

Measuring Ranks via the Complete Laws of Iterated Contraction¹

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Abstract: Ranking theory delivers an account of iterated contraction; each ranking function induces a specific iterated contraction behavior. The paper gives a complete axiomatization of that behavior, i.e., a complete set of laws of iterated contraction. It does so by showing how to reconstruct a ranking function from its iterated contraction behavior uniquely up to multiplicative constant and thus how to measure ranks on a ratio scale.

1. Introduction

Ranking theory, as first presented in Spohn (1983, sect. 5.3, and 1988) is well known by now to offer a complete model of the dynamics of belief, i.e., it allows to state an arbitrarily iterable rule of belief change. By contrast, AGM belief revision theory, as beautifully summarized by Gärdenfors (1988), founders at the problem of iterated belief change, as observed in Spohn (1983, sect. 5.2, and 1988, sect. 3), because it violates the principle of categorical matching, as Gärdenfors, Rott (1995, p. 37) called it. Both theories agree, though, on single belief changes.

There is a price to pay for the greater strength of ranking theory; it makes substantial use of numerical degrees of (dis-)belief. While one can well see how the dynamics of belief works on the basis of these degrees, one may wonder about their meaning; they look arbitrary and seem to lack intuitive access (unlike subjective probabilities, for instance). By contrast, AGM belief revision theory, in order to justify its revision postulates, only appeals to entrenchment orderings, an ordinal and intuitively well manageable notion.

¹ This paper is a short version of the paper „The Measurement of Ranks and the Laws of Iterated Contraction“ by Matthias Hild and me, the first version of which has been under consideration at *Artificial Intelligence Journal* and a second version of which is in work. All further details and proofs are to be found there.

This difference does not weigh much for those only interested in computing, but for the more philosophically minded – recall that both, AGM and ranking theorizing, originated in philosophy – there remains a problem. What do numerical ranks mean? Where exactly is the difference between two numerically different, but ordinally equivalent ranking functions? Just in vague feelings concerning the strength of belief? This would certainly be a poor answer.

Is there really an objection? Yes, to some extent. Cardinal utility became acceptable only after von Neumann, Morgenstern (1944, ch. 3) proved that preferences conforming to certain axioms determine cardinal utilities on an interval scale. Thus, the cardinal concept turned out to be definable by, or reducible to, the ordinal concept; one cannot accept the one and reject the other. Ranks likewise are psychological magnitudes, and hence it appears legitimate to demand a measurement theory for them, too.

Presumably, though, the issue is not about measurement, but about logic. Customarily, any logical calculus is ennobled by a correctness and completeness, i.e., soundness theorem. We need not rehearse here the historic examples for the tremendous insight delivered by such soundness theorems. If the calculus looks sensible, if the semantics is intelligible, and if a soundness theorem proves them to be equivalent, then mutual support makes for a nearly unassailable theory.

AGM belief revision theory has these virtues. Originally, it came in a logical disguise; its beginnings reach back to Gärdenfors' (1978) epistemic approach to the logic of counterfactuals. Its soundness theorem was that the revision postulates ($K*1-8$) (cf. Gärdenfors 1988, sect. 3.3) and the contraction postulates ($K\div 1-8$) (cf. Gärdenfors 1988, sect. 3.4) were proved to be exactly those justified by an underlying entrenchment relation (cf. Gärdenfors 1988, ch. 4). By contrast, ranking theory did not offer a comparable result, thus abetting the appearance of ranks somehow being arbitrary.

There is no need, though, to ponder about the weight of these objections. They simply do not apply, as this paper will constructively show. It will present a rigorous measurement of ranks on a ratio scale in terms of iterated contractions; and in the course of this measurement it will specify a complete set of laws of iterated contraction, something much desired in its own right and in the present context comparable to a soundness theorem in logic.

The basic idea of this paper is quite simple. It is to exploit iterated contractions for getting information about the comparative size of rank differences. If the iter-

ated contractions behave appropriately, these rank difference comparisons will behave appropriately, too, i.e., such that the theory of difference measurement as propounded in Krantz et al. (1971, ch. 4) applies. It requires some skill, though, to find an elaboration of this guiding idea that is intuitively illuminating as well as formally sound.

The first elaboration of this idea is found in Hild (1997) that remained unpublished. Independently, I had the same idea realized in Spohn (1999) in a still awkward and incomplete way. To our knowledge, the present paper is the first mature presentation of the issue.

The plan of the paper is straightforward. In section 2 we shall briefly introduce ranking theory and its details as far as they will be required in the rest of the paper. Section 3 works up to the desired measurement theorem. Section 4, finally, inquires the laws of iterated contraction entailed by this account of rank measurement.

2. A Brief Sketch of Ranking Theory

Ranking theory assumes propositions to be the objects of belief, and not sentences or sentence-like representations. This is an important and debatable decision right at the beginning of all epistemological theorizing. As things presently stand, it is at the same time a decision between being able and not being able to pursue a substantial way of epistemological theorizing. So, let us make this assumption without further discussion. Let W be a set of possibilities, e.g., possible worlds, centered worlds, or small worlds, or what have you, and let \mathcal{A} be any Boolean algebra of subsets of W ; the elements of \mathcal{A} are called *propositions*. Only in section 3 we shall require some further assumptions about the richness of the Boolean algebra considered.

The core notion of ranking theory is this:

Definition 1: κ is a *negative ranking function* for \mathcal{A} iff κ is a function from \mathcal{A} into $\mathbf{R}^+ = \mathbf{R} \cup \{\infty\}$ such that for all $A, B \in \mathcal{A}$:

- (a) $\kappa(A) \geq 0$, $\kappa(W) = 0$, and $\kappa(\emptyset) = \infty$,
- (b) $\kappa(A \cup B) = \min \{\kappa(A), \kappa(B)\}$ [*the law of disjunction (for negative ranks)*].

