belief state by using entrenchment measure. Intuitively, all statements about a bargaining situation are ranked by each negotiator in terms of entrenchment measure but only those which ranks are greater than a certain "cut-off" are the actual negotiation items the negotiator bargains for. Formally, we have the following definition.

Definition 3 A *belief state* is a pair (ρ, e), where ρ is an entrenchment measure and e is a real number, satisfying the following conditions:

(NV) $\rho(\top) > e$. *(Non-vacuity)*
(DB) $\rho(\psi) = e$ for some $\psi \in \mathcal{L}$. *(Descriptive bottom-line)*

If we let $Bel(\rho, e) = \{\varphi \in \mathcal{L} : \rho(\varphi) > e\}$, then $Bel(\rho, e)$ represents all the negotiation items of a negotiator. (NV) guarantees that all tautologies are fully entrenched by each negotiator. (DV) ensures that $Bel(\rho, e)$ is consistent and logically closed, that is, $Bel(\rho, e) \nvdash \perp$ and $Bel(\rho, e) = Cn(Bel(\rho, e))$. In other words, a belief state determines a unique belief set $Bel(\rho, e)$ and an ordering over the belief set (induced by ρ). Therefore the belief state we defined above extends the concept of belief state in belief revision (see [21]).

[3] Without the assumption of the finiteness of logical language, the definition would be invalid. Eliminating the assumption requires rephrasing of several definitions, including the redefinition of this concept by using "$\sup$".

3.3 Bargaining games

Now we are ready to formalize a bargaining situation. We shall consider n-person bargaining situations. We let $N = \{1, 2, \cdots, n\}$ stand for a set of negotiators, called *players*, where $n \geq 2$. Each player $i \in N$ is characterized by her belief state (ρ_i, e_i). $Bel(\rho_i, e_i)$ then represents all her negotiation items. We define an *n-person bargaining game* to be a vector in the form of $((\rho_1, e_1), \cdots, (\rho_n, e_n))$, denoted by $((\rho_i, e_i))_{i \in N}$. $\mathcal{B}^{n,\mathcal{L}}$ denotes the class of all n-person bargaining games described in $\mathcal{L}$.

We might want to compare the concept of bargaining games defined above with the standard definition of bargaining games in game theory. Let us refer them to as *logical games* and *numerical games*, respectively[4]. Besides the difference of representation, there are a few more subtle differences between these two concepts. Firstly, in our definition, the belief states of players are independent each other. A player varying its negotiation items or entrenchment measure does not affect other players' belief states while, in game theory, a change of demands from one player would alter the set of alternatives, thus affects the utility functions of other players. Secondly, the logical games encode more information than the numerical games because a belief state not only quantifies players' preferences but also specifies their beliefs, demands and threats as well as their logical relations.

The following concepts will be used in the rest of the paper.

A bargaining game is *compatible* if there is no conflict among the collective negotiation items, that is, $\bigcup_{i \in N} Bel(\rho_i, e_i)$ is consistent.

A bargaining game is *mutually comparable* if $\rho_i(\top) = \rho_j(\top)$ and $e_i = e_j$ for all $i, j \in N$.

A bargaining game is *normalized* if $\rho_i(\top) = 1$ and $e_i = 0$ for all $i \in N$.

Given two bargaining games $B = ((\rho_i, e_i))_{i \in N}$ and $B' = ((\rho'_i, e'_i))_{i \in N}$, we say B' to be a *subgame* of B, denoted by $B' \sqsubseteq B$, if, for each $i \in N$,

1. $e'_i \geq e_i$;
2. $\rho'_i(\varphi) = \rho_i(\varphi)$ for all $\varphi \in \mathcal{L}$ such that $\rho_i(\varphi) > e'_i$;

From the definition we can easily see that if $B' \sqsubseteq B$, then $Bel(\rho'_i, e'_i) = Cut(\rho_i, e'_i) \subseteq Bel(\rho_i, e_i)$ for all i. In other words, the negotiation items of B' for each player is the most entrenched segment (the top part) of the negotiation items of B, or, B extends B' by appending more weaker entrenched negotiation items[5].

Example 2 Consider a 2-person bargaining game in which player 1 demands p and player 2 asks for q. Both players firmly believe that p and q can never be true at the same time, *i.e.*, $\neg(p \wedge q)$. We depict the game with Figure 1.

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[4] More precisely, they should be called as *logically represented bargaining games* and *numerically represented bargaining games*, respectively, because both of them can basically address the same domain of problems (assuming any continuous domain can be discretized in a certain method).

[5] The reader is invited to compare the concept of subgame with the set-inclusion over feasible sets in game theory. You might find that the subgame is much more restrictive. This indicates that our axiom of monotonicity is much weaker than its game-theoretic counterpart (see Section 4.1 for more details).

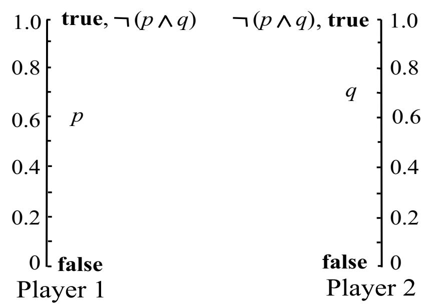

Fig. 1. *An example of two person bargaining game.*

In this figure, the player 1's entrenchment measure is: $\rho_1(\top) = \rho_1(\neg(p \wedge q)) = 1.0$, $\rho_1(p) = 0.6$ and $\rho_1(\bot) = 0$. Player 2's entrenchment measure is similar except for $\rho_2(q) = 0.7$. Note that we did not provide the entrenchment values for the whole language. We can generate the other values by using the procedure provided in Appendix. Then the bargaining game can be denoted by $((\rho_1, 0), (\rho_2, 0))$. Examples of subgames of the game can be easily given, such as $((\rho_1, 0.5), (\rho_2, 0.5))$ and $((\rho_1, 0.8), (\rho_2, 0.6))$.

3.4 Possible deals and bargaining solutions

Different from the traditional bargaining theory, possible agreements are not given in a bargaining game. However, we can generate the set of possible agreements according to players' belief states. We consider that the outcome of a negotiation is a collective concessions made by all the players. Therefore a possible agreement can be easily represented by a vector of subsets of negotiation items. This idea is inspired by Zhang *et al.*'s work in [22].

Definition 4 Let $B = ((\rho_i, e_i))_{i \in N}$ be a bargaining game. A *deal* of B is a vector $D = (D_1, \cdots, D_n)$ satisfying:

1. $D_i \subseteq Bel(\rho_i, e_i)$ for all $i \in N$.
2. $\bigcup_{i \in N} D_i$ is consistent.
3. D_i is comprehensive w.r.t. ρ_i for all $i \in N$.

The set of all deals of B is denoted by $\Omega(B)$, *i.e.*, the feasible set of B.

The first two requirements in the definition are purely logical and intuitive. The third one relies on entrenchment measure of each player. In order to reach an agreement, a player might have to make a concession, *i.e.*, gives up some negotiation items she originally holds. It is reasonable to assume that when a player does so, she always abandons those less entrenched items and try to keep the higher entrenched items as many as possible. Therefore a concession is actually a contraction of negotiation items possibly with a higher cut-off. This idea is exactly the same as "cut revision" [28] and is also implicitly assumed by the AGM theory [21]. Note that $(\emptyset, \cdots, \emptyset)$ belongs to any feasible set of bargaining games. We call this deal the disagreement deal. Therefore $\Omega(B)$ is always non-empty and finite for any $B \in \mathcal{B}^{n,\mathcal{L}}$.

We would like to remark here that the game-theoretic abstraction of bargaining starts from given feasible sets in which inconsistent negotiation items have been resolved. This explains why the game-theoretic characterization of bargaining solutions does not include a mechanism of conflict resolving. This fact seemingly has been largely ignored in the game theory literature.

Definition 5 A *bargaining solution* on $\mathcal{B}^{n,\mathcal{L}}$ is a function f that assigns to a bargaining game $B \in \mathcal{B}^{n,\mathcal{L}}$ a unique deal of the game, *i.e.*, $f(B) \in \Omega(B)$.

Since $\Omega(B)$ can never be empty, a solution always exists. The agreement can then be defines as:

$$A(B) = Cn(\bigcup_{i \in N} f_i(B)) \tag{3}$$

Note that it is not necessary that all the items in the final agreement are visible to every player[6]. The actual contract, which is visible to everybody will be a subset of the final agreement, containing all the demands that have been agreed by all the parties.

Another comment we would like to make here is that a bargaining solution is purely represented by logical statements even though the determination of a solution relies on the numerical measure of entrenchment. It returns the agreement itself rather than its utility values. This is a significant difference between our approach and the game-theoretic approach.

3.5 Comparison of possible deals and measurement of satisfaction

There are a few ways to compare different possible agreements. The first approach is based on set-inclusion over vectors. For any two deals $D, D' \in \Omega(B)$, we write

- $D \succeq D'$ iff $D_i \supseteq D'_i$ for all i;
- $D \succsim D'$ iff $D \succeq D'$ but $D \neq D'$;
- $D \succ D'$ iff $D_i \supset D'_i$ for all i.

Similar to the standard notations in game theory, we define the *weak Pareto frontier* of a game B as

$$WP(B) = \{D \in \Omega(B) : D' \succ D \text{ implies } D' \notin \Omega(B)\} \tag{4}$$

and the *strong Pareto frontier* of B as

$$P(B) = \{D \in \Omega(B) : D' \succsim D \text{ implies } D' \notin \Omega(B)\}. \tag{5}$$

The above comparison criteria and the concepts of Pareto frontiers are defined purely in logical form without using the numerical measure of entrenchment. Since a deal is determined by each player's cut-off, we can also compare different deals in numerical measurement.

[6] We assume that there is a virtual arbitrator who can see all the items in the final agreement.

