Topological Complexity of ω-Powers : Extended Abstract

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1 Introduction

The operation $V \rightarrow V^\omega$ is a fundamental operation over finitary languages leading to ω-languages. It produces ω-powers, i.e. ω-languages in the form V^ω, where V is a finitary language. This operation appears in the characterization of the class REG_ω of ω-regular languages (respectively, of the class CF_ω of context free ω-languages) as the ω-Kleene closure of the family REG of regular finitary languages (respectively, of the family CF of context free finitary languages) [Sta97a].
Since the set Σ^ω of infinite words over a finite alphabet Σ can be equipped with the usual Cantor topology, the question of the topological complexity of ω-powers of finitary languages naturally arises and has been posed by Niwinski [Niw90], Simonnet [Sim92], and Staiger [Sta97a]. A first task is to study the position of ω-powers with regard to the Borel hierarchy (and beyond to the projective hierarchy) [Sta97a,PP04].

It is easy to see that the ω-power of a finitary language is always an analytic set because it is either the continuous image of a compact set $\{0, 1, \ldots, n\}^\omega$ for $n \geq 0$ or of the Baire space ω^ω.

It has been recently proved, that for each integer $n \geq 1$, there exist some ω-powers of context free languages which are $\mathbf{\Pi}^0_n$-complete Borel sets, [Fin01], and that there exists a context free language L such that L^ω is analytic but not Borel, [Fin03]. Notice

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that amazingly the language L is very simple to describe and it is accepted by a simple 1-counter automaton.

The first author proved in [Fin04] that there exists a finitary language V such that V^ω is a Borel set of infinite rank. It was also proved in [DF07] that there is a context free language W such that W^ω is Borel above $\mathbf{\Delta}^0_\omega$.
We proved in [FL07] the following very surprising result which shows that ω-powers exhibit a great topological complexity: for each non-null countable ordinal ξ, there exist some $\mathbf{\Sigma}^0_\xi$-complete ω-powers, and some $\mathbf{\Pi}^0_\xi$-complete ω-powers.

We consider also the Wadge hierarchy which is a great refinement of the Borel hierarchy. We get many more Wadge degrees of ω-powers, showing that for each ordinal $\xi \geq 3$, there are uncountably many Wadge degrees of ω-powers of Borel rank $\xi + 1$.

We show also, using some tools of effective descriptive set theory, that the main result of [FL07] has some effective counterparts.

All the proofs of the results presented here may be found in the conference paper [FL07] or in the preprint [FL08] which contains also some additional results.

2 Topology

We first give some notations for finite or infinite words, assuming the reader to be familiar with the theory of formal languages and of ω-languages, see [Tho90,Sta97a,PP04]. Let Σ be a finite or countable alphabet whose elements are called letters. A non-empty finite word over Σ is a finite sequence of letters: $x = a_0.a_1.a_2 \ldots a_n$ where $\forall i \in [0; n]$ $a_i \in \Sigma$. We shall denote $x(i) = a_i$ the $(i{+}1)^{th}$ letter of x. The length of x is $|x| = n{+}1$. The empty word has 0 letters. Its length is 0. The set of finite words over Σ is denoted $\Sigma^{<\omega}$. A (finitary) language L over Σ is a subset of $\Sigma^{<\omega}$. The usual concatenation product of u and v will be denoted by $u^\frown v$ or just uv.

The first infinite ordinal is ω. An ω-word over Σ is an ω -sequence $a_0 a_1 \ldots a_n \ldots$, where for all integers $i \geq 0 \quad a_i \in \Sigma$. When σ is an ω-word over Σ, we write $\sigma = \sigma(0)\sigma(1) \ldots \sigma(n) \ldots$. The set of ω-words over the alphabet Σ is denoted by Σ^ω. An ω-language over an alphabet Σ is a subset of Σ^ω. The concatenation product is also extended to the product of a finite word u and an ω-word v: the infinite word $u.v$ or $u^\frown v$ is then the ω-word such that: $(uv)(k) = u(k)$ if $k < |u|$, and $(u.v)(k) = v(k - |u|)$ if $k \geq |u|$.
The prefix relation is denoted $\prec$: the finite word u is a prefix of the finite word v (respectively, the infinite word v), denoted $u \prec v$, if and only if there exists a finite word w (respectively, an infinite word w), such that $v = u^\frown w$.