Spohn (1983, 1988) originally referred to such functions as ordinal conditional functions. Later on, they were mostly called ranking functions. We have now added the adjective “negative”. The reason is their standard interpretation: negative ranks (that are non-negative numbers) are degrees of *disbelief*. Thus, $\kappa(A) = 0$ says that A is not disbelieved at all according to κ ; $\kappa(A) > 0$ says that A is disbelieved, and the stronger the larger $\kappa(A)$. Hence, A is *believed* iff \bar{A} is disbelieved to some degree, i.e., iff $\kappa(\bar{A}) > 0$. So, the axioms (1a) and (1b) say that \emptyset is maximally disbelieved and W thus maximally believed and in any case not disbelieved, and that a disjunction is exactly as disbelieved as its less disbelieved disjunct. (1a) and (1b) entail:

$$(2) \quad \text{either } \kappa(A) = 0 \text{ or } \kappa(\bar{A}) = 0 \text{ or both} \quad [\textit{the law of negation}].$$

We shall neither need nor presuppose the strengthening of axiom (1b) to infinite disjunctions (without weakening minimum to infimum). This would force the range to be well-ordered; and then ordinal or natural numbers are a natural choice. For the more general theory without this strengthening the a range of real numbers is more natural. This is also the more suitable choice for the present purpose of developing a theory of measurement.

The structure introduced so far is well known in the literature under varying labels, in the present negative version about disbelief or in the easily definable positive version about belief. The distinctive feature of ranking functions is that they are supplemented by a reasonable notion of conditional ranks; this is their advantage, e.g., over Shackle’s (1961) functions of potential surprise or Cohen’s (1970) operators of inductive support as well as over the possibility measures of Dubois, Prade (1988) who had difficulties to intuitively motivate conditional degrees of possibility:

Definition 3: Let κ be a negative ranking function for \mathcal{A} , and $A \in \mathcal{A}$ with $\kappa(A) < \infty$. Then, for any $B \in \mathcal{A}$ the *conditional negative rank of B given A* is defined as $\kappa(B \mid A) = \kappa(A \cap B) - \kappa(A)$.

This is tantamount to:

$$(4) \quad \kappa(A \cap B) = \kappa(A) + \kappa(B \mid A) \quad [\textit{the law of conjunction (for negative ranks)}]$$

The notion of conditional ranks helps us to various further notions of deep significance. (Just think of the importance of conditional probabilities.) One such notion is that of confirmation or of a reason, a terminological choice intended to maintain the connection with traditional epistemology. A is a *reason* for B if A supports or speaks for B or if A strengthens the belief in B , that is, if the belief in B given A is firmer (or the disbelief weaker) than given \bar{A} , or, in still other words, if A is positively relevant to B . Of course, positive relevance is accompanied by the derivative notion of negative relevance and irrelevance and their conditional versions. All this is directly expressed in terms of ranking theory:

Definition 5: Let κ be a negative ranking function for \mathcal{A} , and $A, B \in \mathcal{A}$. Then A is a *reason for* B or *positively relevant to* B w.r.t. κ iff $\kappa(\bar{B} \mid A) > \kappa(\bar{B} \mid \bar{A})$ or $\kappa(B \mid A) < \kappa(B \mid \bar{A})$. A is a *reason against* B or *negatively relevant to* B w.r.t. κ iff $\kappa(\bar{B} \mid A) < \kappa(\bar{B} \mid \bar{A})$ or $\kappa(B \mid A) > \kappa(B \mid \bar{A})$. Finally, A is *irrelevant to* or *independent of* B w.r.t. κ iff A is a reason neither for nor against B w.r.t. κ . Conditional versions of these notions are defined in a straightforward way.

The formal behavior of these notions is quite remarkable. Trivially, the reason or positive relevance relation is reflexive. It is easy to see, moreover, that positive relevance (like the other relevance notions) is symmetric, but not transitive, in sharp contrast to what we are used from deductive reasons. So, reasons rather yield mutual support and not arbitrarily extendible chains of inference. It is obviously an important task to describe and defend the philosophical significance of this notion of a reason, though not a task for this paper. Here, we must be content with the fact that we have an excellent intuitive grasp of positive relevance, i.e., of reasons thus explained, a fact heavily exploited by the subsequent method of measuring ranks.

The next important point is that conditional ranks allow us to state a general dynamic law for ranking functions. The idea is not that upon receiving information A you move to the ranks conditional on A . Since the rank of \bar{A} would then rise to ∞ , this would make sense only if you were absolutely certain of A . The idea is rather to copy generalized probabilistic conditionalization as proposed by Jeffrey (1965, ch. 11), that is, to assume that upon directly receiving information only about A you assign to A and \bar{A} new degrees of belief depending on the firm-

ness of information, while your ranks conditional on A and \bar{A} remain the same. This suffices to completely determine the dynamic law:

Definition 6: Let κ be a negative ranking function for \mathcal{A} and $A \in \mathcal{A}$ such that $\kappa(A), \kappa(\bar{A}) < \infty$, and $x \in \mathbf{R}^+$. Then the $A \rightarrow x$ -conditionalization $\kappa_{A \rightarrow x}$ of κ is defined by $\kappa_{A \rightarrow x}(B) = \min \{ \kappa(B \mid A), \kappa(B \mid \bar{A}) + x \}$.

Thus, the effect of the $A \rightarrow x$ -conditionalization is to shift the possibilities in A (upwards) so that $\kappa_{A \rightarrow x}(A) = 0$ and the possibilities in \bar{A} (downwards or maybe upwards) so that $\kappa_{A \rightarrow x}(\bar{A}) = x$. The parameter x characterizes the information process (and its interaction with the prior doxastic state); no fixed value of x is the right one for all cases. The crucial point is that this dynamic law is iterable; this kind of conditionalization may be arbitrarily repeated as long as the condition of definition 6 is satisfied.