Given a deal D, for each player i, we know that $\rho_i(D_i)$ is the cut-off of ρ_i for set D_i (see Lemma 2). Therefore the lower $\rho_i(D_i)$ is, the bigger D_i is. We can then use the value of $\rho_i(\top) - \rho_i(D_i)$ to measure D_i. However, this value relies on the individual scale of entrenchment measure, so it is not interpersonally comparable. A better measurement to assess the gain of a player is $\lambda_i = \frac{\rho_i(\top)-\rho_i(D_i)}{\rho_i(\top)-e_i}$. We call λ_i the player i's *degree of satisfaction*. Obviously, $\lambda_i \in [0, 1]$.

We remark that the concept of satisfaction is different from the concept of entrenchment measure. The former indicates the gain of a player from a deal while the latter represents the preference of a bargainer over her negotiation items.

Different from Zhang in [25], we do not view $\rho_i(\top) - \rho_i(D_i)$ or $\frac{\rho_i(\top)-\rho_i(D_i)}{\rho_i(\top)-e_i}$ as the utility of player i because we are not going to apply lotteries over the set of possible deals. Therefore such a definition will not result in a von Neumann-Morgenstern utility. However, it is wroth mentioning that the degree of satisfaction is invariant under positive affine transformations on entrenchment measure, so is satisfies part of the von Neumann-Morgenstern assumptions.

3.6 Numerical mapping

We have seen the differences between the logical representation and game-theoretic representation of bargaining situations. There are also close relationships between them. Given a logical game B, we let

$$S(B) = \{x \in \Re^n : \exists D \in \Omega(B) \forall i \in N.\ D_i = Cut(\rho_i, \rho_i(\top) - x_i)\} \tag{6}$$

It is not hard to see that each point in $S(B)$ uniquely determines a deal in $\Omega(B)$ (not true inversely). In particular, the point $\mathbf{0} = (0, \cdots, 0)$ corresponds to the empty deal $(\emptyset, \cdots, \emptyset) \in \Omega(B)$. Therefore, for each logical game B, there exists a unique numerical game $(S(B), \mathbf{0})$ that corresponds to B. We call $S(B)$ the *numerical feasible set* of B. Unfortunately, $S(B)$ is not necessarily convex unless B is compatible. Figure 2 in Section 4.2 depicts a typical example of such a numerical game. The following observation shows that such a numerical feasible set is comprehensive and compact.

Observation 2 *For any bargaining game B, $S(B)$ is $\mathbf{0}$-comprehensive and compact.*

Proof: Assume that $x \in S(B)$ and $x \geqslant y \geqslant \mathbf{0}$. There exists $D \in \Omega(B)$ such that $D_i = Cut(\rho_i, \rho_i(\top) - x_i)$ for all i. Since $x \geqslant y$, we have $Cut(\rho_i, \rho_i(\top) - y_i) \subseteq D_i$. It turns out that $(Cut(\rho_i, \rho_i(\top) - y_i))_{i \in N}$ is a deal of B, which implies that $y \in S(B)$. Therefore $S(B)$ is $\mathbf{0}$-comprehensive.

$S(B)$ is obviously bounded. To show it is also closed, i.e., $S(B)$ contains its boundary, let $x^k \to x$, where $x^k \in S(B)$. For each k, there is $D^k \in \Omega(B)$ such that $D_i^k = Cut(\rho_i, \rho_i(\top) - x_i^k)$. According to Lemma 2, $D_i^k = Cut(\rho_i, \rho_i(D_i^k))$. It follows by the definition of $\rho_i(D_i^k)$ that $\rho_i(\top) - x_i^k \geq \rho_i(D_i^k)$, which implies that $x_i^k \leq \rho_i(\top) - \rho_i(D_i^k)$. Since $\mathcal{L}$ is finite, the total number of the possible different values of $\rho_i(D_i^k)$ is finite. Therefore there exists a big enough k_i such that for all $k \geq k_i$, $x_i \leq \rho_i(\top) - \rho_i(D_i^k)$. Let $\hat{k} = \max_i k_i$. We then have $x_i \leq \rho_i(\top) - \rho_i(D_i^{\hat{k}})$ for all i. We yield that $Cut(\rho_i, \rho_i(\top) - x_i) \subseteq D^{\hat{k}}$. It follows that $(Cut(\rho_i, \rho_i(\top) - x_i))_{i \in N} \in \Omega(B)$, that is, $x \in S(B)$. Therefore $S(B)$ is closed, so is compact. □

With this numerical mapping, one may think that our problem could be easily solved by applying any game-theoretic solution to its corresponding numerical game and converting the solution back to logical form. There are both technical and methodological obstacles that block us to do so.

From the technical point of view, the corresponded numerical games are not necessarily convex. All the classical bargaining solutions that require convexity are not applicable to our domain [7]. Even those solutions for non-convex domain fail to apply to our problem because these solutions requires to convexify a feasible set or alter a feasible set by using convex hell or set operations [29,30,31,32]. Any such "artificial changes" on feasible sets could result in the numerical games losing its connection to the original logical games.

More importantly, the game-theoretic characterization of bargaining situations reflects only the numerical properties of bargaining solutions. It cannot capture the logical reasoning behind bargaining activities. The ultimate goal of the work is to identify the logical characteristics of bargaining situations. Simply defining a solution does not meet the goal.

4 Logical characterization of bargaining solutions

In this section, we present a logical characterization of bargaining solutions. Similar to the non-convex extensions of bargaining solutions in game theory, it is impossible to keep the uniqueness of solution while requiring the solution to be symmetric and Pareto efficient. We shall offer two different characterizations: unique solution with *weak Pareto optimality* and multiple solutions with Pareto optimality.

4.1 Basic assumptions and axioms

We first investigate the properties we expect a bargaining solution to hold and then try to find the solution that satisfies these properties. Before we start, let us review the properties which have been built in the definition of bargaining solution (see Definition 4):

(IR) $f_i(B) \subseteq Bel(\rho_i, e_i)$ for all i. *(Individual Rationality)*
(Con) $\bigcup_{i \in N} f_i(B)$ is consistent. *(Consistency)*
(Com) $f_i(B)$ is comprehensive w.r.t ρ_i for all i. *(Comprehensiveness)*

The first two assumptions originate from the AGM postulates for single belief revision [33]. Zhang *et al.* and Booth extend them to mutilagent belief revision [34,22]. **IR** confines the bargaining theme to the participants' interests since the only concern of each player is her own negotiation items. **Con** prevents inconsistent agreements. As we remarked in Section 3.1, the logical comprehensiveness has been exclusively used in the construction of belief revision operators. We emphasize that the numerical comprehensiveness is also important for numerical bargaining games, especially for non-convex domains. Roth in [35] has shown that *weak Pareto optimality*, *symmetry*, and *restricted monotonicity* are incompatible if the comprehensiveness is not imposed on the numerical domain of problems. We have seen in last section that the logical comprehensiveness guarantees the comprehensiveness of its corresponding numerical domain (Observation 2). In fact, the assumption **Com** provides a natural link between the logical and numerical representations of bargaining situations.

Now let us consider another set of fundamental properties of bargaining. We shall refer them to as *axioms*.

(CR) If B is compatible, $f_i(B) = Bel(\rho_i, e_i)$ for all i. *(Collective Rationality)*

(Inv) For any positive affine transformation τ over $\Re^n$, $f(\tau(B)) = f(B)$, where $\tau(B) = ((\tau_i \circ \rho_i, \tau_i(e_i)))_{i \in N}$. *(Scale Invariance)*

(Sym) There exists a real number $\lambda \in [0, 1]$ such that for all i, $f_i(B) = \{\varphi \in \mathcal{L} : \rho_i(\varphi) > \lambda e_i + (1 - \lambda)\rho_i(\top)\}$ *(Symmetry)*

(MCM) For any mutually comparable bargaining games B and B', $B' \sqsubseteq B$ implies $f(B') \preceq f(B)$. *(Mutually Comparable Monotonicity)*

CR says that if there is no conflict of interest among the players, all negotiation items from all players should be mutually accepted. This assumption is fundamental to bargaining reasoning for two reasons:

– It reflects the cooperative aspect of bargaining in the sense that every player has incentive to cooperate unless there is a contradiction to her interest.
– It sets out the termination condition for any bargaining procedure. Negotiation is a mean to resolve conflicts. Once the conflicts are settled, there is no need for further compromise.

This axiom is purely logical and is also a fundamental assumption in belief revision, known as *Expansion* [21].

Inv states that the scaling of players' entrenchment measures does not affect bargaining outcomes. Obviously this axioms is an analogue of its game-theoretic counterpart even though there is a subtle difference between them. The axiom says that the solution is *invariant* under the transformation, *i.e.*, $f(\tau(B)) = f(B)$ rather than *co-shifting* under the transformation, *i.e.*, $f(\tau(B)) = \tau(f(B))$.

Sym is to impose fairness on bargaining outcomes. This can be easily done with game theory model. If the players cannot be differentiated on the basis of the description of a bargaining situation after numerical abstraction, we can safely assume that all players gain the same from the bargaining [1]. However, the approach is not applicable to the logically represented bargaining problems because we cannot simply assume that all players have identical logical description about their bargaining situations. If it were so, there would be no conflict between players. In fact, a "fair bargain" does not mean each player receives the same gain from the bargain but means that "*the bargain gives each player equal amount of satisfaction she expect to get from the negotiation*" (otherwise, the less satisfied player could refuse the bargain) [7]. **Sym** expresses the idea exactly.

MCM says that expanding a bargaining game by allowing every player to append more negotiation items will not affect the previously reached agreement provided the appended negotiation items are less entrenched than the existing ones and the expansion is done evenly by all players. It is easy to see that this axiom is an analogue of Roth's *Restricted Monotonicity* [35]. However, **MCM** is much weaker than *Restricted Monotonicity* because the relationship of subgames is more restrictive than set-inclusion over feasible sets.

