

Directed Feedback Vertex Set is Fixed-Parameter Tractable

Igor Razgon, Barry O'Sullivan

Computer Science Department, University College Cork
{i.razgon,b.o'sullivan}@cs.ucc.ie

Abstract. We resolve positively a long standing open question regarding the fixed-parameter tractability of the parameterized Directed Feedback Vertex Set problem. In particular, we propose an algorithm which solves this problem in $O(8^k k! * poly(n))$.

Keywords. Directed Feedback Vertex Set, Fixed-Parameter Tractability

1 Introduction

In this paper we address the following problem. Given a directed graph G and a parameter k . Find a subset S of vertices of G of size at most k such that any directed cycle of G intersects with S or, if there is no such a subset, answer 'NO'. This is the parameterized Directed Feedback Vertex Set (DFVS) problem. The fixed-parameter tractability of this problem is a long-standing open question in the area of parameterized complexity. In this paper we resolve this question positively by proving the following theorem.

Theorem 1. *The parameterized DFVS problem can be solved in time $O(8^k k! * poly(n))$ where n is the number of vertices of G and $poly(n)$ is a polynomial on n whose degree is a constant independent of k .*

1.1 Overview of the Proposed Method

First of all, we define a graph separation problem on a directed *acyclic* graph (DAG) D as follows. Given two disjoint sets $X = \{x_1, \dots, x_l\}$ and $Y = \{y_1, \dots, y_l\}$ of vertices of D called the *terminals*. A subset R of non-terminal vertices *orderly separates* X from Y if $D \setminus R$ has no path from x_i to y_j for each x_i, y_j such that $i \geq j$. Find a subset R as above of size at most k or, if there is no such a subset, answer 'NO'. We call this problem parameterized ORDERED MULTICUT in a DAG (ORD-MC-DAG). Now, the proof of Theorem 1 consists of two stages.

On the first stage we assume that the parameterized ORD-MC-DAG problem is fixed-parameter tractable (FPT). Under this assumption we prove that the parameterized DFVS problem is FPT as well. In order to show this, we design an algorithm solving the parameterized DFVS problem in time $O(2^k k! * poly(n))$.

$f(k, n) * poly(n)$), where $f(k, n)$ is the runtime of an algorithm solving the parameterized ORD-MC-DAG problem. The proposed algorithm for the parameterized DFVS problem is based on the principle of iterative compression, which recently attracted a considerable attention from researchers in the field [1,2,3]. The proposed algorithm appears in [4] as a part of the proof that the parameterized DFVS is FPT-equivalent to the parameterized ORD-MC-DAG problem.

On the second stage we propose an algorithm solving the parameterized ORD-MC-DAG problem in time $O(4^k * poly(n))$, thus proving that the parameterized ORD-MC-DAG problem is FPT. In order to design the algorithm we considered the $O(4^k * poly(n))$ algorithm for the MULTIWAY CUT problem on *undirected* graph proposed in [5]. The resulting algorithm for the ORD-MC-DAG problem is obtained by adaptation of the method proposed in [5] to the terms of the ORD-MC-DAG problem.

Theorem 1 immediately follows from combination of the above two stages.

1.2 Related Work

Currently it is known that DFVS problem is FPT for a number of classes of directed graphs [6,7,8]. These classes are amenable to the *short cycle* approach, according to which a cycle of length $f(k)$ is identified and the branching is performed on the vertices of the cycle with recursive invocation of the algorithm to the corresponding *residual* graph. However, as noted in [8], the shortest cycle approach is unlikely to lead to a parameterized algorithm for the general DFVS problem.

The connection between DFVS and the graph separation problem has been noticed in [9], where a polynomial transformation of DFVS to a version of the MULTICUT problem on directed graphs has been described. This connection has been refined in [4] where the parameterized ORD-MC-DAG problem has been introduced and proven to be FPT-equivalent to the parameterized DFVS problem. As said in the previous subsection, a part of the proof serves as the first stage of the proof of Theorem 1 of the present paper.

There has been a considerable attention from the parameterized complexity community to the separation problems on *undirected* graphs. FPT-algorithms for the MULTIWAY CUT problem and a restricted version of the MULTICUT problem were proposed in [10]. An improved algorithm for the MULTIWAY CUT problem has been proposed in [5]. As mentioned above, an adaptation of this algorithm to the ORD-MC-DAG problem serves as the second stage of the proof of Theorem 1. Improved algorithms solving the MULTICUT problem for a number of special classes of graphs are proposed in [11].

For the parameterized DFVS problem on *undirected* graphs, the challenging questions were to design an algorithm solving this problem in $O(c^k * poly(n))$ where c is a constant and to obtain a polynomially bounded *kernel* for this problem. The former problem has been solved independently in [1,12], the size of the constant has been further improved in [13]. The latter problem has been solved first in [14]. The size of the kernel has been drastically improved in [15].

Finally, non-trivial exact exponential algorithms for non-directed and directed FVS problems appear in [16,17,18].

1.3 Notations

Let G be a directed graph. We denote its sets of vertices and edges by $V(G)$ and $E(G)$, respectively. Let $(u, v) \in E(G)$. Then (u, v) is a *leaving* edge of u and an *entering* edge of v . Accordingly, u is an *entering neighbor* of v and v is a *leaving neighbor* of u . Also, u is the *tail* of (u, v) and v is the *head* of (u, v) . A vertex u is *minimal* if it has no entering neighbors and *maximal* if it has no leaving neighbors.