Intuitively, $A \rightarrow x$ -conditionalization comprises expansion, revision, and contraction, the three kinds of belief changes studied in AGM belief revision theory. For any $x > 0$, the $A \rightarrow x$ -conditionalization of κ is an expansion, if $\kappa(A) = \kappa(\bar{A}) = 0$, i.e., if A is initially neutral, and a revision, if $\kappa(A) > 0$, i.e., if A is initially disbelieved. And the $A \rightarrow 0$ -conditionalization of κ , after which neither A nor \bar{A} is believed, is a (genuine or vacuous) contraction by A , if $\kappa(A) = 0$, i.e., if A is initially not disbelieved, and a contraction by \bar{A} , if $\kappa(\bar{A}) = 0$. Obviously, not all ways of $A \rightarrow x$ -conditionalization are thereby exhausted; conditionalization is a substantially more general notion.

The conceptions of expansion, revision, and contraction indeed agree also formally (as already noted in Spohn 1988, footnote 20, and elaborated in Gärdenfors 1988, sect. 3.7). It will be useful to state this precisely:

First, within our propositional framework a *belief set* \mathcal{K} is any subset of \mathcal{A} containing W , but not \emptyset and closed under intersection and the superset relation, i.e.: $W \in \mathcal{K}$; $\emptyset \notin \mathcal{K}$, if $A, B \in \mathcal{K}$, then $A \cap B \in \mathcal{K}$; and if $A \in \mathcal{K}$ and $A \subseteq B \in \mathcal{A}$, then $B \in \mathcal{K}$. In other words, \mathcal{K} is a filter in the mathematical sense. Let $\mathcal{F}(\mathcal{A})$ denote the set of filters or belief sets in \mathcal{A} . Moreover, we say that the filter or belief set \mathcal{K} is *generated* by $\mathcal{B} \subseteq \mathcal{A}$ iff \mathcal{K} is the smallest filter in $\mathcal{F}(\mathcal{A})$ comprising \mathcal{B} .

The complementary notion is that of an ideal. For any set $\mathcal{B} \subseteq \mathcal{A}$ of propositions, let $\mathcal{B}^c = \{ \bar{A} \mid A \in \mathcal{B} \}$. Then $\mathcal{I} \subseteq \mathcal{A}$ is an *ideal* iff \mathcal{I}^c is a filter, i.e., if \mathcal{I} con-

tains \emptyset , but not W and is closed under union and the subset relation. Let $\mathcal{I}(\mathcal{A})$ denote the set of ideals in \mathcal{A} .

AGM belief revision theory defines their belief change operators for all belief sets. As often noticed, it is better to have them defined for a single belief set only; this allows the belief change disposition to change with the belief set. With this in mind we may define:

Definition 7: Let $\mathcal{N} \in \mathcal{I}(\mathcal{A})$ be an ideal in \mathcal{A} . Then $*$ is a *single revision* for $\mathcal{A} - \mathcal{N}$ iff $*$ is a function assigning to each proposition $A \in \mathcal{A} - \mathcal{N}$ a belief set $*(A) \in \mathcal{F}(\mathcal{A})$ such that:

- (a) $A \in *(A)$,
- (b) if $\bar{B} \notin *(A)$, then $*(A \cap B)$ is the belief set generated by $*(A) \cup \{B\}$.

It is obvious that (7a+b) is equivalent to the revision postulates (K*1) – (K*8) of Gärdenfors (1988, sect. 3.3). Gärdenfors prefers to have revisions defined for all propositions; some revisions then result in the universal or contradictory belief set. In the present context it is slightly preferable to deny to the whole of \mathcal{A} the status of a belief set and thus to have revision undefined for the exceptional set \mathcal{N} (of ‘null’ propositions).

Likewise, we may define:

Definition 8: Let $\mathcal{N} \in \mathcal{I}(\mathcal{A})$ be an ideal in \mathcal{A} . Then \div is a *single contraction* for $\mathcal{A} - \mathcal{N}^c$ iff \div is a function assigning to each proposition $A \in \mathcal{A} - \mathcal{N}^c$ a belief set $\div(A) \in \mathcal{F}(\mathcal{A})$ such that:

- (a) $A \notin \div(A) \subseteq \div(\emptyset)$,
- (b) if $A \notin \div(A \cap B)$, then $\div(A) \cap \div(B) \subseteq \div(A \cap B) \subseteq \div(A)$.

Again, it is obvious that (8a+b) is equivalent to the contraction postulates (K \div 1) – (K \div 8) of Gärdenfors (1988, sect. 3.4), when contraction remains undefined for W (the belief in W cannot be given up) and some other propositions forming a filter \mathcal{N}^c .

Moreover, single revisions and contractions are related by the Levi and the Harper identity: if \div is a single contraction and $*(A)$ is the belief set generated by $\div(\bar{A}) \cup \{A\}$, then $*$ is a single revision; and conversely, if $\div(A)$ is defined as $*(W) \cap *(\bar{A})$.

All this is directly related to ranking theory. Obviously, a belief set $\mathcal{K}(\kappa) = \{A \in \mathcal{A} \mid \kappa(\bar{A}) > 0\}$ is associated with each negative ranking function κ for \mathcal{A} . This allows us to define and observe:

- (9) Let κ be a negative ranking function for \mathcal{A} . Then the *single revision* $*_{\kappa}$ induced by κ is defined by $*_{\kappa}(A) = \mathcal{K}(\kappa_{A \rightarrow x})$ for all A with $\kappa(A) < \infty$ and some $x > 0$. $*_{\kappa}$ is indeed a single revision for $\mathcal{A} - \mathcal{N}$, where $\mathcal{N} = \{A \mid \kappa(A) = \infty\}$.