For a finitary language $V \subseteq \Sigma^{<\omega}$, the ω-power of V is the ω-language

$$V^\omega = \{u_1 \ldots u_n \ldots \in \Sigma^\omega \mid \forall i \geq 1 \ \ u_i \in V\}$$

We recall now some notions of topology, assuming the reader to be familiar with basic notions which may be found in [Kur66,Mos80,Kec95,LT94,Sta97a,PP04].
There is a natural metric on the set Σ^ω of infinite words over a countable alphabet Σ which is called the prefix metric and defined as follows. For $u, v \in \Sigma^\omega$ and $u \neq v$ let $d(u, v) = 2^{-l_{pref(u,v)}}$ where $l_{pref(u,v)}$ is the first integer n such that the $(n+1)^{th}$ letter of u is different from the $(n+1)^{th}$ letter of v. The topology induced on Σ^ω by this metric is just the product topology of the discrete topology on Σ. For $s \in \Sigma^{<\omega}$, the set $N_s := \{\alpha \in \Sigma^\omega \mid s \prec \alpha\}$ is a basic clopen (i.e., closed and open) set of Σ^ω. More generally open sets of Σ^ω are in the form $W^\frown \Sigma^\omega$, where $W \subseteq \Sigma^{<\omega}$.

When Σ is a finite alphabet, the prefix metric induces on Σ^ω the usual Cantor topology and Σ^ω is compact.
The Baire space ω^ω is equipped with the product topology of the discrete topology on ω. It is homeomorphic to $P_\infty := \{\alpha \in 2^\omega \mid \forall i \in \omega\ \exists j \geq i\ \ \alpha(j) = 1\} \subseteq 2^\omega$, via the map defined on ω^ω by $H(\beta) := 0^{\beta(0)}10^{\beta(1)}1\ldots$

We define now the **Borel Hierarchy** on a topological space X:

Definition 1. *The classes $\mathbf{\Sigma}^0_\xi(X)$ and $\mathbf{\Pi}^0_\xi(X)$ of the Borel Hierarchy on the topological space X are defined as follows:*
$\mathbf{\Sigma}^0_1(X)$ is the class of open subsets of X.
$\mathbf{\Pi}^0_1(X)$ is the class of closed subsets of X.
And for any countable ordinal $\xi \geq 2$:
$\mathbf{\Sigma}^0_\xi(X)$ is the class of countable unions of subsets of X in $\cup_{\gamma<\xi}\mathbf{\Pi}^0_\gamma$.
$\mathbf{\Pi}^0_\xi(X)$ is the class of countable intersections of subsets of X in $\cup_{\gamma<\xi}\mathbf{\Sigma}^0_\gamma$.
As usual the ambiguous class $\mathbf{\Delta}^0_\xi$ is the class $\mathbf{\Sigma}^0_\xi \cap \mathbf{\Pi}^0_\xi$.

Suppose now that $X \subseteq Y$; then $\mathbf{\Sigma}^0_\xi(X) = \{A \cap X \mid A \in \mathbf{\Sigma}^0_\xi(Y)\}$, and similarly for $\mathbf{\Pi}^0_\xi$, see [Kec95, Section 22.A]. Notice that we have defined the Borel classes $\mathbf{\Sigma}^0_\xi(X)$ and $\mathbf{\Pi}^0_\xi(X)$ mentioning the space X. However when the context is clear we will sometimes omit X and denote $\mathbf{\Sigma}^0_\xi(X)$ by $\mathbf{\Sigma}^0_\xi$ and similarly for the dual class.