To capture the intuition behind these axioms, let us consider a simple sequential bargaining model[8]. Suppose that any negotiation is carried out within an allocated time slot. At the beginning of a negotiation, all players submit their negotiation items to an arbitrator. After the game started, each player can withdraw some of her own items from the arbitrator any time during the negotiation period. Once the arbitrator finds that all the remaining items are consistent, she will terminate the procedure and announce the final agreement. We assume that players are impatient with the unproductive passage of time so that they may drop some of less entrenched items with the elapse of time.

Now we can provide a more intuitive interpretation for each of the above axioms. **CR** establishes the termination condition for the arbitrator to apply. **Inv** allows the players each to adjust their entrenchment measures so that the entrenchment measure coincides with the time interval (starting from the same e and ending with the same $\rho(\top)$). **Sym** says that at the time when the negotiation terminates, all the remaining items are included in the final agreement. **MCM** expresses the ides that a longer negotiation time would result in a richer agreement.

4.2 Maxmin solutions

We now search for a solution that satisfies the above axioms. One of the most intuitive idea to construct a bargaining solution is that a solution should be able to balance and maximize the degrees of satisfaction for all players. This leads to the following concept.

Definition 6 For any bargaining game $B = ((\rho_i, e_i))_{i\in N}$, let

$$\Lambda(B) = arg \max_{D\in\Omega(B)} \min_{i\in N} \frac{\rho_i(\top) - \rho_i(D_i)}{\rho_i(\top) - e_i} \quad (7)$$

We call any solution f on $\mathcal{B}^{n,\mathcal{L}}$ satisfying $f(B) \in \Lambda(B)$ a *maxmin solution*.

7 A similar idea has been already expressed in the Nash's original work : "*A solution means a determination of the amount of satisfaction each individual should expect to get from the situation. ... It is reasonable to assume that the two, being rational, would simply agree to that anticipation, or to an equivalent one.*" [1] .

8 Fully developing a sequential model for the logically represented bargaining problems is out of the scope of this paper even though it can be of great help for a deep understanding of the axioms.

Recall that "$arg\max$" denotes the arguments of max. Thus $\Lambda(B)$ contains all the deals in $\Omega(B)$ that maximizes the satisfaction of the least satisfied players. In other words, a maxmin solution tries to maximize and balance the degrees of satisfaction of all players. Apparently, the maxmin solutions of a game are not necessarily unique. The following function defines a unique solution on $\mathcal{B}^{n,\mathcal{L}}$.

Definition 7 Let

$$F(B) = (\bigcap_{D\in\Lambda(B)} D_1, \cdots, \bigcap_{D\in\Lambda(B)} D_n) \tag{8}$$

We call F *the modest maxmin solution*.

Figure 2 illustrates the numerical domain of a typical logical bargaining game and its maxmin solutions (efficient maxmin solutions will be defined in Section 4.4).

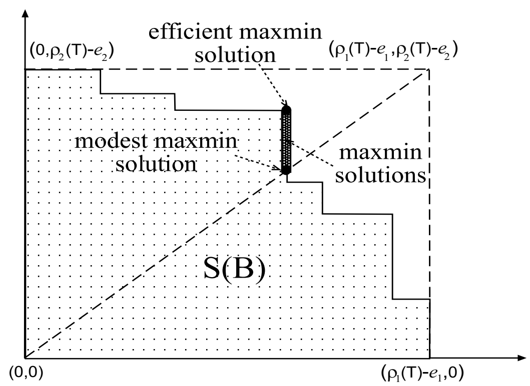

Fig. 2. *Numerical domain of a bargaining game and its maxmin solutions.*

Example 3 Consider the game in Example 2. There is only one maxmin solution to this game $(Cn\{\neg(p\wedge q)\}, Cn\{\neg p, q\})$, which is also the modest maxmin solution. However, if $\rho_1(p) = \rho_2(q) < 1.0$, there will be two maxmin solutions: $(Cn\{p, \neg q\}, Cn\{\neg(p \wedge q)\})$ and $(Cn\{\neg(p\wedge q)\}, Cn\{\neg p, q\})$. The modest maxmin solution is then $(Cn\{\neg(p\wedge q)\}, Cn\{\neg(p\wedge q)\})$. If $\rho_1(p) = \rho_2(q) = 1.0$, then the modest maxmin solution is $(\emptyset, \emptyset)$, the disagreement deal.

The following result gives a more concrete construction of F. It also shows that F itself is a maxmin solution.

Lemma 3 *Given* $B = ((\rho_i, e_i))_{i\in N}$, *let*

$$\lambda = \max_{D\in\Omega(B)} \min_{i\in N} \frac{\rho_i(\top) - \rho_i(D_i)}{\rho_i(\top) - e_i}. \tag{9}$$

Then $F_i(B) = \{\varphi \in \mathcal{L} : \rho_i(\varphi) > \lambda e_i + (1-\lambda)\rho_i(\top)\}$ *for all* i. *Moreover,* $F(B) \in \Lambda(B)$.

Proof: For each i, let

$$\hat{D}_i = \{\varphi \in \mathcal{L} : \rho_i(\varphi) > \lambda e_i + (1-\lambda)\rho_i(\top)\}.$$

Assume that $D \in \Lambda(B)$. Since $\min\limits_{i \in N} \frac{\rho_i(\top)-\rho_i(D_i)}{\rho_i(\top)-e_i} = \lambda$, we have $\rho_i(D_i) \leq \lambda e_i + (1-\lambda)\rho_i(\top)$ for all i. It implies that for any φ, if $\varphi \in \hat{D}_i$, then $\varphi \in D_i$ (otherwise, $\rho_i(D_i) \geq \rho_i(\varphi) > \lambda e_i + (1-\lambda)\rho_i(\top)$, a contradiction). This means that $\hat{D}_i \subseteq D_i$. We then yield that $\hat{D}_i \subseteq F_i(B)$ for all i. It also implies that $\hat{D} \in \Omega(B)$.

On the other hand, for each i, we have $\rho_i(\hat{D}_i) \leq \lambda e_i + (1-\lambda)\rho_i(\top)$. Thus $\frac{\rho_i(\top)-\rho_i(\hat{D}_i)}{\rho_i(\top)-e_i} \geq \lambda$. It follows that $\min\limits_{i} \frac{\rho_i(\top)-\rho_i(\hat{D}_i)}{\rho_i(\top)-e_i} \geq \lambda$. Since $\hat{D} \in \Omega(B)$ and λ is maximal to all deals, we have $\min\limits_{i} \frac{\rho_i(\top)-\rho_i(\hat{D}_i)}{\rho_i(\top)-e_i} = \lambda$, which implies that $\hat{D} \in \Lambda(B)$. Therefore for each i, $F_i(B) \subseteq \hat{D}_i$. We conclude that $F(B) = \hat{D}$. Consequently we have $F(B) \in \Lambda(B)$. □

The following observation shows that the modest maxmin solution is a logical version of KS-solution according to the numerical mapping introduced in Section 3.6.

Observation 3 *Given a bargaining game $B = ((\rho_i, e_i))_{i \in N}$, let $S(B)$ the numerical feasible set of B (defined by (6)). Then for any $i \in N$,*

$$F_i(B) = Cut(\rho_i,\ \rho_i(\top) - KS_i(S(B), \mathbf{0})) \tag{10}$$

*where $KS(S(B), \mathbf{0})$ is the KS-solution of the numerical game (S(B), **0**).*

Proof: We first calculate the ideal point. Notice that for any $x \in S(B)$, $0 \leq x_i \leq \rho_i(\top) - e_i$ for any $i \in N$. On the other hand, we know $(\emptyset, \cdots, \emptyset, Bel(\rho_i, e_i), \emptyset, \cdots, \emptyset) \in \Omega(B)$, which corresponds to the point $(0, \cdots, 0, \rho_i(\top) - e_i, 0, \cdots, 0) \in S(B)$. Therefore the ideal point $a(S, \mathbf{0})$ is $(\rho_1(\top) - e_1, \cdots, \rho_n(\top) - e_n)$.

Now we prove (10). According to Lemma 3, it suffices to show that $KS_i(S(B), \mathbf{0}) = \lambda(\rho_i(\top) - e_i)$ for all i, where λ is defined by (9). Let $x^* = (\lambda(\rho_i(\top) - e_i))_{i \in N}$. Firstly, since $\lambda \in [0,1]$, the point x^* is on the segment from $\mathbf{0}$ to $a(S(B), \mathbf{0})$. Secondly, $F(B) \in \Omega(B)$ implies that $x^* \in S(B)$ because of Lemma 3. Finally we need to prove that x^* is the maximizer of $S(B)$ on the segment from $\mathbf{0}$ to $a(S, \mathbf{0})$. If it is not, there exists $\lambda' \in [0,1]$ such that $\lambda' > \lambda$ and $(\lambda'(\rho_i(\top) - e_i))_{i \in N} \in S(B)$. It follows that there is a deal $D \in \Omega(B)$ such that

$$D_i = Cut(\rho_i, \lambda' e_i + (1-\lambda')\rho_i(\top)) \text{ for all } i.$$

Therefore $\rho_i(D_i) \leq \lambda' e_i + (1-\lambda')\rho_i(\top)$, i.e., $\frac{\rho_i(\top)-\rho_i(D_i)}{\rho_i(\top)-e_i} \geq \lambda'$ for all i. It turns out that $\min\limits_{i \in N} \frac{\rho_i(\top)-\rho_i(D_i)}{\rho_i(\top)-e_i} \geq \lambda' > \lambda$, which contradicts the definition of λ. □

4.3 Characterization

The following theorem shows that the modest maxmin solution is characterized by the four axoms: *Collective Rationality*, *Scale Invariance*, *Symmetry* and *Mutually Comparable Monotonicity*.