Similarly, we may define and observe:

- (10) Let κ be a negative ranking function for \mathcal{A} and $A \in \mathcal{A}$ such that $\kappa(\bar{A}) < \infty$. Then the *contraction* $\kappa_{\div A}$ of κ by A is defined as

$$\kappa_{\div A} = \left\{ \begin{array}{l} \kappa, \text{ if } \kappa(\bar{A}) = 0, \\ \kappa_{A \rightarrow 0}, \text{ if } \kappa(\bar{A}) > 0 \end{array} \right\}.$$

The *single contraction* \div_{κ} induced by κ is then defined as the function assigning to each $A \in \mathcal{A}$ such that $\kappa(\bar{A}) < \infty$ the belief set $\div_{\kappa}(A) = \mathcal{K}(\kappa_{\div A})$. \div_{κ} is indeed a single contraction for $\mathcal{A} - \mathcal{N}^c$, where $\mathcal{N}^c = \{A \mid \kappa(A) = \infty\}$.

Again, $*_{\kappa}$ and \div_{κ} are related by the Levi and the Harper identity.

There is a salient difference between (9) and (10). In (10) it made sense to define contraction on the level of ranking functions, since this contraction is unique on this level; it then induces contraction on the level of belief sets. By contrast, there is no unique revision on the level of ranking functions; for each $x > 0$ $A \rightarrow x$ -conditionalization gives a different result. However, on the level of belief sets it does not matter on which $x > 0$ we base the revision; hence (9) is well-defined.

This difference has an important consequence. If revision and contraction are special cases of conditionalization and if the latter is iterable, then, one might think, (9) and (10) help us to notions of iterated revision and contraction. This is indeed true for contraction. Contraction on the level of ranking functions is clearly iterable; it thus induces a unique behavior of iterated contraction on the level of belief sets. It is this feature that we shall exploit in the rest of the paper for a measurement of ranks and a complete axiomatization of iterated contraction as induced by a ranking function. The same does not work for revision, however, since only the first, but not the subsequent revisions are independent of the condi-

tionalization parameter x . Does iterated contraction induce iterated revision via the Levi identity? No, since expansion, too, is not unique on the level of ranking functions.

Now we have collected all the material we shall need, and we may immediately proceed to the proposed measurement theory for ranks.

3. Measuring Ranks by Iterated Contractions

To begin with, let us make explicit our talk about iterated contraction. The ranking theoretic terminology is fixed in

Definition 11: Let κ be a negative ranking function for \mathcal{A} and $A_1, \dots, A_n \in \mathcal{A}$ ($n \geq 0$) such that $\kappa(\bar{A}_i) < \infty$ ($i = 1, \dots, n$). Then the *iterated contraction* $\kappa_{+\langle A_1, \dots, A_n \rangle}$ of κ by $\langle A_1, \dots, A_n \rangle$ is defined as $\kappa_{+\langle A_1, \dots, A_n \rangle} = (\dots(\kappa_{+A_1})\dots)_{+A_n}$; this includes the iterated contraction $\kappa_{+\langle \rangle} = \kappa$ by the empty sequence $\langle \rangle$. The *iterated contraction induced by κ* is defined as that function which assign to any finite sequence $\langle A_1, \dots, A_n \rangle$ of propositions with $\kappa(\bar{A}_i) < \infty$ the belief set $\div_{\kappa}\langle A_1, \dots, A_n \rangle = \mathcal{K}(\kappa_{+\langle A_1, \dots, A_n \rangle})$. Hence, $\div_{\kappa}\langle \rangle = \mathcal{K}(\kappa)$.

Let us note right away that iterated contraction as induced by a ranking function is not commutative. It is so only under special conditions:

Theorem 12: Let κ be a negative ranking function for \mathcal{A} and $A, B \in \mathcal{A}$. Then $\div_{\kappa}\langle A, B \rangle \neq \div_{\kappa}\langle B, A \rangle$ if and only if $A, B \in \mathcal{K}(\kappa)$, $\kappa(B \mid \bar{A}) = 0$ or $\kappa(A \mid \bar{B}) = 0$, and $\kappa(\bar{B} \mid \bar{A}) < \kappa(\bar{B} \mid A)$ (which is equivalent to $\kappa(\bar{A} \mid \bar{B}) < \kappa(\bar{A} \mid B)$).

This may at first be surprising. However, Hansson (1993, p. 648) gives an intuitively compelling example showing that this is exactly what we should expect. Let us sketch the gist of the matter. It is, roughly, that the last condition requires the positive relevance of A to B (and vice versa) and that the first conditions then have the effect either that $A \cap \bar{B}$ is disbelieved (or, “if A , then B ” believed) after contracting first by A and then by B , but not after the reverse contraction, or that $\bar{A} \cap B$ is disbelieved (or “if B , then A ” believed) after contracting first by B and then by A , but not after the reverse contraction (or that both is the case). This is exactly

how non-commutativity of contraction may come about. Indeed, the survival of material implication in iterated contractions will play a crucial role below.

The general format of our discussion is fixed in

Definition 13: Let \mathcal{A} be an algebra of propositions over W and $\mathcal{N} \in \mathcal{I}(\mathcal{A})$ an ideal in \mathcal{A} . Let $\mathcal{A}_{\mathcal{N}}$ denote the set of all finite (possibly empty) sequences of propositions from $\mathcal{A} - \mathcal{N}^c$. Then \div is a *potential iterated contraction*, a *potential IC*, for $(\mathcal{A}, \mathcal{N})$ iff \div is a function from $\mathcal{A}_{\mathcal{N}}$ into $\mathcal{F}(\mathcal{A})$. A potential IC \div is an *iterated ranking contraction*, an *IRC*, for $(\mathcal{A}, \mathcal{N})$ iff there is a negative ranking function κ such that $\mathcal{N} = \{A \in \mathcal{A} \mid \kappa(A) = \infty\}$ and $\div = \div_{\kappa}$.