Let $ES \subseteq E(G)$. We denote by $G[ES]$ the subgraph of G created by the edges of ES and the vertices incident to them. We denote by $G \setminus ES$ the graph obtained from G by removal of the edges of ES . For a set $R \subseteq V(G)$, $G \setminus R$ denotes the graph obtained from G by removal the vertices of R and their incident edges

In our discussion we frequently mention a *path*, a *cycle*, or a *walk* in a directed graph. By default, we mean that they are *directed* ones.

A directed feedback vertex set (DFVS) of G is a subset S of $V(G)$ such that $G \setminus S$ is a directed acyclic graph (DAG). Let A and B be disjoint subsets of vertices of $V(G)$. A set $R \subseteq V(G) \setminus (A \cup B)$ *separates* A from B if $G \setminus R$ has no path from any vertex of A to any vertex of B .

The parameterized problems considered in this paper get as input an additional parameter k and their task is to find an output of size at most k or to answer 'NO' if there is no such an output. A parameterized problem is fixed-parameter tractable (FPT) if it can be solved in time $O(g(k) * poly(n))$, where n is the size of the problem (in this paper, the number of vertices of the underlying graph), $poly(n)$ is a polynomial on n whose degree is a constant independent of k . Sometimes we call the time $O(g(k) * poly(n))$ an *FPT-time* and an algorithm solving the given problem in an FPT-time an *FPT-algorithm*.

1.4 Organization of the paper

The rest of the paper is a proof of Theorem 1. Section 2 presents the first stage of the proof and Section 3 presents the second stage of the proof as outlined in the above overview.

2 Parameterized DFVS problem is FPT if Parameterized ORD-MC-DAG problem is FPT

Let D be a DAG and let $X = \{x_1, \dots, x_l\}$, $Y = \{y_1, \dots, y_l\}$ be two disjoint subsets of its vertices called *the terminals*. We say that a subset R of non-terminal vertices of D *orderly separates* X from Y if $D \setminus R$ has no path from x_i to y_j for all i, j from 1 to l such that $i \geq j$. We call the corresponding problem of finding the smallest set of non-terminal vertices orderly separating X from Y ORDERED

MULTICUT in a DAG and abbreviate it as ORD-MAC-DG.¹ The *parameterized* ORD-MC-DAG problem gets as an additional parameter an integer $k \geq 0$, its task is to find a set R orderly separating X from Y of size at most k or to say ‘NO’ if there is no such a set. In this section we assume that the parameterized ORD-MC-DAG problem is FPT and let $SolveORDMCDAG(D, X, Y, k)$ be a procedure solving this problem in an FPT-time. Based on this assumption, we design an FPT-algorithm for the parameterized DFVS problem.

The proposed algorithm for DFVS is based on the principle of *iterative compression* which recently proved successful for the design of parameterized algorithms for a number of problems. In particular, let v_1, \dots, v_n be the vertices of the input graph G . The algorithm iteratively generates a sequence of graphs G_0, \dots, G_n where G_0 is the empty graph and G_i is the subgraph of G induced by $\{v_1, \dots, v_i\}$. For each generated graph the algorithm maintains a DFVS S_i of this graph having size at most k or returns ‘NO’ if for some G_i it turns out to be impossible. If the algorithm succeeds to construct S_n it is returned because this is a DFVS of $G = G_n$ having size at most k .

The sets S_i are computed recursively. In particular, $S_0 = \emptyset$. For each S_i , $i > 0$, if S_{i-1} is a DFVS for G_i then $S_i = S_{i-1}$. Otherwise, if $|S_{i-1}| \leq k - 1$, then $S_i = S_{i-1} \cup \{v_i\}$. Finally, if none of the above two cases is satisfied then we denote $S_{i-1} \cup \{v_i\}$ by S'_i (observe that $|S'_i| = k + 1$) and try to get a DFVS S_i of G_i of size smaller than S'_i . In particular, for each subset F of S'_i , the algorithm applies procedure $ReplaceDFVS(G_i \setminus F, S'_i \setminus F)$ whose output is a DFVS F' of $G_i \setminus F$ of size smaller than $S'_i \setminus F$ and disjoint with $S'_i \setminus F$ or ‘NO’ if none exists. If we succeed to find at least one such F' then $S_i = F \cup F'$. Otherwise, ‘NO’ is returned. In other words, the algorithm guesses all possibilities of $F = S'_i \cap S_i$ and for each guessed set F the algorithm tries to find an appropriate set $S_i \setminus S'_i$. Clearly the desired set S_i exists if and only if at least one of these attempts is successful.

The pseudocode of the $ReplaceDFVS$ function is shown below.

$ReplaceDFVS(G, S)$

Parameters: a directed graph G and a DFVS S of G , $|S|$ denoted by m .

Output: a DFVS R of G which is disjoint with S and having size smaller than S or ‘NO’ if no such R exists.

1. **If** G is acyclic **then** return the empty set.
2. **If** S induces cycles **then** return ‘NO’.
3. Let ES be the set of all edges of G entering to the vertices of S .
4. **For each** possible ordering s_1, \dots, s_m of the vertices of S **do**
5. For each s_i , let T_i be the set of vertices w of $G \setminus S$ such that $G[ES]$ has a path from w to s_i .