The class of **Borel sets** is $\mathbf{\Delta}^1_1 := \bigcup_{\xi<\omega_1}\ \mathbf{\Sigma}^0_\xi = \bigcup_{\xi<\omega_1}\ \mathbf{\Pi}^0_\xi$, where ω_1 is the first uncountable ordinal.

For a countable ordinal α, a subset of Σ^ω is a Borel set of *rank* α iff it is in $\mathbf{\Sigma}^0_\alpha \cup \mathbf{\Pi}^0_\alpha$ but not in $\bigcup_{\gamma<\alpha}(\mathbf{\Sigma}^0_\gamma \cup \mathbf{\Pi}^0_\gamma)$.

We now define completeness with regard to reduction by continuous functions. For a countable ordinal $\alpha \geq 1$, a set $F \subseteq \Sigma^\omega$ is said to be a $\mathbf{\Sigma}^0_\alpha$ (respectively, $\mathbf{\Pi}^0_\alpha$)-*complete set* iff for any set $E \subseteq Y^\omega$ (with Y a finite alphabet): $E \in \mathbf{\Sigma}^0_\alpha$ (respectively, $E \in \mathbf{\Pi}^0_\alpha$) iff there exists a continuous function $f : Y^\omega \to \Sigma^\omega$ such that $E = f^{-1}(F)$. $\mathbf{\Sigma}^0_n$ (respectively, $\mathbf{\Pi}^0_n$)-complete sets, with n an integer ≥ 1, are thoroughly characterized in [Sta86].
Recall that a set $X \subseteq \Sigma^\omega$ is a $\mathbf{\Sigma}^0_\alpha$ (respectively $\mathbf{\Pi}^0_\alpha$)-complete subset of Σ^ω iff it is in $\mathbf{\Sigma}^0_\alpha$ but not in $\mathbf{\Pi}^0_\alpha$ (respectively in $\mathbf{\Pi}^0_\alpha$ but not in $\mathbf{\Sigma}^0_\alpha$), [Kec95].

For example, the singletons of 2^ω are $\mathbf{\Pi}^0_1$-complete subsets of 2^ω. The set P_∞ is a well known example of a $\mathbf{\Pi}^0_2$-complete subset of 2^ω.

If $\mathbf{\Gamma}$ is a class of sets, then $\check{\mathbf{\Gamma}} := \{\neg A \mid A \in \mathbf{\Gamma}\}$ is the class of complements of sets in $\mathbf{\Gamma}$. In particular, for every non-null countable ordinal α, $\check{\mathbf{\Sigma}}^0_\alpha = \mathbf{\Pi}^0_\alpha$ and $\check{\mathbf{\Pi}}^0_\alpha = \mathbf{\Sigma}^0_\alpha$.

We now introduce the Wadge hierarchy, which is a great refinement of the Borel hierarchy defined via reductions by continuous functions, [Wad83,Dup01].

Definition 2 (Wadge [Wad83]). *Let X, Y be two finite alphabets. For $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$, L is said to be Wadge reducible to L' ($L \leq_W L'$) iff there exists a continuous function $f : X^\omega \to Y^\omega$, such that $L = f^{-1}(L')$.*
L and L' are Wadge equivalent iff $L \leq_W L'$ and $L' \leq_W L$. This will be denoted by $L \equiv_W L'$. And we shall say that $L <_W L'$ iff $L \leq_W L'$ but not $L' \leq_W L$.
A set $L \subseteq X^\omega$ is said to be self dual iff $L \equiv_W L^-$, and otherwise it is said to be non self dual.