Given this terminology, our principal aim is to measure ranks with the help of iterated contraction on a ratio scale. This means to reconstruct a ranking function κ from its iterated contraction \div_{κ} , indeed uniquely up to a multiplicative constant. This is what we shall do in this section. The further aim, completing the investigation in the next section, is to state which properties a potential IC must have in order to be an IRC, i.e., an IC suitable for measuring ranks. Of course, (13) does not count as an answer; we shall be searching for informative properties not referring to ranking functions.

We shall reach our principal aim in four simple steps. The first step is familiar; it consists in the observation already made in AGM belief revision theory that the ordering of negative ranks, i.e., of disbelief, may be inferred from single contractions. In our terms, this means that we have for each negative ranking function κ and all $A, B \in \mathcal{A}$:

$$(14) \quad \kappa(A) \leq \kappa(B) \text{ iff } \div_{\kappa} \text{ is not defined for } \langle \bar{B} \rangle \text{ or } \bar{A} \notin \div_{\kappa} \langle \bar{A} \cap \bar{B} \rangle.$$

That is, B is at least as disbelieved as A , or \bar{B} is at least as firmly believed as \bar{A} , either if \bar{B} is maximally believed or if giving up the belief in $\bar{A} \cap \bar{B}$ entails giving up the belief in \bar{A} (cf. Gärdenfors 1988, p. 96). Let us fix this connection without reference to ranks:

Definition 15: Let \div be a potential IC for $(\mathcal{A}, \mathcal{N})$. Then the *potential disbelief comparison* \leq_{\div} associated with \div is the binary relation on \mathcal{A} such that for all A, B

$\in \mathcal{A}$: $A \preceq_+ B$ iff $B \in \mathcal{N}$ or $\bar{A} \notin \div\langle \bar{A} \cap \bar{B} \rangle$. The associated *disbelief equivalence* \triangleq_+ and the *strict disbelief comparison* \prec_+ are defined in the usual way. The disbelief comparison associated with the IRC \div_κ is denoted by \preceq_κ , so that (14) entails that $A \preceq_\kappa B$ iff $\kappa(A) \leq \kappa(B)$.

Of course, such a potential disbelief comparison \preceq_+ is well-behaved and thus a proper disbelief comparison only if the associated potential IC \div is well-behaved. For instance, \preceq_+ is a weak order only if the potential IC \div restricted to one-term sequences is a single contraction according to (8) (cf. Gärdenfors 1988, sect. 4.6). Let us defer, though, the systematic inquiry what good behavior amounts to in the end. At present, the relevant point is that single contractions yield no more than a measurement of ranks on an ordinal scale.

Hence, we must take further steps. The second step is the crucial one. Hild (1997) was the first to invent it, Spohn (1999) independently had the same idea. It consists in the observation that the reason relations as defined in (5), i.e., positive relevance, negative relevance, and irrelevance, can also be expressed in terms of contractions, albeit only iterated ones. For our measurement purposes the most convenient notion is non-negative relevance from which we may define the other ones. Moreover, we have to more generally refer to conditional relevance. These two points are taken care of in the next

Theorem 16: Let κ be a negative ranking function for \mathcal{A} and $A, B, C \in \mathcal{A}$ such that $\kappa(C) < \infty$. Then A is not a reason against B , or non-negatively relevant to B , given C w.r.t. κ iff $\kappa(A \mid C)$ or $\kappa(\bar{A} \mid C)$ or $\kappa(B \mid C)$ or $\kappa(\bar{B} \mid C)$ is infinite or $\kappa(A \cap B \cap C) - \kappa(A \cap \bar{B} \cap C) \leq \kappa(\bar{A} \cap B \cap C) - \kappa(\bar{A} \cap \bar{B} \cap C)$, i.e., iff neither $(C \cap A) \rightarrow \bar{B}$ nor $(C \cap \bar{A}) \rightarrow B$ is a member of $\div_\kappa\langle C \rightarrow A, C \rightarrow \bar{A}, C \rightarrow B, C \rightarrow \bar{B} \rangle$ or the latter is undefined.

For a proof see the full paper. Note that we can express the equivalence of (16) also in the following way: For any four mutually disjoint propositions $A, B, C, D \in \mathcal{A}$ with finite ranks $\kappa(A) - \kappa(B) \leq \kappa(C) - \kappa(D)$ iff $A \cup B$ is non-negatively relevant to $A \cup C$ given $A \cup B \cup C \cup D$ w.r.t. κ , i.e., iff neither \bar{A} nor \bar{D} is a member of $\div_\kappa\langle \bar{A} \cap \bar{B}, \bar{C} \cap \bar{D}, \bar{A} \cap \bar{C}, \bar{B} \cap \bar{D} \rangle$.

Thus, we can now make the same transition as we did from (14) to (15) and adopt the following

Definition 17: Let \div be a potential IC for $(\mathcal{A}, \mathcal{N})$. Then the *potential disjoint difference comparison (potential DisDC)* \preceq_+^d associated with \div is the four-place relation defined for all quadruples of mutually disjoint propositions in $\mathcal{A} - \mathcal{N}$ such that for all such propositions A, B, C, D $(A - B) \preceq_+^d (C - D)$ iff $\bar{A}, \bar{D} \notin \div(\bar{A} \cap \bar{B}, \bar{C} \cap \bar{D}, \bar{A} \cap \bar{C}, \bar{B} \cap \bar{D})$ – where the ordered pair of A and B is denoted by $(A - B)$ simply for mnemonic reasons. The associated *disjoint difference equivalence* \approx_+^d and the *strict disjoint difference comparison* \prec_+^d are defined in the usual way. The potential DisDC associated with the IRC \div_κ is denoted by \preceq_κ^d , so that (19) entails that $(A - B) \preceq_\kappa^d (C - D)$ iff $\kappa(A) - \kappa(B) \leq \kappa(C) - \kappa(D)$.