¹ For the sake of convenience of the analysis, we admit some abuse of notation treating sets as ordered sequences. To circumvent this problem we can consider that the vertices are assigned with names so that (x_1, \dots, x_l) is the lexicographic ordering of the names of X and (y_1, \dots, y_l) is the lexicographic ordering of the names of Y .

6. Let G' be a graph obtained from $G \setminus ES$ by introducing a set $T = \{t_1, \dots, t_m\}$ of new vertices and for each t_i introducing an edge (w, t_i) for each $w \in T_i$ ²
7. **If** $SolveORDMCDAG(G', S, T, |S| - 1)$ does not return 'NO' **then** return the output of $SolveORDMCDAG(G', S, T, |S| - 1)$
8. **endfor**
9. Return 'NO'

Denote by $f(k, n)$ the time complexity of $SolveORDMCDAG$ applied to a graph of n vertices and parameter k and let us evaluate the time complexity of the above algorithm for the parameterized DFVS problem. For each of n iterations, the algorithm checks at most 2^{k+1} subsets of vertices of the current DFVS. Each check involves the run of the $ReplaceDFVS$ function with the size of its second parameter bounded by $k + 1$. Accordingly, the number of distinct orderings explored by the main cycle of the function is at most $(k + 1)!$ For each ordering, the function $SolveORDMCDAG$ is called exactly once and the size of its last parameter is bounded by k . The resulting runtime is $O(2^k * k! * f(k, n) * poly(n))$, where $poly(n)$ takes into account the $O(n)$ iterations of the iterative compression method, auxiliary operations such as checking whether the given set is indeed a DFVS of G , and factor $k + 1$ of the above factorial.

The non-trivial part of the analysis is the correctness proof of $ReplaceDFVS$, which is provided by the following theorem.

Theorem 2. *If $ReplaceDFVS(G, S)$ returns a set R , it satisfies the output specification and conversely, if 'NO' is returned, then there is no set satisfying the output specification.*

Proof. Assume first that $ReplaceDFVS(G, S)$ returns a set R . This means that there is an ordering s_1, \dots, s_m of S such that R orderly separates S from T in G' where T and G' are as defined by the algorithm. By definition of an orderly separating set, $R \subseteq V(G) \setminus S$. Assume by contradiction that R is not a DFVS of G and let C be a cycle of $G \setminus R$.

By definition of ES , the graph $G \setminus ES$ is acyclic therefore C contains edges of ES . Partition the edges of ES in C into maximal paths. Let P_1, \dots, P_l be these paths listed by the order of their appearance in C . It follows from definition of ES that each P_i ends with a vertex s_{j_i} for some j_i . Since line 2 of $ReplaceDFVS(G, S)$ rules out the possibility that the edges of ES may induce cycles and due to the maximality of P_i , path P_i begins with a vertex which does not belong to S that is, with some $w_i \in T_{j_i}$. Considering again that $G[ES]$ is acyclic, in order to connect P_1, \dots, P_l into a cycle, C includes a path in $G \setminus R \setminus ES$ from s_{j_1} to a vertex of T_{j_2} , \dots , from $s_{j_{l-1}}$ to a vertex of T_{j_l} , from s_{j_l} to T_{j_1} . Clearly $(j_1 \geq j_2) \vee \dots \vee (j_{l-1} \geq j_l) \vee (j_l \geq j_1)$ because otherwise we get a contradictory inequality $j_1 < j_1$. Thus $G \setminus R \setminus ES = (G \setminus ES) \setminus R$ has a path from

² Note that $G \setminus ES$ is a DAG because any cycle of G includes a vertex of S and hence an edge of ES . By construction, G' is DAG as well. Note also that graphs G' are isomorphic for all possible orders, we introduce the operation within the cycle for convenience only.

some s_i to a vertex of T_j such that $i \geq j$. By definition of G' , graph $G' \setminus R$ has a path from s_i to t_j in contradiction to our assumption that R orderly separates S from T in G' . This contradiction proves that R is a DFVS of G .

Now, consider the opposite direction. We prove that if R is a DFVS of G disjoint from S and of size at most $|S| - 1$ then it orderly separates S from T in G' for *at least one ordering* s_1, \dots, s_m of S . It will immediately follow that if *SolveORDMCDAG* function returns 'NO' for *all* possible orders then there is no DFVS of G with the desired property and the answer 'NO' returned by *ReplaceDFVS*(G, S) in this case is valid.

So, let R be a DFVS of G with the desired properties and fix an arbitrary ordering s_1, \dots, s_m of S . Let t_1, \dots, t_m and G' be as in the description of *ReplaceDFVS*(G, R). Then the following two claims hold.

Claim 1 *For each i , $G' \setminus R$ has no path from s_i to t_i .*

Proof. Assume that this is not true and let P be such a path, let w be the immediate predecessor of t_i in this path. By definition of G' , the prefix P'' of P ending by w is a path of $G' \setminus R$. Taking into account the definition of G' , $w \in T_i$ and G has a path P' from w to s_i including the edges of ES only. Observe that the vertices of P' do not intersect with R . Really, the heads of all edges of P' belong to S which is disjoint from R by definition, the first vertex w does not belong to R because w participates in a path of $G' \setminus R$. Thus path P' is a subgraph of $G' \setminus R$. The concatenation of P' and P'' creates a closed walk in $G' \setminus R$, which, of course, contains a cycle obtained by taking the closest repeated vertices. This is a contradiction to our assumption that R is a DFVS of G . \square

Claim 2 *Fix an arbitrary l such that $1 \leq l \leq m$. Then there is p such that $1 \leq p \leq l$ such that $G' \setminus R$ no path from s_p to any other t_i from 1 to l .*

Proof. Intuitively, the argument we use in this proof is analogous to the argument one uses to demonstrate existence of minimal vertices in a DAG.