Theorem 1 *A bargaining solution on $\mathcal{B}^{n,\mathcal{L}}$ satisfies **CR**, **Inv**, **Sym** and **MCM** if and only if it is the modest maxmin solution F.*

Proof: "⇐" Lemma 3 has shown that F is a bargaining solution. We are to verify that F satisfies the four axioms. **Sym** is implied by Lemma 3. **CR** is true because if B is compatible, $(Bel(\rho_1, e_1), \cdots, Bel(\rho_n, e_n))$ is the unique maximizer of $\min_{i\in N} \frac{\rho_i(\top)-\rho_i(D_i)}{\rho_i(\top)-e_i}$. **Inv** holds because $\frac{\rho_i(\top)-\rho_i(F_i(B))}{\rho_i(\top)-e_i}$ is invariant under any positive affine transformation. We now prove **MCM**. Consider two mutually comparable bargaining games $B = ((\rho_i, e_i))_{i\in N}$ and $B' = ((\rho_i', e_i'))_{i\in N}$. Let

$$\lambda = \max_{D\in\Omega(B)} \min_{i\in N} \frac{\rho_i(\top)-\rho_i(D_i)}{\rho_i(\top)-e_i}, \quad \lambda' = \max_{D\in\Omega(B')} \min_{i\in N} \frac{\rho_i'(\top)-\rho_i'(D_i)}{\rho_i'(\top)-e_i'}$$

Since both B and B' are mutually comparable, for each i, we have

$$\lambda e_i + (1-\lambda)\rho_i(\top) = \min_{D\in\Omega(B)} \max_{i\in N} \rho_i(D_i) \tag{11}$$

$$\lambda' e_i' + (1-\lambda')\rho_i'(\top) = \min_{D\in\Omega(B')} \max_{i\in N} \rho_i'(D_i) \tag{12}$$

As $B' \sqsubseteq B$ implies $\Omega(B') \subseteq \Omega(B)$, it is not hard to prove that for any $D \in \Omega(B')$ and $i \in N$, $\rho_i'(D_i) \geq \rho_i(D_i)$. It then follows that $\min_{D\in\Omega(B')} \max_{i\in N} \rho_i'(D_i) \geq \min_{D\in\Omega(B)} \max_{i\in N} \rho_i(D_i)$. By using (11) and (12), we yield that $\lambda' e_i' + (1-\lambda')\rho_i'(\top) \geq \lambda e_i + (1-\lambda)\rho_i(\top)$. According to Lemma 3, we conclude that $F(B') \preceq F(B)$.

"⇒" Given a bargaining game $B = ((\rho_i, e_i))_{i\in N}$, we can find a positive affine transformation $\tau = (\tau_1, \cdots, \tau_n)$ such that $\tau_i(\rho_i(\top)) = \tau_j(\rho_j(\top))$ and $\tau_i(e_i) = \tau_j(e_j)$ for any i, j. By **Inv**, we have $f(\tau(B)) = f(B)$. Therefore we can simply assume that B itself is mutually comparable. According to **Sym**, there exists a real number $\bar{\lambda} \in [0, 1]$ such that

$$f_i(B) = \{\varphi : \rho_i(\varphi) > \bar{\lambda} e_i + (1-\bar{\lambda})\rho_i(\top)\} \quad \text{for each } i \in N. \tag{13}$$

For each i, we know that $\rho_i(f_i(B)) \leq \bar{\lambda} e_i + (1-\bar{\lambda})\rho_i(\top)$. Thus $\bar{\lambda} \leq \frac{\rho_i(\top)-\rho_i(f_i(B))}{\rho_i(\top)-e_i}$ for all i. It follows that $\bar{\lambda} \leq \min_{i\in N} \frac{\rho_i(\top)-\rho_i(f_i(B))}{\rho_i(\top)-e_i}$.

Let $\lambda = \max_{D\in\Omega(B)} \min_{i\in N} \frac{\rho_i(\top)-\rho_i(D_i)}{\rho_i(\top)-e_i}$. Since $f(B) \in \Omega(B)$, we have $\bar{\lambda} \leq \lambda$. By Lemma 3,

$$F_i(B) = \{\varphi : \rho_i(\varphi) > \lambda e_i + (1-\lambda)\rho_i(\top)\} \text{ for all } i. \tag{14}$$

Comparing (13) and 14, we know that $\bar{\lambda} \leq \lambda$ implies $f_i(B) \subseteq F_i(B)$ for all i. It turns out that $f(B) \preceq F(B)$.

To show $F(B) \preceq f(B)$, let $B' = ((\rho_i', e_i'))_{i\in N}$ be a bargaining game such that, for each $i \in N$,

1. $e_i' = \lambda e_i + (1-\lambda)\rho_i(\top)$;
2. $\rho_i'(\varphi) = \begin{cases} \rho_i(\varphi), & \text{if } \varphi \in F_i(B); \\ \lambda e_i + (1-\lambda)\rho_i(\top), & \text{otherwise.} \end{cases}$

where λ is defined above. It is easy to see that $B' \sqsubseteq B$ and B' is mutually comparable. Moreover, B' is compatible. According to **CR** and Lemma 3, $f(B') = F(B)$. By **MCM**, we have $f(B') \preceq f(B)$. Therefore $f(B) = F(B)$. □

The above result is similar to but different from Conley and Wilkie's characterization of KS-solution to non-convex problems [36]. In that work, the KS-solution is characterized by weak Pareto optimality, symmetry, scale invariance and restricted monotonicity with comprehensive but not necessarily convex domain of problems. We will show that replacing **CR** with weak Pareto optimality does not lead to a characterization of the modest maxmin solution (see Section 4.5 Observation 5).

4.4 Pareto efficient solutions

Whenever a negotiation comes to a standstill, the modest maxmin solution assumes that an balanced concession will be made by all the parties in order to reach an agreement. Therefore the outcome is not necessarily Pareto efficient.

Definition 8 A maxmin solution f on $\mathcal{B}^{n,\mathcal{L}}$ is *efficient* if it satisfies the following property:

(PO) $f(B) \in P(B)$. *(Pareto Optimality)*

where $P(B)$ is the strong Pareto frontier of B defined by (5).

PO requires a solution to maximize the gains of all players. To achieve this goal, strong cooperation among all players is needed. Consider the scenario in Example 2. If both players entrench their demands with the same strength, i.e., $\rho_1(p) = \rho_2(q)$, the negotiation will fall into a standstill. If nobody is willing to give in, the outcome of the negotiation will reach a lose-lose situation, which is not Pareto optimal. However, if one of the player agrees to give way to the other, a Pareto optimal solution will be achieved (see Example 3).

A few issues arise when we try to characterize the efficient maxmin solutions. Firstly, the efficient maxmin solutions are not necessarily unique. A solution maximizing one player's gain might not be able to maximize the gain of other players. Secondly, **PO** is not compatible with **Sym**. A counterexample can be easily constructed by using Example 2. **MCM** is also no long true for efficient maxmin solutions. This is because one efficient solution that favors one player in one game might not do the same in its subgames. However, the efficient maxmin solutions satisfy the following natural weakening of **MCM**[9]:

(R. MCM) For any mutually comparable bargaining games B and B', if $B' \sqsubseteq B$ and B' is compatible, then $f(B') \preceq f(B)$. *(Restricted Mutually Comparable Monotonicity)*

[9] Interestingly, Hougaard and Tvede in [30] also introduce a weak version of monotonicity in a similar way by requiring the "small" problem to be convex. However, no intuitive remark is given. In fact, R.MCM provides a good interpretation for their axiom because a compatible game corresponds to a convex numerical feasible set.

The following theorem shows that an efficient maxmin solution is characterized by the three axioms: *Scale Invariance*, *Pareto Optimality* and *Restricted Mutually Comparable Monotonicity*.

Theorem 2 *A bargaining solution on $\mathcal{B}^{n,\mathcal{L}}$ is an efficient maxmin solution if and only if it satisfies* ***Inv****,* ***PO*** *and* ***R.MCM****.*

Proof: "$\Leftarrow$" The only non-trivial proof is the one for **R.MCM**. Suppose that $B', B \in \mathcal{B}^{n,\mathcal{L}}$ are both mutually comparable. Assume that $B' \sqsubseteq B$ and B' is compatible. Let $e' = e'_i$ for all i. Then for each i, $f_i(B') = Bel(\rho'_i, e'_1) = \{\varphi : \rho'_i(\varphi) > e'_i\} = \{\varphi : \rho_i(\varphi) > e'\}$. It follows that $\rho_i(f_i(B')) \leq e'$. Thus $\max\limits_{i \in N} \rho_i(f_i(B')) \leq e'$. Since $f(B') \in \Omega(B)$, we then have $\min\limits_{D \in \Omega(B)} \max\limits_{i \in N} \rho_i(D_i) \leq e'$. Let $\lambda = \max\limits_{D \in \Omega(B)} \min\limits_{i \in N} \frac{\rho_i(\top) - \rho_i(D_i)}{\rho_i(\top) - e_i}$. Because B is mutually comparable, we can entail that $e' \geq \lambda e_i + (1-\lambda)\rho_i(\top)$ for all i. By Lemma 3, we know $F_i(B) = \{\varphi : \rho_i(\varphi) > \lambda e_i + (1-\lambda)\rho_i(\top)\}$. Therefore $f_i(B') \subseteq F_i(B) \subseteq f_i(B)$ for each i, as desired.