Now, one can already see what we are heading for. On the one hand, we have shown how to derive such a difference comparison from iterated contractions. On the other hand, we know that if such a difference comparison behaves in the appropriate way, we can use it for a difference measurement of ranking functions. Put the pieces together and you have a measurement of ranks in terms of iterated contractions.

However, we are not yet fully prepared for this final step. If we want to apply the general theory of difference measurement to the present case, the difference comparison must hold for any four propositions, not only for any four mutually disjoint propositions. The required extension is the third step of our measurement procedure.

There are various options at this point, and the details are not so important. Probably the simplest idea is to straightforwardly require that for each proposition with a finite rank there are at least n mutually disjoint equally ranked propositions for some $n \geq 4$; let us call this assumption *n-richness*. With this assumption we can extend any potential DisDC to all quadruples of propositions. Such an extension will be called a *potential doxastic difference comparison (potential DoxDC)* and denoted by \preceq_+ if associated with the potential IC \div . The associated *doxastic difference equivalence* \approx_+ and the *strict doxastic difference comparison* \prec_+ are again defined in the usual way.

After this auxiliary move, we can take the fourth step and complete our measurement procedure. If potential DoxDC's have the right properties, we can construct a ratio scale from them. What this means and how this goes can be directly read off from the standard theory of difference measurement; cf. Krantz et al.

(1971, ch. 4). We have to do no more than copy definition 3 and theorem 2 from there, p. 151, and slightly adapt it for our purposes:

Definition 18: \preceq is a *doxastic difference comparison (DoxDC)* for $(\mathcal{A}, \mathcal{N})$ (with \approx being the associated equivalence and \prec the associated strict comparison) iff \preceq is a quarternary relation on $\mathcal{A} - \mathcal{N}$ such that for all $A, B, C, D, E, F \in \mathcal{A}$:

- (a) \preceq is a weak order on $(\mathcal{A} - \mathcal{N}) \times (\mathcal{A} - \mathcal{N})$ [weak order],
- (b) if $(A - B) \preceq (C - D)$, then $(D - C) \preceq (B - A)$ [sign reversal],
- (c) if $(A - B) \preceq (D - E)$ and $(B - C) \preceq (E - F)$, then $(A - C) \preceq (D - F)$ [monotonicity],
- (d) if $(A - W) \preceq (B - W)$, then $(A - W) \approx (A \cup B - W)$ [law of disjunction].

The DoxDC \preceq is *Archimedean* iff, moreover, for any sequence A_1, A_2, \dots in $\mathcal{A} - \mathcal{N}$:

- (e) if A_1, A_2, \dots is a strictly bounded standard sequence, i.e., if for all i $(A_1 - A_1) \prec (A_2 - A_1) \approx (A_{i+1} - A_i)$ and if there is a $D \in \mathcal{A} - \mathcal{N}$ such that for all i $(A_i - A_1) \prec (D - W)$, then the sequence A_1, A_2, \dots is finite.

Finally, the DoxDC \preceq is *full* iff for all $A, B, C, D \in \mathcal{A} - \mathcal{N}$:

- (f) if $(A - A) \preceq (A - B) \preceq (C - D)$, then there exist $C', D' \in \mathcal{A}$ such that $(A - B) \approx (C' - D) \approx (C - D')$.

It is obvious that (18a-d) are axioms necessary for the measurement of ranks. The Archimedean axiom (18e) is also necessary. (18f), finally, is a structural axiom which is not entailed by ranking theory, but which is required for the solvability of all the measurement inequalities. (Likewise, the move from disjoint to doxastic difference comparisons also required a certain structural richness of the underlying algebra of propositions.)

We are not claiming that DoxDC's are intuitively well accessible. Indeed, they are not, we find. Disbelief comparisons or their positive counterparts, entrenchment relations, are highly accessible. By contrast, we have no safe intuitive assessment of doxastic differences between four arbitrary propositions, even if they are mutually disjoint. Therefore we did not start this section with (18) that plays only a mediating role, but rather explained how difference judgments reduce to well accessible relevance judgments and how all these assessments reduce to even better accessible iterated contractions.

The theorem appertaining to (18) finally establishes the measurability of ranking functions.

Theorem 19: Let \preceq be a full Archimedean DoxDC for $(\mathcal{A}, \mathcal{N})$. Then there is a negative ranking function κ for \mathcal{A} such that for all $A \in \mathcal{A}$ $\kappa(A) = \infty$ iff $A \in \mathcal{N}$ and for all $A, B, C, D \in \mathcal{A} - \mathcal{N}$ $(A - B) \preceq (C - D)$ iff $\kappa(A) - \kappa(B) \leq \kappa(C) - \kappa(D)$. If κ' is another negative ranking function with these properties, then there is an $x > 0$ such that $\kappa' = x \cdot \kappa$.

To sum up: We have seen how potential IC's induce potential DisDC's, how rich potential IC's induce potential DoxDC's, that the right kind of potential IC's, for instance, IRC's, indeed induce DoxDC's, and that DoxDC's, if they are full and Archimedean, determine ranking functions uniquely up to a multiplicative constant. In particular, this means that we may start from a rich ranking function κ , then consider only the rich IRC \div_{κ} associated with κ , and finally reconstruct the whole of κ from \div_{κ} in the unique way indicated. This appears to be a most satisfying representation result. The only data we need are the beliefs under various iterated contractions. These data reflect not only the comparative strength of beliefs, they also reflect the comparative nature of reasons and relevance. And these inferred comparisons suffice to fix the cardinal structure of ranking functions.