Assume that the claim is not true. Fix an arbitrary i , $1 \leq i \leq l$. Since according to claim 1, $G' \setminus R$ has no path from s_i to t_i , there is some $z(i)$, $1 \leq z(i) \leq l$, $z(i) \neq i$ such that $G' \setminus R$ has a path P_i from s_i to $t_{z(i)}$.

Consider a sequence i_0, \dots, i_l , where $i_0 = i$, $i_j = z(i_{j-1})$ for each j from 1 to l . This is a sequence of length $l + 1$ whose elements are numbers from 1 to l . Clearly there are at least two equal elements in this sequence. We may assume w. l. o. g. that these are elements i_0 and i_y where $1 \leq y \leq l$ (if these elements are i_q and i_r where $0 < q < r$ we can just set $i_0 = i_q$ and rebuild the above sequence). For each j from 0 to $y - 1$, consider the path P'_{i_j} obtained from path P_{i_j} by removal of its last vertex. By definition of G' , P'_{i_j} is a path in $G' \setminus R$ and finishing by a vertex $w_{i_{j+1}} \in T_{i_{j+1}}$.

Let P''_1, \dots, P''_y be paths in $G[ES]$ such that each P''_j is a path from w_{i_j} to s_{i_j} (such a path exists by the definition of w_{i_j}). Arguing as in Claim 1, one can see that each P''_j is a path in $G' \setminus R$. Consequently, $G' \setminus R$ has a directed walk obtained by the following concatenation of paths: $P'_{i_0}, P''_1, \dots, P'_{i_{y-1}}, P''_y$. This walk begins

with s_{i_0} and finishes with s_{i_y} . Since we assumed that $i_0 = i_y$, we have a closed walk in $G \setminus R$ which contains a cycle in contradiction to the definition of R as a DFVS of G . \square

Now, we construct the desired ordering by a process that resembles the topological sorting. Fix an index p such that s_p does not have a path to any t_i in G' as guaranteed by Claim 2. If $p \neq m$ then interchange s_p and s_m in the ordering being constructed (of course if two terminals of S interchange, then the corresponding terminals of T , t_p and t_m in the considered case, interchange as well). Assume that the last $m - l$ vertices in the ordering of S have been fixed. If $l = 1$ then, taking into account that $G' \setminus R$ has no path from s_1 to t_1 in $G' \setminus R$ by Claim 1, the resulting ordering is ready. Otherwise, fix p , $1 \leq p \leq l$ as stated by Claim 2. If $p \neq l$, interchange s_l and s_p in the ordering. Proceed until all the elements of the order are fixed. \blacksquare

Thus, in this section we have proved the following theorem.

Theorem 3. *The parameterized DFVS problem can be solved in time of $O(2^k * k! * f(k, n) * \text{poly}(n))$, where $f(k, n)$ is the time of solving the parameterized ORD-MC-DAG problem on a graph with $O(n)$ vertices.*

3 Parameterized ORD-MC-DAG problem is FPT

In this section we provide an FPT algorithm for the parameterized ORD-MC-DAG problem whose input is a DAG G , the sets $X = \{x_1, \dots, x_l\}$ and $Y = \{y_1, \dots, y_l\}$ of terminals, and a parameter $k \geq 0$. First of all, we notice that we may assume that all vertices of X are minimal ones and all vertices of Y are maximal ones. In particular, we show that graph G can be efficiently transformed into a graph G' , $V(G) = V(G')$, for which this assumption is satisfied so that a set R orderly separates X from Y in G if and only if R orderly separates X from Y in G' .

Let G' be a graph obtained from G by the following 2-stages transformation. On the first stage, remove all entering edges of each x_i and all leaving edges of each y_i . On the second stage we introduce new edge (u, v) for each pair of non-terminal vertices u, v such that G has edges $(u, x_i), (x_i, v)$ or $(u, y_i), (y_i, v)$ for some terminal x_i or y_i (of course, new edges are introduced only for those pairs that do not have edges (u, v) in G). Let G' be the resulting graph. Note that G' is a DAG because it is a subgraph of the transitive closure of G .

Proposition 1. *A set $R \subseteq V(G) \setminus (X \cup Y)$ orderly separates X from Y in G if and only if it orderly separates X from Y in G' .*

Proof. Assume that R orderly separates X from Y in G but does not do this in G' and let P be a path from x_i to y_j ($i \geq j$) in $G' \setminus R$. Replace each edge (u, v) which is not present in G by the pair of edges of G which are replaced by (u, v) according to the above transformation. The resulting sequence P' of vertices form a walk in G . Since G is a DAG, vertex repetitions (and cycles as a result) cannot occur, hence P' is a path in G . The vertices of $V(P') \setminus V(P)$ are

terminal ones, hence they do not belong to R . Consequently, P' is a path from x_i to y_j in $G \setminus R$, in contradiction to our assumption regarding R .