The relation $\leq_W$ is reflexive and transitive, and $\equiv_W$ is an equivalence relation.
The *equivalence classes* of $\equiv_W$ are called *Wadge degrees*.
The Wadge hierarchy WH is the class of Borel subsets of a set X^ω, where X is a finite set, equipped with $\leq_W$ and with $\equiv_W$.
For $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$, if $L \leq_W L'$ and $L = f^{-1}(L')$ where f is a continuous function from X^ω into Y^ω, then f is called a continuous reduction of L to L'. Intuitively it means that L is less complicated than L' because to check whether $x \in L$ it suffices to check whether $f(x) \in L'$ where f is a continuous function. Hence the Wadge degree of an ω-language is a measure of its topological complexity.
Notice that in the above definition, we consider that a subset $L \subseteq X^\omega$ is given together with the alphabet X.
We can now define the *Wadge class* of a set L:

Definition 3. *Let L be a subset of X^ω. The Wadge class of L is :*

$$[L] = \{L' \mid L' \subseteq Y^\omega \text{ for a finite alphabet } Y \text{ and } L' \leq_W L\}.$$

Recall that each *Borel class* $\mathbf{\Sigma}^{\mathbf{0}}_\alpha$ and $\mathbf{\Pi}^{\mathbf{0}}_\alpha$ is a *Wadge class*. A set $L \subseteq X^\omega$ is a $\mathbf{\Sigma}^{\mathbf{0}}_\alpha$ (respectively $\mathbf{\Pi}^{\mathbf{0}}_\alpha$)-*complete set* iff for any set $L' \subseteq Y^\omega$, L' is in $\mathbf{\Sigma}^{\mathbf{0}}_\alpha$ (respectively $\mathbf{\Pi}^{\mathbf{0}}_\alpha$) iff $L' \leq_W L$.

Theorem 4 (Wadge). *Up to the complement and $\equiv_W$, the class of Borel subsets of X^ω, for a finite alphabet X, is a well ordered hierarchy. There is an ordinal $|WH|$, called the length of the hierarchy, and a map d^0_W from WH onto $|WH| - \{0\}$, such that for all $L, L' \subseteq X^\omega$:*
$d^0_W L < d^0_W L' \leftrightarrow L <_W L'$ and
$d^0_W L = d^0_W L' \leftrightarrow [L \equiv_W L'$ or $L \equiv_W L'^-]$.

The Wadge hierarchy of Borel sets of **finite rank** has length ${}^1\varepsilon_0$ where ${}^1\varepsilon_0$ is the limit of the ordinals α_n defined by $\alpha_1 = \omega_1$ and $\alpha_{n+1} = \omega_1^{\alpha_n}$ for n a non negative integer, ω_1 being the first non countable ordinal. Then ${}^1\varepsilon_0$ is the first fixed point of the ordinal exponentiation of base ω_1. The length of the Wadge hierarchy of Borel sets in $\mathbf{\Delta^0_\omega} = \mathbf{\Sigma^0_\omega} \cap \mathbf{\Pi^0_\omega}$ is the ω_1^{th} fixed point of the ordinal exponentiation of base ω_1, which is a much larger ordinal. The length of the whole Wadge hierarchy of Borel sets is a huge ordinal, with regard to the ω_1^{th} fixed point of the ordinal exponentiation of base ω_1. It is described in [Wad83,Dup01] by the use of the Veblen functions.

There are some subsets of the topological space Σ^ω which are not Borel sets. In particular, there exists another hierarchy beyond the Borel hierarchy, called the projective hierarchy. The first class of the projective hierarchy is the class $\mathbf{\Sigma}^1_1$ of **analytic** sets. A set $A \subseteq \Sigma^\omega$ is analytic iff there exists a Borel set $B \subseteq (\Sigma \times Y)^\omega$, with Y a finite alphabet, such that $x \in A \leftrightarrow \exists y \in Y^\omega$ such that $(x, y) \in B$, where $(x, y) \in (\Sigma \times Y)^\omega$ is defined by: $(x, y)(i) = (x(i), y(i))$ for all integers $i \geq 0$.
A subset of Σ^ω is analytic if it is empty, or the image of the Baire space by a continuous map. The class of analytic sets contains the class of Borel sets in any of the spaces Σ^ω. Notice that $\mathbf{\Delta}^1_1 = \mathbf{\Sigma}^1_1 \cap \mathbf{\Pi}^1_1$, where $\mathbf{\Pi}^1_1$ is the class of co-analytic sets, i.e. of complements of analytic sets.