"$\Rightarrow$" Given a game $B = ((\rho_i, e_i))_{i \in N}$, by **Inv**, we can simply assume that B is mutually comparable. Similar to the proof of Theorem 1, we let $B' = ((\rho'_i, e'_i))_{i \in N}$ be a bargaining game such that, for each $i \in N$,

1. $e'_i = \lambda e_i + (1-\lambda)\rho_i(\top)$;
2. $\rho'_i(\varphi) = \begin{cases} \rho_i(\varphi), & \text{if } \varphi \in F_i(B); \\ \lambda e_i + (1-\lambda)\rho_i(\top), & \text{otherwise.} \end{cases}$

where $\lambda = \max\limits_{D \in \Omega(B)} \min\limits_{i \in N} \frac{\rho_i(\top) - \rho_i(D_i)}{\rho_i(\top) - e_i}$. Thus $B' \sqsubseteq B$ and $Bel(\rho'_i, e'_i) = F_i(B)$ for all i. In addition, $\bigcup\limits_{i \in N} F_i(B)$ is consistent. Hence $F(B) \in \Omega(B')$ and $F(B)$ is the only Pareto optimal deal in $\Omega(B')$. Applying **PO** on $\Omega(B')$, we get $f(B') = F(B)$. Since B' is mutually comparable and compatible, by **R.MCM**, we have $f(B') \preceq f(B)$, i.e., $F(B) \preceq f(B)$, which implies that $f(B) \in \Lambda(B)$. Therefore f is a maxmin solution. By **PO**, we conclude that f is an efficient maxmin solution. $\square$

In [30], Hougaard and Tvede characterize the efficient KS-solutions to non-convex problems with the axioms similar to the original KS axioms. Xu and Yoshihara in [37] also presents a characterization of the set-valued KS-solution with an axiomatization that is more close to the original KS axioms than Hougaard and Tvede's but assuming comprehensiveness. Interestingly, both characterizations are more complicated than ours even though they are based on a higher level abstraction than the logical abstraction. Notice that we do not require symmetry because symmetry is not a characteristic of the Pareto efficient solutions.

4.5 Other axioms and their relationships

In this section, we consider a few other properties of bargaining solutions that have been exclusively investigated in game theory, such as weak Pareto optimality, Independence of Irrelevant Alternatives and Independence of Alternative Other Than the Disagreement Point and Ideal Point.

Weak Pareto Optimality Weak Pareto Optimality (**WPO**) was firstly introduced by Nash in [1]. It says that a negotiation will not end up with an agreement if there is available outcome in which they are all better off. In most situations, **WPO** is more acceptable than **PO** because it requires less cooperation among players. The logical version of **WPO** can be easily expressed as follows:

(WPO) $f(B) \in WP(B)$. *(Weak Pareto Optimality)*

The following result shows that the modest maxmin solution is weakly efficient.

Observation 4 ***CR**, **Inv**, **Sym** and **MCM** implies **WPO**:*

Proof: According to Theorem 1, we only have to show F satisfies WPO. Suppose that there exists a deal $D \in \Omega(B)$ such that $D \succ F(B)$. Then for each $i \in N$, there is a sentence $\varphi \in D_i$ such that $\varphi \notin F_i(B)$. It follows that $\rho_i(F_i(B)) \geq \rho_i(\varphi)$. Since D_i is comprehensive, $\rho_i(\varphi) > \rho_i(D_i)$. It turns out that $\rho_i(F_i(B)) > \rho_i(D_i)$, which implies that $\frac{\rho_i(\top)-\rho_i(F_i(B))}{\rho_i(\top)-e_i} < \frac{\rho_i(\top)-\rho_i(D_i)}{\rho_i(\top)-e_i}$. Therefore $F(B) \notin \Lambda(B)$, a contradiction. □

One might think that we can replace **CR** by using **WPO** in Theorem 1. However, this is not true.

Observation 5 ***CR** is independent of **Inv**, **Sym**, **MCM** and **WPO**.*

Proof: We know that all the five axioms together induce a solution to be the modest maxmin solution. To show the independence of **CR**, it suffices to exhibit a solution that satisfies all the other four axioms and is different from the modest maxmin solution F (see Definition 6). We only consider the case when $n = 2$. The proof can be easily extended to the cases when $n > 2$.

We construct a bargaining solution F^c on $\mathcal{B}^{2,\mathcal{L}}$ in the following. For any game $B = ((\rho_1, e_1), (\rho_2, e_2))$,

Case 1: if B satisfies the condition $Bel(\rho_1, e_1) = Cn(\top)$ and $Bel(\rho_2, e_2) \supset \{\varphi : \rho_2(\varphi) = \rho_2(\top)\}$ (*i.e.*, $\exists\psi \in Bel(\rho_2, e_2).\rho_2(\psi) < \rho_2(\top)$), we let
$F^c_1(B) = Cn(\top)$ and
$F^c_2(B) = \{\varphi : \rho_2(\varphi) = \rho_2(\top)\}$;

Case 2: otherwise, we let $F^c(B) = F(B)$.

Note that $Cn(\top)$ represents the set of all tautologies. $\{\varphi : \rho_2(\varphi) = \rho_2(\top)\}$ represents all the statements that are entrenched by player 2 as firmly as a tautology. It is not hard to construct such a game that satisfies the condition of case 1. Since any game B in case 1 is compatible, we have $F_2(B) = Bel(\rho_2, e_2)$. Therefore F^c diverges from F in case 1. Now we only need to prove that F^c satisfies all the four axioms. Obviously, F^c satisfies **WPO** because $F^c_1(B) = F_1(B)$ for any $B \in \mathcal{B}^{2,\mathcal{L}}$.

To verify **Inv**, notice that the condition of case 1 is invariant under any positive affine transformation. It is sufficient to show that the solution F^c is invariant for any game that satisfies the condition of case 1. This is obviously true because $a\rho_2(\varphi)+b = a\rho_2(\top)+b$ iff $\rho_2(\varphi) = \rho_2(\top)$ for any φ, $a(> 0)$ and b.

To show **Sym**, given a game B, if it is in case 2, we have $F^c(B) = F(B)$. If B belongs to case 1, we let $\lambda = \frac{\rho_2(\top)-\rho_2(F^c_2(B))}{\rho_2(\top)-e_2}$, or, $\rho_2(F^c_2(B)) = \lambda e_2 + (1-\lambda)\rho_2(\top)$.

Since $F_2^c(B)$ is comprehensive, we have $F_2^c(B) = Cut(\rho_2, \rho_2(F_2^c(B)))$. Therefore we yield that $F_1^c(B) = Cn(\top) = Bel(\rho_1, e_1) = Cut(\rho_1, \lambda e_2 + (1-\lambda)\rho_2(\top))$ and $F_2^c(B) = Cut(\rho_2, \rho_2(F_2^c(B))) = Cut(\rho_2, \lambda e_2 + (1-\lambda)\rho_2(\top))$.

Finally we prove **MCM**. Given two mutually comparable games B and B' such that $B' \sqsubseteq B$, if B and B' are both in case 2, then the result is obviously true. If both are in case 1, then $F_1^c(B') = F_1^c(B)$. To compare $F_2^c(B')$ and $F_2^c(B)$, notice that for any $\varphi \in Bel(\rho'_2, e'_2)$, we have $\rho'(\varphi) = \rho(\varphi)$. Therefore $F_2^c(B') = \{\varphi \in Bel(\rho'_2, e'_2) : \rho'_2(\varphi) = \rho'_2(\top)\} = \{\varphi \in Bel(\rho'_2, e'_2) : \rho_2(\varphi) = \rho_2(\top)\} \subseteq \{\varphi \in Bel(\rho_2, e_2) : \rho_2(\varphi) = \rho_2(\top)\}$. It then follows that $F_2^c(B') \subseteq F_2^c(B)$. We get $F^c(B') \preceq F^c(B)$. If B' in case 1 and B in case 2, we have $F^c(B') \preceq F(B') \preceq F(B) = F^c(B)$. Therefore $F^c(B') \preceq F^c(B)$. If B' is in case 2 and B in case 1, it must be in the situation that $Bel(\rho_1, e_1) = Bel(\rho'_1, e'_1) = Cn(\top)$ and $Bel(\rho'_2, e'_2) = \{\varphi : \rho'_2(\varphi) = \rho'_2(\top)\}$. Therefore $F_1^c(B) = F_1^c(B') = Cn(\top)$ and $F_2^c(B) = \{\varphi : \rho_2(\varphi) = \rho_2(\top)\} = \{\varphi : \rho'_2(\varphi) = \rho'_2(\top)\} = F_2(B') = F_2^c(B')$. Surely we have $F^c(B') \preceq F^c(B)$.

To conclude, we have proved that there exists a solution that satisfies the axioms **WPO**, **Inv**, **Sym** and **MCM** but the solution is not identical to the modest maxmin solution. Therefore these axioms do not imply **CR**. □

This result shows that the four axioms **Inv**, **Sym**, **MCM** and **WPO** are not enough to characterize the modest maxmin solution. According to Conley and Wilkie's result, we can conclude that the axioms **Inv**, **Sym**, and **MCM** are weaker than their game-theoretic counterparts. The logical axiom **CR** plays an irreplaceable role in the characterization of the logical bargaining solution.

Contraction independence Independence of Irrelevant Alternatives (IIA) is another fundamental assumption in Nash's bargaining theory. The idea can be expressed in our terminology as follows:

(UC) *If $B' \sqsubseteq B$ and $f(B) \in \Omega(B')$, then $f(B') = f(B)$. (Unbalanced Contraction)*

Nash's IIA is the most controversial one among his axioms [7]. Consider a simple example. A young couple were deciding how to spend the coming weekend. The boy suggested to see the new released movie "Casino Royale" ($\{seeing\_a\_movie$, $Casino\_Royale\}$). The girl agreed on seeing a movie but does not like 007 ($\{seeing\_a\_movie$, $\neg Casino\_Royale\}$). After a unpleasant skirmish, they failed to decide which movie they will see but still agreed on going to cinema ($\{seeing\_a\_movie\}$). One day later before the weekend, the girl said: "I heard that *Casino Royale* was not that bad" (she gave up her belief $\neg Casino\_Royale$). A new agreement was formed accordingly ($\{seeing\_a\_movie$, $Casino\_Royale\}$). We see that the new situation is a subgame of the old situation and the previously reached agreement is a feasible deal of the subgame but the previously reached agreement is not the agreement of the subgame.