Assume now that R has the orderly separation property regarding G' but fails to orderly separate the specified pairs of terminals in G . Let P be a path from x_i to y_j in $G \setminus R$ such that $i \geq j$. Replace each appearance of an intermediate terminal vertex in P by an edge from its predecessor to its successor in P . As a result we obtained a path from x_i to y_j in $G' \setminus R$ in contradiction to our assumption. ■

Proposition 1 justifies the validity of our assumption that the vertices of X are minimal in G and the vertices of Y are maximal ones.

In order to proceed, we extend our notation. We denote by $OrdSep(G, X, Y)$ the size of the smallest set of vertices of $G \setminus (X \cup Y)$ orderly separating X from Y in G . If $(x_i, y_j) \in E(G)$ for some i and j such that $i \geq j$, we set $OrdSep(G, X, Y) = \infty$ because even the removal of all nonterminal vertices will not orderly separate X from Y . For two disjoint subsets A and B of $V(G)$, we denote by $Sep(G, A, B)$ the size of the smallest subset of $V(G) \setminus (A \cup B)$ separating A from B . If for some $u \in A$ and $v \in B$, $(u, v) \in E(G)$ we set $Sep(G, A, B) = \infty$. If A consists of a single vertex u , we write $Sep(G, u, B)$ instead $Sep(G, \{u\}, B)$. We denote by $G^C(u)$ the graph obtained from G by removal of u and adding all possible edges (u_1, u_2) such that u_1 is an entering neighbor of u , u_2 is a leaving neighbor of u and there is no edge (u_1, u_2) in G .

The method of solving the ORD-MC-DAG problem presented below is an adaptation to the ORD-MC-DAG problem of the algorithm for the MULTIWAY CUT problem in undirected graphs [5]. In particular, the following theorem, which is the cornerstone of the proposed method, is an adaptation of Theorem 3.2. of [5].

Theorem 4. *Assume that $OrdSep(G, X, Y) < \infty$. Let u be a leaving neighbor of x_l and assume that $Sep(G, x_l, Y) = Sep(G^C(u), x_l, Y)$. Then $OrdSep(G, X, Y) = OrdSep(G^C(u), X, Y)$.*

Proof. Let S_m be the set of vertices of $G^C(u) \setminus (X \cup Y)$ of size $Sep(G^C(u), x_l, Y)$ which separates x_l from Y in $G^C(u)$. Observe that S_m separates x_l from Y in G . Really, let P be a path from x_l to some y_j in G . If it does not include u then the same path is present in $G^C(u)$, hence it includes a vertex of S_m . Otherwise, P includes u . Since $OrdSep(G, X, Y) < \infty$, $u \notin Y$, hence it has a predecessor u_1 and a successor u_2 . It follows that $G^C(u)$ has a path obtained from P by removing u and adding edge (u_1, u_2) , this new path includes a vertex of S_m , hence P itself does.

Consider the graph $G \setminus S_m$. Let $C_1 \subseteq V(G \setminus S_m)$ including x_l and all the vertices reachable from x_l in $G \setminus S_m$. Let C_2 be the rest of vertices of $G \setminus S_m$. Note that $u \in C_1$ because otherwise $u \in S_m$ in contradiction to our assumption.

Let S_k be the smallest subset of vertices of $V(G) \setminus (X \cup Y)$ that orderly separates X from Y in G . The sets C_1, S_m, C_2 impose a partition of S_k into sets $A = S_k \cap C_1$, $B = S_k \cap S_m$ and $C = S_k \cap C_2$.

Consider now the graph $G \setminus C_1$. Let S'_m be the subset of S_m consisting of vertices v such that $G \setminus C_1$ has a path from v to some y_j which does not include any vertex of $B \cup C$. We are going to prove that $|S'_m| \leq |A|$.

Since S_m separates x_l from Y in G and is a smallest one subject to this property (by the assumption of the lemma), G has $|S_m|$ internally vertex-disjoint paths from x_l to Y each includes exactly one vertex of S_m (by Menger's Theorem). Consider the prefixes of these paths which end up at the vertices of S_m . As a result we have a subset \mathbf{P} of $|S_m|$ internally vertex-disjoint paths, each starts at x_l ends up at a distinct vertex of S_m . Consider the subset \mathbf{P}' of those $|S'_m|$ paths of \mathbf{P} which end up at the vertices of S'_m .

Observe that each of these paths includes a vertex of A . Really let P_1 be a path of \mathbf{P}' which does not include a vertex of A . Let s be the final vertex of P_1 . Observe that all vertices of P_1 except s belong to C_1 : as witnessed by $P_1 \setminus s$ they are reachable from x_l by a path that does not meet any vertex of S_m . Since B and C are subsets of C_2 , $P_1 \setminus s$ does not intersect with B and C . Let P_2 be a path in $G \setminus C_1$ from s to y_j which does not include the vertices of B and C , which exists by definition of S'_m . Taking into account that $A \subseteq C_1$, P_2 does not include the vertices of A as well. Let P be the concatenation of P_1 and P_2 . Clearly, P is a path (vertex repetition is impossible in a DAG) from x_l to y_j which intersects with neither of A , B , C , that is, it does not intersect with S_k in contradiction to the fact that S_k orderly separates X from Y in G . Thus we obtain that $|S'_m| \leq |A|$.