4. The Laws of Iterated Contraction

We are not yet finished, though. We just stated that the right kind of potential IC's measure ranks and that IRC's are the right kind. However, we still miss a general characterization of what the right kind is avoiding reference to ranking functions. The required information is, of course, implicit in (18) and the auxiliary steps leading to (18). Yet, the implicit information needs to be explicitly elaborated in a perspicuous way, so that content and import of our measurement result becomes intelligible. Spohn (1999) still neglected the issue, whereas Hild (1997) started to answer it.

Thereby we can also close a gap in the current literature. Despite 15 years of discussion, it was not possible to find and agree upon a stronger or even complete set of laws of iterated contraction (cf. the overview in Rott 2006). In our view, this is so because there was no accepted semantics or no model to guide the search for

such laws. As explained, ranking theory provides such a model, and hence these laws will fall right into our lap as a consequence of our measurement theory. That is, theorem 24 shows that the laws we shall find are complete given the structural conditions of richness and fullness.

So, let us start with the required characterization and let us then work out its consequences up to their intended goal:

Definition 20: Let \mathcal{A} be an algebra of propositions over W and $\mathcal{N} \in \mathcal{I}(\mathcal{A})$ an ideal in \mathcal{A} . Let again be $\mathcal{N}^c = \{\bar{A} \mid A \in \mathcal{N}\}$ and $\mathcal{A}_{\mathcal{N}}$ the set of all finite sequences of propositions from $\mathcal{A} - \mathcal{N}^c$. We shall use S as a variable for elements of $\mathcal{A}_{\mathcal{N}}$. Then \div is an *iterated contraction (IC)* for $(\mathcal{A}, \mathcal{N})$ iff \div is a potential IC for $(\mathcal{A}, \mathcal{N})$ such that for all $A, B, C \in \mathcal{A} - \mathcal{N}^c$, and $S \in \mathcal{A}_{\mathcal{N}}$:

- (IC1) the function $A \mapsto \div\langle A \rangle$ is a single contraction [*single contraction*],
- (IC2) if $A \notin \div\langle \emptyset \rangle$, then $\div\langle A, S \rangle = \div\langle S \rangle$ [*strong vacuity*],
- (IC3) if $\bar{A} \cap \bar{B} = \emptyset$, then $\div\langle A, B, S \rangle = \div\langle B, A, S \rangle$ [*restricted commutativity*],
- (IC4) if $A \subseteq B$ and $A \cup \bar{B} \notin \div\langle A \rangle$, then $\div\langle A \cup \bar{B}, B, S \rangle = \div\langle A, B, S \rangle$
[*path independence*],
- (IC5) if $A \subseteq \bar{C}$ or $A, B \subseteq C$ and $A \triangleleft_+ B$, then $A \triangleleft_{\div\langle C \rangle} B$, and if the inequality in the antecedent is strict, that of the consequent is strict, too
[*order preservation*],
- (IC6) $\div_{\langle S \rangle}$ is an IC [*iterability*].

A notational slip occurs in (IC5). Disbelief comparisons, difference comparisons, etc., are explained relative to potential IC's; thus, strictly taken, the notation " $\triangleleft_{\div\langle C \rangle}$ " is nonsense. However, for typographic reasons we shall always write " $\triangleleft_{\div\langle S \rangle}$ " instead of the more correct " $\triangleleft_{\div\langle C \rangle}$ "; there is no danger of confusion.

Before proceeding to the formal business, we should first look at the intuitive and formal content of these axioms. At the same time, it is interesting to examine the extent to which they go beyond the existing efforts to come to grips with iterated contraction.

Iterated contractions must, of course, behave like single contractions at each single step; therefore (IC1). Strong vacuity (IC2) goes beyond vacuity for single contractions (expressed by (8b) for $B = \emptyset$), since it says that a vacuous contraction does not only leave the beliefs unchanged, but indeed the entire doxastic state as reflected in possible further contractions. This is certainly how vacuous con-

tractions were intended, even though it was not expressible in terms of single contractions. (IC6) goes without saying; it lies at the heart of iteration that it can be carried out without limit. Of course, (IC6) does not make definition 25 circular; it only states succinctly what we could have attained by stating all the other axioms more clumsily for all $\div\langle S \rangle$.

Obviously, hence, (IC3) – (IC5) are the proper laws of iterated contraction. We find them intuitively convincing, though, of course, our intuitions are already shaped by ranking theory. Let us briefly discuss them.

A first important point to note is that (IC5) may be seen to be equivalent to the postulates proposed by Darwiche, Pearl (1997) or rather, since these are postulates of iterated revision, to the postulates of iterated contraction corresponding to them. Hence, (IC5) may be assumed to be as widely accepted as the Darwiche/Pearl postulates.

Thus, it is exactly (IC3) and (IC4) by which our axiomatization of iterated contraction goes beyond the present state of the art. As to (IC3), *restricted commutativity*, we had mentioned that iterated contractions cannot be expected to always commute, and (12) described the conditions under which they do not do so. One condition was that the two contracted propositions A and B are positively relevant to each other (and the point then was that, though at least one of the two material implications expressing the positive relevance must survive the two contractions, it may be the one after $\div\langle A, B \rangle$ and the other after $\div\langle B, A \rangle$). However, in (IC3) we assumed an extreme negative relevance, i.e., that “if A , then *not* B ” is logically true. This deductive relation holds across all doxastic states whatsoever. So, (12) can never apply to such A and B , whatever the doxastic state, and we may accept *restricted commutativity* as an axiom. In other words, if two disbeliefs are logically incompatible, there can be no interaction between giving up these disbeliefs, and hence it seems also intuitively convincing that the order in which they are given up should not matter at all.