The following result shows that **UC** is not compatible with the other two axioms that are considered to be much more fundamental.

Observation 6 *There is no bargaining solution over $\mathcal{B}^{2,\mathcal{L}}$ satisfying **CR**, **Sym** and **UC**.*

Proof: Consider a 2-person bargaining game $B = ((\rho_1, e_1), (\rho_2, e_2))$ described in the language $\mathcal{L}$ in which p is one of the propositional variables. Assume that ρ_1 and ρ_2 are the induced entrenchment measures by the procedure described in Appendix based on the the initial assignments $\rho_1(\top) = \rho_1(p) = \rho_2(\top) = \rho_2(\neg p) = 1$, $\rho_1(\bot) = \rho_2(\bot) = 0$ and $e_1 = e_2 = 0$. Then there are three possible deals in $\Omega(B)$: $(\emptyset, \emptyset)$, $(Cn\{p\}, \emptyset)$ and $(\emptyset, Cn\{\neg p\})$. According to **Sym**, only $(\emptyset, \emptyset)$ can be the solution (the satisfaction cannot be both in 1). However, both $((\rho_1, 0), (\rho_2, 1))$ and $((\rho_1, 1), (\rho_2, 0))$ are subgames of B. Since $(\emptyset, \emptyset)$ is a deal of these subgames, by **UC**, $(\emptyset, \emptyset)$ should be the solution of the subgames. On the other hand these two subgames are all compatible. By **CR**, $(Cn\{p\}, \emptyset)$ and $(\emptyset, Cn\{\neg p\})$ is the solution of each subgame, respectively. This leads to a contradiction. □

Interestingly , the following natural weakening of **UC** goes well with our axioms:

(BC) For any mutually comparable bargaining games B and B', if $B' \sqsubseteq B$ and $f(B) \in \Omega(B')$, then $f(B') = f(B)$. *(Balanced Contraction)*

Observation 7 ***CR**, **Inv**, **Sym** and **MCM** implies **BC**.*

Proof: Notice that Lemma 3 implies that for any mutually comparable game B,

$$F_i(B) = \{\varphi \in \mathcal{L} : \rho_i(\varphi) > \eta\} \tag{15}$$

where $\eta = \min_{D \in \Omega(B)} \max_{i \in N} \rho_i(D_i)$. Suppose that $B' \sqsubseteq B$ and B' is mutually comparable. Since $\Omega(B') \subseteq \Omega(B)$, we have $\eta' = \min_{D \in \Omega(B')} \max_{i \in N} \rho'_i(D_i) \geq \min_{D \in \Omega(B)} \max_{i \in N} \rho_i(D_i)$ $= \eta$. However, if $F(B) \in \Omega(B')$, then we must have $\eta' = \eta$. Since for each i, $\eta' \geq e_i$. According to Theorem 1, we have $F(B') = F(B)$. Thus $f(B') = f(B)$. □

It is easy to see that **BC** is the counterpart of Roth's *Independence of Alternative Other Than the Disagreement Point and Ideal Point*, which is satisfied by the KS-solution (see [6] page 107).

4.6 The Nash bargaining solution

The result shown in above subsection seemingly suggests that the Nash bargaining solution is no longer the most desirable solution if we represent a bargaining situation in logic. In fact, this is not true. A bargaining solution is excessive sensitive to the choice of domain (see [38] p. 46). The Nash bargaining solution is built up on the assumption that the bargaining situations are represented in the von Neumann-Morgenstern utilities that are derived from preferences over lotteries which satisfy the expected utility assumptions [39]. Zhang in [25] has shown that if we allow to play lotteries over possible deals, the Nash solution can be extended to the logically represented domain of problems. However, if we do not allow to randomize the feasible sets, the Nash solution is not necessarily the most intuitive solution.

Conley and Wilkie in [29] shows that with non-convex domain, the characterization of the Nash solution requires a significant change on IIA. Conley and Wilkie introduce the following assumption as the replacement of IIA in conjunction with continuity:

– Ethical Monotonicity (E.Mon): If $S' \subseteq S$, $d' = d$, and $E^f(S,d) \in conh(S')$, then $f(S',d) \leq f(S,d)$, where $E^f(S,d) = f(conh(S),d)$.

It is easy to see that E. Mon is much less intuitive than IIA. Mariotti in [31] improves Conley and Wilkie's result by dropping continuity and weakening IIA. Nonetheless, the resulting axiomatization is still not as intuitive as the original Nash's system and much less natural than the axioms for the KS-solution.

It is worth mentioning that if we allow multi-valued solutions, the characterization can be much less complicated. Kaneko [40], Conley and Wilkie [29], Zhou [41], Mariotti [38], Xu and Yoshihara [37] have shown that the original Nash's axiomatization can be mostly retained if we allow multi-valued or set-valued solutions when the domain of problems is non-convex. These results shed some light on the further extension of the Nash bargaining solution to the logically represented domain of problems. Nevertheless, multi-valued solutions or set-valued solution is not always an option for computing-related applications due to the following reasons:

– For most AI-related applications, a single-valued solution is essential, especially for the implementation of automated negotiation systems, which requires a single solution as the output. In such a case, the intuitive axioms lead to the KS-solution.
– If multi-valued or set-valued solutions are allowed, the objections of randomizing possible deals become weak.
– The maxmin rule is one of the most general criteria for decision-making that have been widely used in AI-related applications. However, the applicability of the maximization of the product of utilities heavily relies on the utility representation and its semantics.

In such a sense, the KS-solution is the most preferable option for AI-related applications.

5 The wage negotiation example

In this section, we examine our solution by using the introductory example we presented in Section 1. To gain a better understanding of the methodology we use in the analysis, we first simplify the example by temporally ignoring the strike-related conflicts and concentrating on the conflict about wage raise and unemployment.

Figure 3 depicts a special situation whereby the conflict arises due to the union's demand of wage raise and the management's threat of laying off employees.

As it is shown in the diagram, the initial assignment of the entrenchment measures of the parties is the following:

Union	Management
$\rho_u(\top) = 1.0$; $\rho_u(\textit{jobs}) = 0.9$; $\rho_u(\textit{layoff} \rightarrow \neg \textit{jobs}) = 0.7$; $\rho_u(\textit{raise}) = 0.6$; $\rho_u(\bot) = 0.0$.	$\rho_m(\top) = 1.0$; $\rho_m(\textit{raise} \rightarrow \textit{layoff}) = 0.7$; $\rho_m(\bot) = 0.0$.

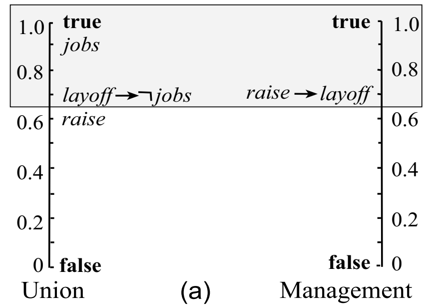

Fig. 3. *A special situation of wage negotiation.*

The conflict between the negotiation parties can be easily identified as follows: $\{$*raise, raise*$\rightarrow$*layoff, layoff*$\rightarrow \neg$*jobs, jobs*$\} \vdash \perp$.

Since the union considers that the members jobs are more important than a wage raise, meanwhile the management's determination of laying off employees is relatively high ($\rho_m(\textit{raise} \rightarrow \textit{layoff}) > \rho_u(\textit{raise})$), the union will probably lose the negotiation. There is only one maxmin solution among 14 possible deals in this situation, which is (see the shaded part in the diagram):

$$(Cn\{jobs, \textit{layoff} \rightarrow \neg jobs\}, Cn\{raise \rightarrow \textit{layoff}\})$$

The corresponding satisfaction for the union and the management is $(0.4, 1.0)$, respectively. The situation leads to a lose-win outcome, where the management is the winner because its threat has taken effect.

Figure 4 shows two different variations of situation (a). Situation (b) is the same as

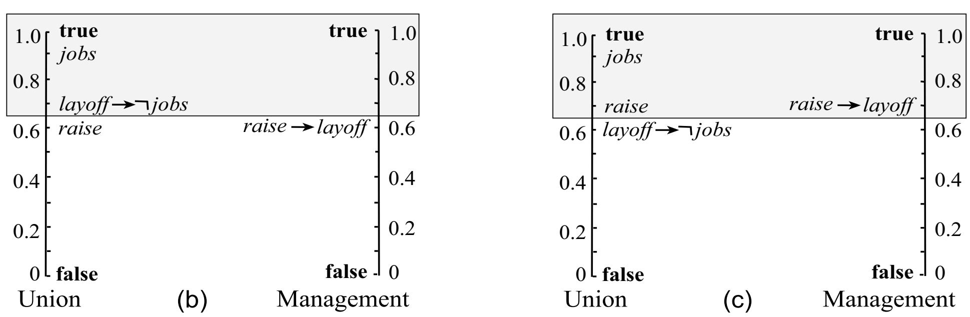

Fig. 4. *Two other special situations of wage negotiation.*

(a) except that the management is slightly less certain on the necessity and feasibility

of laying off employees but still entrenches it as firmly as the union entrenches its appeal. In this situation there are two efficient maxmin solutions: $(Cn\{\textit{jobs}, \textit{layoff} \rightarrow \neg\textit{jobs}, \textit{raise}\}, Cn\{\top\})$ and $(Cn\{\textit{layoff} \rightarrow \neg\textit{jobs}, \textit{jobs}\},$
$Cn\{\textit{raise} \rightarrow \textit{layoff}\})$. The modest maxmin solution is:

$$(Cn\{\textit{layoff} \rightarrow \neg\textit{jobs}, \textit{jobs}\}, Cn\{\top\}).$$

The outcome is a "lose-lose" situation in which the union fails its appeal and the weakness of the firm is exposed (because its threat is not included in the agreement). The corresponding satisfaction for each party is $(0.4, 0.4)$. However, if the management's determination of the layoff is even weaker, say $\rho_m(\textit{raise} \rightarrow \textit{layoff}) = 0.5$, the negotiation will lead to a "win-lose" outcome (the management gives up before the union does).