Consider now the set $S'_k = S'_m \cup B \cup C$. By definition, $|S'_k| = |S'_m| + |B| + |C|$ and $|S_k| = |A| + |B| + |C|$. Taking into account that $|S'_m| \leq |A|$ as proven above, it follows that $|S'_k| \leq |S_k|$. As well, $u \notin S'_k$ just because S'_k does not intersect with C_1 . We are going to prove that S'_k orderly separates X from Y in G , which will finish the proof of the theorem.

Assume by contradiction that this is not so and consider a path P from x_i to y_j in $G \setminus S'_k$ such that $i \geq j$. Assume first that P does not intersect with C_1 . That is, P is a path of $G \setminus C_1$. Since S_k orderly separates X and Y , P includes at least one vertex of S_k or, more precisely, at least one vertex of $V(G \setminus C_1) \cap S_k = B \cup C$. This means that P includes at least one vertex of S'_k in contradiction to our assumption.

Assume now that P includes a vertex w of C_1 . By definition, there is a path P_1 from x_l to w in $G \setminus S_m$. Let P_2 be the suffix of P starting at w . The concatenation of P_1 and P_2 results in a path P' from x_l to y_j . By definition, this path must include vertices of S_m and, since P_1 does not intersect with S_m , P_2 does. Let s be the *last* vertex of S_m which we meet if we traverse P_2 from w to y_j and consider the suffix P'' of P_2 starting at s .

Observe that P'' does not intersect with C_1 because this contradicts our assumption that s is the last vertex of P_2 which belongs to S_m . Really, if there is a vertex $v \in C_1 \cap P''$, draw a path P_3 from x_l to v which does not include any of S_m , take the suffix P_4 of P'' starting at v , concatenate P_3 and P_4 and get a path from x_l to y_j which implies that P_4 must intersect with S_m (because P_3 cannot) and a vertex s' of this intersection is a vertex of P'' . Since $s \notin C_1$, $v \neq s$, that is v is a successor of s in P'' , so is s' . Since $s \neq s'$ (to avoid cycles), s' is a vertex of S_m occurring in P'' , and hence in P_2 , later than s , in contradiction to the definition of s .

Thus P'' belongs to $G \setminus C_1$. Since P'' is a suffix of P which does not intersect with S'_k , P'' does not intersect with S'_k as well, in particular, it does not intersect with $B \cup C$. It follows that $s \in S'_m$ in contradiction to the definition of P . ■

Below we present an FPT-algorithm for the ORD-MC-DAG problem. The algorithm is presented as a function $FindCut(G, X, Y, k)$.

$FindCut(G, X, Y, k)$

1. **If** $|X| = 1$ **then** compute the output efficiently.
2. **If** $Sep(G, x_l, Y) > k$ **then** return 'NO'
3. **If** x_l has no leaving neighbors **then** return $FindCut(G \setminus \{x_l, y_l\}, X \setminus \{x_l\}, Y \setminus \{y_l\}, k)$ (i.e., orderly separate x_1, \dots, x_{l-1} from y_1, \dots, y_{l-1})
4. Select a leaving neighbor u of x_l
5. **If** $Sep(G^C(u), x_l, Y) = Sep(G, x_l, Y)$ **then** return $FindCut(G^C(u), X, Y)$.
6. Let $S_1 = FindCut(G \setminus u, X, Y, k - 1)$ and $S_2 = FindCut(G^C(u), X, Y, k)$.
If $S_1 \neq NO'$, return $\{u\} \cup S_1$. Else, if $S_2 \neq NO'$, return S_2 . Else, return 'NO'.

Before we provide a formal analysis of the algorithm, note the properties of the ORD-MC-DAG problem that make it amenable to the proposed approach. The first useful property is that vertex x_l has to be separated from *all* the vertices of Y . This property ensures the correctness of Theorem 4 and makes possible “shrinking” of the problem if the condition of Step 5 is satisfied. The second property is that if the condition of step 3 is satisfied, i.e. the vertices x_l and y_l are of no use anymore, then, as a result of their deletion, we again obtain an instance of the ORD-MC-DAG problem, i.e. we can again identify a vertex of $X \setminus \{x_l\}$ to be separated from all the vertices of $Y \setminus \{y_l\}$ and hence Theorem 4 applies again.

In order to analyze the algorithm we introduce a definition of a *legal input*. A tuple (G, X, Y, k) is a legal input if G is a DAG, X and Y are subsets of $V(G)$, the vertices of X are minimal, the vertices of Y are maximal, $|X| = |Y|$, $k \geq 0$. Since $FindCut$ is initially applied to a legal input, the following lemma proves correctness of $FindCut$.

Lemma 1. *Let (G, X, Y, k) be a legal input with $|X| = l$. Then $FindCut(G, X, Y, k)$ returns a correct output in a finite amount of recursive applications. Moreover, all tuples to which $FindCut$ is applied recursively during its execution are legal inputs.*

Proof. The proof is by induction on $|V(G)|$. In the smallest possible legal input, graph G consists of 2 vertices x_1 and y_1 , $X = \{x_1\}$, $Y = \{y_1\}$. According to the description of the algorithm, this is a trivial case which is computed correctly without recursive application of $FindCut$. The rest of the proof is an easy, though lengthy, verification of the lemma for all cases of recursive application of $FindCut$.