The intuitive content of (IC4), *path independence*, may be described as follows: Suppose you believe A . Then you also believe the logical consequences of A . Let B be one of them; $B \rightarrow A$ is another. Now you contract by A . This entails that you have to give up at least one of B and $B \rightarrow A$. Suppose you keep B and give up $B \rightarrow A$. What *path independence* claims is that it does not make a difference then whether you give up $A (= (B \rightarrow A) \cap B)$ and then B or whether you give up $B \rightarrow A$ right away and then B . The description is still simpler in terms of disbelief: Sup-

pose you disbelieve two logically incompatible propositions, and you have to contract both of them. Then you can either contract one after the other. Or you can first contract their disjunction, and if you still disbelieve one of them, you then contract it as well. (IC4) says that both ways result in the same doxastic state. This seems entirely right to us. This may suffice as an explanation of the intuitive appeal of our iterated contraction axioms.

Let us turn, at last, to the formal business. What we have to show is that the potential DoxDC \preceq_+ generated by an IC \div is indeed a DoxDC. In order to reach this peak, we have to climb some antecedent hills.

First, we may observe that the disbelief comparison induced by an IC has the expected properties:

(21) For any IC \div \preceq_+ is a weak order on \mathcal{A} [*weak order*],

(22) for any IC \div if $A \preceq_+ B$, then $A \hat{=} \preceq_+ A \cup B$ [*law of disjunction*].

These are the familiar consequences of (IC1), i.e., of the properties of single contractions.

Next, we should note that for IC's we can express the induced potential DisDC in terms of the induced disbelief comparison:

(23) For any IC \div and any four mutually disjoint $A, B, C, D \in \mathcal{A} - \mathcal{N}$, $(A - B) \preceq_+^d (C - D)$ iff $A \preceq_{+(\bar{A} \cap \bar{B}, \bar{C} \cap \bar{D})} C$ and $D \preceq_{+(\bar{A} \cap \bar{B}, \bar{C} \cap \bar{D})} B$; and the strict inequality holds on the left hand side iff at least one of the inequalities of the right hand side is strict.

It may indeed be shown in a series of lemmata that the binary disbelief comparison and the quaternary disjoint difference comparison induced by an IC cohere in the expected way.

These lemmata help us to reach our next goal, i.e., to establish that the disjoint difference comparison \preceq_+^d induced by an IC \div already has the pertinent properties of a DoxDC. That is, we can prove (again for all the missing lemmata and proofs see the full paper):

- (24) For any $\text{IC} \div$ and four mutually disjoint $A, B, C, D \in \mathcal{A} - \mathcal{N}$, either $(A - B) \preceq_+^d (C - D)$ or $(C - D) \preceq_+^d (A - B)$ or both [completeness].
- (25) For any $\text{IC} \div$ and six mutually disjoint $A, B, C, D, E, F \in \mathcal{A} - \mathcal{N}$, if $(A - B) \preceq_+^d (C - D)$ and $(C - D) \preceq_+^d (E - F)$, then $(A - B) \preceq_+^d (E - F)$ [transitivity].
- (26) For any $\text{IC} \div$ and four mutually disjoint $A, B, C, D \in \mathcal{A} - \mathcal{N}$, if $(A - B) \preceq_+^d (C - D)$, then $(D - C) \preceq_+^d (B - A)$ [sign reversal].
- (27) For any $\text{IC} \div$ and six mutually disjoint $A, B, C, D, E, F \in \mathcal{A} - \mathcal{N}$, if $(A - B) \preceq_+^d (D - E)$ and $(B - C) \preceq_+^d (E - F)$, then $(A - C) \preceq_+^d (D - F)$ [monotonicity].

The rest is easy walking. It remains to show that all the properties established for disjoint difference comparisons carry over to unrestricted doxastic difference comparisons. This step is only tedious, but not difficult; it only requires the already mentioned structural assumption of n -richness, more specifically of 6-richness, which gets us from potential DisDC's to potential DoxDC's.

Concerning the remaining properties of IC's we have no ambitions. Let us simply accept

Definition 28: Let \div be an IC for $(\mathcal{A}, \mathcal{N})$. Then \div is called *Archimedean* iff the DoxDC \preceq_+ induced by \div is Archimedean. And \div is called *full* iff the DoxDC \preceq_+ induced by \div is full.

We did not attempt to express the Archimedean property purely in terms of iterated contraction; this appears to be an unilluminating exercise. Likewise, though fullness is easily translated into contractions, its original explanation in terms of difference comparisons is the most perspicuous.

All this can be summed up in the following

Theorem 29: For any IC \div for $(\mathcal{A}, \mathcal{N})$, the potential DoxDC \preceq_+ induced by \div is a DoxDC for $(\mathcal{A}, \mathcal{N})$. And any 6-rich full Archimedean IC \div for $(\mathcal{A}, \mathcal{N})$ is an IRC for $(\mathcal{A}, \mathcal{N})$, i.e., there is a negative ranking function κ for \mathcal{A} with $\div = \div_{\kappa}$. Moreover, for each ranking function κ' with $\div = \div_{\kappa'}$, there is an $x > 0$ such that $\kappa' = x \cdot \kappa$.

Each part of this theorem is already proved. It concludes our presentation of the method of measuring ranking functions through iterated contractions and thus our justification of using *numerical* ranks. It also shows that (IC1) – (IC6) are necessary properties of iterated contraction and (together with the Archimedean property) jointly sufficient ones in the presence of richness and fullness; under such structural assumptions they offer a complete characterization of iterated contractions – at least if iterated contraction is conceived as proposed in the ranking theoretic way.

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