Figure 4 (c) describes a "win-win" situation in which the union considers the job market is not that bad, so it gains more negotiation power than in situation (a). In this case, the agreement is:

$$(Cn\{\textit{jobs}, \textit{raise}\}, Cn\{\textit{raise} \rightarrow \textit{layoff}\}).$$

The union wins the negotiation, notwithstanding some union members would have to faced up with finding other job opportunities. Note that the satisfaction for each party in this situation is $(0.4, 1.0)$, which is exactly the same as in situation (a). In this sense, we cannot differentiate these two situations with the game-theoretic model.

Finally we consider a more complicated situation in which all the negotiation items and threats described in Example 1 are taken into account.

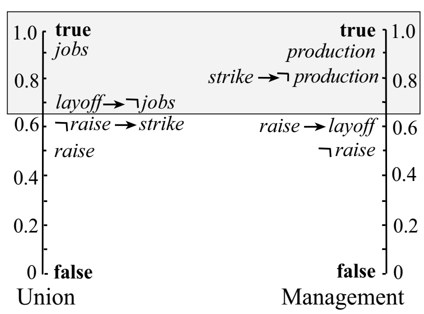

Fig. 5. *A tit-for-tat situation of wage negotiation.*

Figure 5 shows a tit-for-tat bargaining situation in which both sides insists on their demands in the same entrenchment degree. The situation leads to two efficient maxmin solutions among 32 different deals: $(Cn\{\textit{jobs}, \textit{layoff}{\rightarrow}\neg\textit{jobs}, \neg\textit{raise}{\rightarrow}\textit{stike}, \textit{raise}\}, Cn\{\textit{production},$ $\textit{strike}{\rightarrow}\neg\textit{production}\})$ and $(Cn\{\textit{jobs}, \textit{layoff}{\rightarrow}\neg\textit{jobs}\},$

$Cn\{$*production*, *strike* $\rightarrow \neg$*production*, *raise* $\rightarrow$ *layoff*, $\neg$*raise*$\})$. The modest maxmin solution is:

$$(Cn\{\textit{jobs}, \textit{layoff} \rightarrow \neg \textit{jobs}\}, Cn\{\textit{production}, \textit{strike} \rightarrow \neg \textit{production}\})$$

None of the parties is completely satisfied with the outcome but the outcome gives them the same degree of satisfaction $(0.4, 0.4)$.

We might have noticed through the example that the logical model of bargaining offers a different approach of bargaining analysis from the traditional game-theoretic method. The approach allows us to vary a problem with different situations to identify the conflicts among the negotiation parties and to resolve the conflicts through both logical and numerical reasoning. We have seen that the situation (a) and (c) in the example have the same numerical mapping $comp\{(1.0, 0.3), (0.4, 1.0)\}$ but result in totally different outcomes. The situation (b) corresponds to a symmetric numerical game $comp\{(1.0, 0.4), (0.4, 1.0)\}$, whereas the situation and its outcome is not symmetric at all. The refinement of bargaining modeling discloses the roles of various key factors on the bargaining outcome, which is fundamental to the development of an understanding of bargaining reasoning and to the determination of the sources of players' bargaining power[10]. In addition, the example also illustrates that bargaining reasoning is non-monotonic (see, for instance, [43]). The threats $\neg$*raise* $\rightarrow$ *strike* and *raise* $\rightarrow$ *layoff* plays a role as soft rules in the determination of bargaining solution. The interleaving ordering of these rules determines which rules we should use in the inference.

6 Related work

This work draws heavily on three different research areas: *cooperative models of bargaining* in game theory, *belief revision* in Artificial Intelligence (AI) and *automated negotiation* in mutliagent systems. Some ideas are inspired by the author and his colleagues' previous work [44,45,46,26,22,25,42,47]. In this subsection, we briefly summarize the related work in these researches.

6.1 Cooperative model of bargaining in game theory

Although logical reasoning is not a major concern in game theory[11], the game-theoretic bargaining theory has a profound influence on the present work. The bargaining solution we have proposed in this paper is basically a logical version of Kalai-Smorodinsky solution in terms of the numerical mapping of logically represented bargaining games (see Observation 3). Therefore there are some similarities between our results and the

[10] Computational issues could arise due to the refinement of the analysis. We leave these issues for the future research. The reader is referred to Zhang and Zhang [42] for some preliminary investigations on the computational model and computational complexity analysis of logic-based bargaining solutions.

[11] It is worth mentioning that a few pieces of work have been done in game theory that apply logical approach to the analysis of economic phenomena (see [48]).

KS-solution as well as its non-convex extensions, such as Herrero (1989), Conley and Wilkie (1991), Hougaard and Tvede (2003), and Xu and Yoshihara (2006).

Herrero in [49] presented an extension of KS-solution to non-convex 2-person bargaining problems. Her solution is set-valued and the characterisation of the solution requires lower semi-continuity and a variation of IIA. Hougaard and Tvede in [30] refined the result by eliminating the requirement of continuity and extending the domain to n-person games. Xu and Yoshihara in [37] also present a characterization of the set-valued KS-solution with comprehensive but non-convex domain of problems. Different from these results, our axiomatization for multi-valued solutions is more concise, which does not require symmetry.

Conley and Wilkie proposed and characterized an extension of single-valued KS-solution to n-person no-nconvex bargaining problems. The axioms used are equivalent to those used by Kalai and Smorodinsky in [3] except that PO is replaced by WPO.

We remark that although some of our axioms are inspired by the game-theoretic axioms, not all axioms and assumptions in our framework have game-theoretic counterparts. In our model, a bargaining situation is specified in terms of players' bargaining items and their preferences over the items. The feasible set is generated through a conflict resolving mechanism. The game-theoretic model, on the other hand, abstracts a bargaining situation in a higher level that the feasible set is assumed to be given, where conflicts of negotiation items have been resolved. Therefore, there are no game-theoretic axioms that correspond to *Consistency* and *Collective Rationality*. Even though the axioms, such as *Scale Invariance*, *Pareto Optimality* and *Weakly Pareto Optimality*, are clear analogue of the related game-theory axioms, they are by no means equivalent to their counterpart. In fact, most of our axioms and assumptions are weaker than their game-theoretic counterparts in the sense that we cannot arbitrarily generate a numerical domain with particular mathematical properties, say $\{x \in \Re^n : \Sigma x_i \leq n\}$, because it may lose its connection to the original logical representation. Observation 5 has clearly shown the difference between our axioms and their game-theoretic counterparts. More importantly, our axiomatization is much more natural and intuitive than game-theoretic axiomatization.

6.2 Belief revision

Belief revision is a theory dealing with the problem of belief conflicting. Much of the research in the field is based on the work of Alchourrón, Gärdenfors and Makinson (1985 [33]), which has been widely referred to as the *AGM theory*. An axiomatic characterization of belief revision operations is developed in the AGM theory. The assumptions **IR**, **Con**, and **CR** in this work are inspired by the AGM's postulates [21]. The concept of entrenchment measure is also developed based on the notion of epistemic entrenchment in belief revision as the name suggests [24]. The difference between them is that an epistemic entrenchment ordering ranks logical statements in pre-order while an entrenchment measure maps a sentence to a real number. As we have mentioned in Section 3.1, the notion of comprehensiveness comes from the idea of cut revision [27,28], in which whenever a revision of belief state is carried out, the sentences that are given up are those having the lowest degrees of epistemic entrenchment.

Recently, the AGM theory has been applied to model mutual belief revision, belief merging, information fusion and belief arbitration [34,44,50,51,22]. The work on belief merging, information fusion and arbitration uses **IR**, **Con**, and **CR** as part of axiomatization [34,51,50]. However, the basic tenet of that work diverges from negotiation in the sense that belief merging maximizes the contributions from different information resources (preserving consistency) while negotiation requires a "fair outcome" that balances players' gains. Therefore their axiomatic characterizations are different. The work on mutual belief revision and the belief-revision-based model of negotiation specify the generic logical properties of multiagent belief revision and negotiation [44,46,45,22,42]. Opposite to the game theoretic approach, that work is limited to purely qualitative analysis, therefore the information about bargainers' preferences cannot be fully specified within that framework.

6.3 Automated negotiation

There has been increasing interest in recent years in automated negotiation [12,8,16,10,11]. Game-theoretic approach has been successfully applied to the modeling and evaluation of interaction and cooperation among autonomous agents [10]. Logical approach has been also taken into account in negotiation and bargaining agent modeling. The most influential work in this research is the argumentation-based framework of negotiation [16,14,17]. Kraus *et al* has proposed a logical model specifying the procedure of argumentation. The framework enables explicit representation of negotiation items, promises, threats and more importantly, arguments (negotiating communications) by using BDI logic. However, the focus of that work is on the representation of negotiation procedures and communications between negotiating agents rather than the characteristics of bargaining situations. Therefore the motivations and conclusions are different from the present work. It is worth mentioning that we do not model agent's goals, desires and intentions not only because we try to achieve simple and clear-cut exploration but also a cooperative model of bargaining does not need to model bargaining procedures. Therefore the actions driven by agent's goals, desires and intentions are not the major concern of our framework.

Zhang in [25] has proposed a logical model of the Nash solution by representing bargaining situations in propositional logic. Although the Nash solution and its characterization are retained in his framework, logical reasoning is not clearly identified through the characterization. This is due to the use of lotteries over possible agreements, which causes the logically represented possible agreements being completely quantified. Therefore the bargaining solution relies on the numerical representation. More precisely, the outcome of bargaining is represented in a form with the combinational of probabilities and logical statements. The current work does not allow to randomize possible deals. Thus the outcome of bargaining can be purely represented in logical language. Since the Nash solution is not the theme of this work, that framework can be viewed as a complement of the current work.