Assume now that $|V(G)| > 2$. If $l = 1$ or $Sep(G, x_l, Y) > k$, the output is correct according to the description of the algorithm (the correctness of the latter

case follows from the obvious inequality $Sep(G, x_l, Y) \leq OrdSep(G, X, Y)$). If x_l has no leaving neighbors then $FindCut$ is recursively applied to the tuple $(G \setminus \{x_l, y_l\}, X \setminus \{x_l\}, Y \setminus \{y_l\}, k)$. Clearly, this tuple is a legal input, hence the lemma holds regarding this input by the induction assumption, in particular the output of $FindCut(G \setminus \{x_l, y_l\}, X \setminus \{x_l\}, Y \setminus \{y_l\}, k)$ is correct. Since x_l has no leaving neighbors, it has no path to the vertices of Y . Hence, any subset of vertices orderly separating $X \setminus \{x_l\}$ from $Y \setminus \{y_l\}$, orderly separates X from Y and vice versa. It follows that the output of $FindCut(G \setminus \{x_l, y_l\}, X \setminus \{x_l\}, Y \setminus \{y_l\}, k)$ is a correct output of $FindCut(G, X, Y, k)$ and hence the lemma holds regarding (G, X, Y, k) .

Assume that the algorithm selects such a leaving neighbor u of x_l such that $Sep(G, x_l, Y) = Sep(G^C(u), x_l, Y)$. Then $FindCut$ is recursively applied to $(G^C(u), X, Y, k)$. Observe that u is a non-terminal vertex because if $u = y_i$ (u cannot be x_i because all the vertices of X are minimal ones) then $Sep(G, x_l, Y) = \infty > k$ and 'NO' would be returned on an earlier stage. It follows that $(G^C(u), X, Y, k)$ is a legal input. Taking into account that $|V(G^C(u))| < |V(G)|$, the lemma holds regarding (G, X, Y, k) by the induction assumption, in particular, the output R of $FindCut(G^C(u), X, Y, k)$ is correct. Assume that $R \neq NO'$. Then R is subset of non-terminal vertices of size at most k , which orderly separates X from Y in $G^C(u)$. Assume that R does not orderly separate X from Y in G . Then $G \setminus R$ has a path P from x_i to y_j such that $i \geq j$. If P does not include u then this path is present in $G^C(u)$. Otherwise, taking into account that u is non-terminal vertex, this path can be transformed into a path in $G^C(u)$ by removal u and introducing edge (u_1, u_2) where u_1 and u_2 are the immediate predecessor and the immediate successor of u in P , respectively. In both cases P intersects with R , a contradiction. This contradiction shows that R orderly separates X from Y in G . If $FindCut(G^C(u), X, Y, k)$ returns 'NO' this means that $OrdSep(G^C(u), X, Y) > k$. By Theorem 4, in the considered case $OrdSep(G^C(u), X, Y) = OrdSep(G, X, Y)$, that is $OrdSep(G, X, Y) > k$ and hence the answer 'NO' returned by $FindCut(G, X, Y)$ is correct. It follows that the lemma holds for the considered case.

Assume now that none of the previous cases holds. In this case the algorithm selects a leaving neighbor u of x_l such that $Sep(G, x_l, Y) < Sep(G^C(u), x_l, Y)$ and applies itself recursively to $(G \setminus u, X, Y, k-1)$ and $(G^C(u), X, Y, k)$. Observe that u is not a terminal vertex because if $u = y_i$ (u cannot be x_i because all the vertices of X are minimal ones) then $Sep(G, x_l, Y) = \infty > k$, hence an earlier condition is satisfied. Note also that $k > 0$. Really if $k = 0$ then $Sep(G, x_l, Y) = 0$ to avoid satisfaction of an earlier condition. But this means that there is no path from x_l to the vertices of Y hence either x_l has no leaving neighbors or for any leaving neighbor of u , $Sep(G^C(u), x_l, Y) = Sep(x_l, Y) = 0$, in any case one of the earlier conditions is satisfied. It follows that both $(G \setminus u, X, Y, k-1)$ and $(G^C(u), X, Y, k)$ are legal inputs. Since the graphs involved in these inputs have less vertices than G , the recursive applications of $FindCut$ to these tuples are correct by the induction assumption. Assume that the output R of $FindCut(G \setminus u, X, Y, k-1)$ is not 'NO'. Then R is a set of nonterminal

vertices of size at most $k - 1$ which separates X from Y in $G \setminus u$. Clearly that $R \cup \{u\}$ returned by $FindCut(G, X, Y, k)$ in this case is correct. Assume now that $FindCut(G \setminus u, X, Y, k - 1)$ returns 'NO'. Clearly this means that there is no subset R separating X and Y in G such that $|R| \leq k$ and $u \in R$. Assume in this case that the output R of $FindCut(G^C(u), X, Y, k)$ is not 'NO'. Arguing as in the previous paragraph, we see that R orderly separates X from Y in G , hence the output R returned by $FindCut(G, X, Y, k)$ in the considered case is correct. Finally assume that $FindCut(G^C(u), X, Y, k)$ returns 'NO'. Clearly, this means that there is no subset R of non-terminal vertices orderly separating X from Y in G such that $|R| \leq k$ and $u \notin R$. Thus, any decision regarding u does not result in getting the desired orderly separating subset. Hence, such a subset does not exist and the answer 'NO' returned by $FindCut(G, X, Y, k)$ in the considered case is correct. ■

Lemma 1 allows us to define a search tree whose nodes are associated with the legal inputs to which $FindCut(G, X, Y, k)$ is recursively applied during its execution. The root of the tree is associated with (G, X, Y, k) . Let (G', X', Y', k') be a node of this tree where $X' = \{x'_1, \dots, x'_l\}$, $Y' = \{y'_1, \dots, y'_l\}$ (for convenience we identify a node with the tuple associated with this node). If $FindCut(G', X', Y', k')$ does not apply itself recursively then (G', X', Y', k') is a leaf. Otherwise, depending on the particular branching decision, (G', X', Y', k') has the child $(G' \setminus \{x'_l, y'_l\}, X' \setminus \{x'_l\}, Y' \setminus \{y'_l\})$ or the child $(G'^C(u), X', Y', k')$ or children $(G' \setminus u, X', Y', k' - 1)$ and $(G'^C(u), X', Y', k')$, where u is a leaving neighbor of x'_l .

Lemma 2. *The number $L(G, X, Y, k)$ of leaves of the tree rooted by (G, X, Y, k) is $O(4^k)$.*

Proof. For the legal input (G, X, Y, k) with $|X| = l$, let $m = \max(2k + 1 - Sep(G, x_l, Y), 0)$. We are going to prove that the number of leaves of the search tree is at most 2^m . Taking into account that $m \leq 2k + 1$, the result will immediately follow.

The proof is by induction on the number $N(G, X, Y, k)$ of nodes of the tree rooted by (G, X, Y, k) . If $N(G, X, Y, k) = 1$ then, taking into account that $m \geq 0$, the statement immediately follows. Consider the situation where $N(G, X, Y, k) > 1$. Assume first that (G, X, Y, k) has exactly one child (G', X', Y', k) with $|X'| = l'$. Clearly $L(G, X, Y, k) = L(G', X', Y', k)$. Let $m' = \max(2k + 1 - Sep(G', x_{l'}, Y'), 0)$. Observe that $m' \leq m$. Really, if $(G', X', Y', k) = (G^C(u), X, Y, k)$, then $m' = m$ by the description of the algorithm. Otherwise, $(G', X', Y', k) = (G \setminus \{x_l, y_l\}, X \setminus \{x_l\}, Y \setminus \{y_l\}, k)$. This type of child is created only if $Sep(G, x_l, Y) = 0$. Clearly, in this case $m' \leq m$. Taking into account the induction assumption, we get $N(G, X, Y, k) = N(G', X', Y', k) \leq 2^{m'} \leq 2^m$, as required.

Consider the case where (G, X, Y, k) has two children $(G \setminus u, X, Y, k - 1)$ and $(G^C(u), X, Y, k)$ where u is a leaving neighbor of x_l . Observe that in this case $m > 0$. Really, if $m = 0$ then $Sep(G, x_l, Y) > k$ which corresponds to an earlier non-recursive case. Thus $m = 2k + 1 - Sep(G, x_l, Y)$. Let $m_1 = \max(2(k - 1) + 1 - Sep(G \setminus u, x_l, Y), 0)$. Taking into account that $Sep(G \setminus u, x_l, Y) \geq Sep(G, x_l, Y) - 1$, $m_1 < m$. Let $m_2 = \max(2k + 1 - Sep(G^C(u), x_l, Y), 0)$. By the description

of the algorithm, $Sep(G^C(u), x_l, Y) > Sep(G, x_l, Y)$, hence $m_2 < m$. We obtain $L(G, X, Y, k) = L(G \setminus u, X, Y, k-1) + L(G^C(u), X, Y, k) \leq 2^{m_1} + 2^{m_2} \leq 2^{m-1} + 2^{m-1} = 2^m$, the second inequality follows by the induction assumption. ■

According to Lemma 1, each node (G', X', Y', k') of the search tree is a valid input and hence $|V(G')| \geq 2$. On the other hand if (G', X', Y', k') is a non-leaf node and (G'', X'', Y'', k'') is its child then $|V(G'')| < |V(G')|$ by description of the algorithm. It follows that each path from the root to a leaf in the search tree has length $O(n)$. Considering the statement of Lemma 2, we get that the search tree has $O(4^k n)$ nodes. The runtime of $FindCut(G, X, Y, k)$ can be represented as a number of nodes of the search tree multiplied by the runtime spent by the algorithm *per* node. The heaviest operations performed by the algorithm at the given node (G, X, Y, k) are checking whether $Sep(G, x_l, Y) > k$ and, if not, checking whether $Sep(G^C(u), x_l, Y) = Sep(G, x_l, Y)$ for a particular leaving neighbor u of x_l . Clearly these operations can be performed in a time polynomial in n , where the degree of the polynomial is a constant independent on k (by applying a network flow algorithm). Thus the runtime of $FindCut(G, X, Y, k)$ is $O(4^k * poly(n))$. Since the input graph G_{IN} may not satisfy our assumptions regarding the minimality of the vertices of X and the maximality of the vertices of Y , the entire algorithm for the ORD-MC-DAG problem includes also the transformation shown in the beginning of the section. However the transformation can be performed in a polynomial time and hence is taken into consideration by the expression $O(4^k * poly(n))$. Thus we have proved the following theorem.

Theorem 5. *There is an FPT-algorithm solving the parameterized ORD-MC-DAG problem in time $O(4^k * poly(n))$*

Theorem 1 immediately follows from the combination of Theorems 3 and 5.

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