Zhang and Zhang in [42,47] present a computational model for the belief-revision-based negotiation model. They also use the maxmin rule to construct bargaining solutions. Different from the present work, that work is based on strategic model of bargaining in non-cooperative environment. However, that work shed light on the computational properties of logic-based bargaining solutions.

7 Conclusion and discussion

We have defined and characterized a logical solution to n-person bargaining problems. We express a bargaining situation in propositional logic with numerical representation of bargainers' preferences. The bargaining solution is constructed based on the idea of balancing and maximizing bargainers' satisfaction by applying the maxmin rule to possible deals. We have shown that the solution is uniquely characterized by four axioms: *collective rationality*, *scale invariance*, *symmetry* and *mutually comparable monotonicity* in conjunction with three other fundamental assumptions: *individual rationality*, *consistency* and *comprehensiveness*. We characterize the Pareto efficient solutions by the axioms *scale invariance*, *Pareto optimality* and *restricted mutually comparable monotonicity* along with the basic assumptions.

The axiomatization specifies two basic elements of bargaining reasoning: resolution of conflicts and impartiality of bargains. The purely logical axioms *individual rationality*, *consistency* and *collective rationality* models the reasoning of conflict resolving. The axioms with game-theoretic counterparts, such as *scale invariance*, *symmetry* and *mutually comparable monotonicity*, are used for balancing and maximizing the bargainers' gains (satisfaction) from the ultimate bargains. It is not surprising that these two sets of axioms can deal with each of the problems separately because all the purely logical axioms are sourced from belief revision, a formalism of conflict resolving, and the rest of axioms have deep roots in game theory. However, it is surprising that these two sets of axioms cooperate each other very well that we cannot see any gap between them. We have shown that none of them can stand alone in the characterization. The whole axiomatization is natural and intuitive, which clearly exhibits a coherent combination of the logical (so qualitative) and quantitative characteristics of bargaining reasoning.

The ultimate goal of the work is to offer a new methodology of bargaining analysis with logical reasoning. However, the logical theory of bargaining is not a rival of the game-theoretic bargaining theory. We view them as complementary. The game-theoretic approach has certainly an advantage to the continuous domain of problems. The logical theory is basically designed for the discrete domains even though any continuous problem can be discretized and represented in logical language theoretically.

As Rubinstein points out, "*the language of utility allows the use of geometrical presentations and facilitates analysis; in contrast, the numerical presentation results in an unnatural statement of the axioms and the solutions* ([52] p.85). The language of logic offers a different level of abstraction on bargaining situations, which allows an explicit representation of bargaining items and a natural modeling of bargaining reasoning. More importantly, the switch to the logical language leads to a new methodology of bargaining analysis. As we have shown in Section 5, a "horizontal" analysis of different situations under one physical bargaining problem can be conducted, which differentiates itself from the classical game-theoretic approach.

It is clear-cut from the AI perspective that the logical framework of bargaining provides a foundation for the implementation of intelligent negotiating agents with reasoning components. It also makes it possible to incorporate other AI approaches, such BDI logic[53], defeasible logic[54], belief revision and belief merging, into automated negotiation systems. We can also perceive that the modeling of bargaining reasoning would facilitate more comprehensive analysis of human interactions, therefore would broaden the applications of bargaining theory that already exist in the fields of sociology, politics, economics as well as computer science.

Appendix: Completion of entrenchment measure

In this appendix, we present a supplementary result which shows that any partial entrenchment ranking on a non-empty subset of $\mathcal{L}$ can be extended to an entrenchment measure on the whole language.

As we have mentioned earlier, beliefs are private information. A player may not know the other players' beliefs, therefore it is impractical to assume that each player has an entrenchment measure over the whole underlying language. However, each player must have a personal preferences on her own negotiation items. We can then assume that each player submits a partial entrenchment measure on her negotiation items to an arbitrator. The arbitrator extends the partial entrenchment measure for each player to the whole language based on the assumption that if an item is not specified in the partial entrenchment measure, the least value is assigned to the item as long as the overall function satisfies LR. For instance, if $\rho(p) = r$, we can simply assume $\rho(p \vee q) = r$ unless a higher value has been assigned to $p \vee q$ in the partial entrenchment measure (with LR, we always have $\rho(p) \leq \rho(p \vee q)$ for any q). The following procedure implements the idea.

Let X be any non-empty subset of $\mathcal{L}$. Suppose that ρ is a function that maps X to real numbers satisfying LR (see Definition 1). We shall extend the function to an entrenchment measure $\hat{\rho}$ on the language $\mathcal{L}$ so that $\hat{\rho}$ preserves the values of ρ on X and satisfies LR.

Firstly, we construct a hierarchy, $\{X^i\}_{i=1}^{+\infty}$, of X as follows:

1. $T^0 = X$;
2. For $k \geq 0$,
   $X^{k+1} = \{\varphi \in T^k : \rho(\varphi) = \max_{\psi \in T^k} \rho(\psi)\}$;
   $T^{k+1} = T^k \backslash X^{k+1}$.

Secondly, we extend the hierarchy of X to a hierarchy, $\{\bar{X}^i\}_{i=1}^{+\infty}$, of $Cn(X)$ with the following procedure:

1. $\bar{X}^1 = Cn(X^1)$.
2. $\bar{X}^{k+1} = Cn(\bigcup_{i=1}^{k+1} X^i) \backslash \bigcup_{i=1}^{k} \bar{X}^i$ for $k > 0$.

Finally, we define a function $\bar{\rho}$ from $\mathcal{L}$ to $\Re$ as follows:

1. for each $\varphi \in \bar{X}^k$, $\bar{\rho}(\varphi) = \rho(\psi)$ for some $\psi \in X^k$;
2. for each $\varphi \in \mathcal{L} \backslash Cn(X)$, $\bar{\rho}(\varphi) = \min\{\rho(\psi) : \psi \in X\} - 1$.

Note that we could assume that each player gives a value to $\bot$, indicating the lowest degree to the least entrenched item. In this case we have $\bot \in X$, hence $Cn(X) = \mathcal{L}$. If it is not the case, the arbitrator creates a lowest degree for the player as shown in above case 2.

Observation 8 *$\bar{\rho}$ defined above is an entrenchment measure. Moreover, $\bar{\rho}$ is a conservative extension of ρ that preserves the values of ρ over X, that is, for any $\varphi \in X$, $\bar{\rho}(\varphi) = \rho(\varphi)$.*

Proof: According to LR, all logically equivalent statements have the same degree of entrenchment. Therefore both the hierarchies $\{X^i\}_{i=1}^{+\infty}$ and $\{\bar{X}^i\}_{i=1}^{+\infty}$ are well defined. In fact, there are a number M such that $X = \bigcup_{i=1}^{M} X^i$ and $Cn(X) = \bigcup_{i=1}^{M} \bar{X}^i$. It is easy to see that for any $\varphi \in X^i$ and $\psi \in X^j$, $i < j$ implies $\rho(\varphi) > \rho(\psi)$. The similar assertion is also true for $\bar{\rho}$.

Now we prove that $\bar{\rho}$ is an entrenchment measure, that is, $\bar{\rho}$ satisfies LR. Suppose that $\varphi_1, \cdots, \varphi_m \vdash \varphi$. Let $r = \min\{\bar{\rho}(\varphi_1), \cdots, \bar{\rho}(\varphi_m)\}$. If $r = \min\{\rho(\psi) : \psi \in X\} - 1$, according to the definition of $\bar{\rho}$, we always have $\bar{\rho}(\varphi) \geq r$. If $r > \min\{\rho(\psi) : \psi \in X\} - 1$, then we know $\varphi_1, \cdots, \varphi_m \in Cn(X)$. It follows that there is a number i_0 such that $\varphi_1, \cdots, \varphi_m \in \bigcup_{i=1}^{i_0} \bar{X}^i$ and $r = \rho(\psi)$ for some $\psi \in X^{i_0}$. Since $\bigcup_{i=1}^{i_0} \bar{X}^i = Cn(\bigcup_{i=1}^{i_0} X^i)$, it follows that $\varphi \in \bigcup_{i=1}^{i_0} \bar{X}^i$. We then have $\bar{\rho}(\varphi) \geq \rho(\psi) = r$.

Finally, we prove that $\bar{\rho}$ is a conservative extension of ρ. It is sufficient to show that for each k, $X^k \subseteq \bar{X}^k$. Obviously, $X^1 \subseteq \bar{X}^1$. Assume that for all $i \leq k$, $X^i \subseteq \bar{X}^i$. We prove that $X^{k+1} \subseteq \bar{X}^{k+1}$. If it is not true, then $X^{k+1} \cap \bigcup_{i=1}^{k} \bar{X}^i \neq \emptyset$. It turns out that there exists $\varphi \in X^{k+1}$ such that $\varphi \in \bigcup_{i=1}^{k} \bar{X}^i$. Thus there is $i_0 \leq k$ such that $\varphi \in \bar{X}^{i_0}$, or, $\varphi \in Cn(\bigcup_{i=1}^{i_0} X^i)$. It follows that there are $\varphi_1, \cdots, \varphi_m \in \bigcup_{i=1}^{i_0} X^i$ such that $\varphi_1, \cdots, \varphi_m \vdash \varphi$. By LR, $\min\{\rho(\varphi_1), \cdots, \rho(\varphi_m)\} \leq \rho(\varphi)$. According to the construction of the hierarchy $\{X^i\}_{i=1}^{+\infty}$, it is not hard to show that $\varphi \in \bigcup_{i=1}^{i_0} X^i$, which contradicts the fact that $\varphi \in X^{k+1}$. Therefore $X^{k+1} \subseteq \bar{X}^{k+1}$, as desired. $\square$

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