The ω-power of a finitary language V is always an analytic set because if V is finite and has n elements then V^ω is the continuous image of a compact set $\{0, 1, \ldots, n-1\}^\omega$ and if V is infinite then there is a bijection between V and ω and V^ω is the continuous image of the Baire space ω^ω, [Sim92].

3 Topological complexity of ω-powers

We now state our first main result, showing that ω-powers exhibit a very surprising topological complexity.

Theorem 5 ([FL07]). *Let ξ be a non-null countable ordinal.*
(a) There is $A \subseteq 2^{<\omega}$ such that A^ω is $\mathbf{\Sigma}^0_\xi$-complete.
(b) There is $A \subseteq 2^{<\omega}$ such that A^ω is $\mathbf{\Pi}^0_\xi$-complete.

To prove Theorem 5, we use in [FL07] a level by level version of a theorem of Lusin and Souslin stating that every Borel set $B \subseteq 2^\omega$ is the image of a closed subset of the Baire space ω^ω by a continuous bijection, see [Kec95, p.83]. It is the following theorem, proved by Kuratowski in [Kur66, Corollary 33.II.1]:

Theorem 6. *Let ξ be a non-null countable ordinal, and $B \in \mathbf{\Pi}^0_{\xi+1}(2^\omega)$. Then there is $C \in \mathbf{\Pi}^0_1(\omega^\omega)$ and a continuous bijection $f : C \to B$ such that f^{-1} is $\mathbf{\Sigma}^0_\xi$-measurable (i.e., $f[U]$ is $\mathbf{\Sigma}^0_\xi(B)$ for each open subset U of C).*

The existence of the continuous bijection $f : C \to B$ given by this theorem (without the fact that f^{-1} is $\mathbf{\Sigma}^0_\xi$-measurable) has been used by Arnold in [Arn83] to prove that every Borel subset of Σ^ω, for a finite alphabet Σ, is accepted by a non-ambiguous finitely branching transition system with Büchi acceptance condition. Notice that the sets

of states of these transition systems are countable.
Our first idea was to code the behaviour of such a transition system. In fact this can be done on a part of ω-words of a special compact set $K_{0,0}$. However we have also to consider more general sets $K_{N,j}$ and then we need the hypothesis of the $\mathbf{\Sigma}^0_\xi$-measurability of the function f. The complete proof can be found in [FL07,FL08].

Notice that for the class $\mathbf{\Sigma}^0_2$, we need another proof, which uses a new operation which is very close to the erasing operation defined by Duparc in his study of the Wadge hierarchy, [Dup01]. We get the following result.

Theorem 7. *There is a context-free language $A \subseteq 2^{<\omega}$ such that $A^\omega \in \mathbf{\Sigma}^0_2 \setminus \mathbf{\Pi}^0_2$.*

Notice that it is easy to see that the set $2^\omega \setminus P_\infty$, which is the classical example of $\mathbf{\Sigma}^0_2$-complete set, is not an ω-power. The question is still open to know whether there exists a *regular* language L such that L^ω is $\mathbf{\Sigma}^0_2$-complete.

Recall that, for each non-null countable ordinal ξ, the class of $\mathbf{\Sigma}^0_\xi$-complete (respectively, $\mathbf{\Pi}^0_\xi$-complete) subsets of 2^ω forms a single *non self-dual* Wadge degree. Thus Theorem 5 provides also some Wadge degrees of ω-powers. More generally, it is natural to ask for the Wadge hierarchy of ω-powers. In the long version [FL08] of the conference paper [FL07] we get many more Wadge degrees of ω-powers.

In order to state these new results, we now recall the notion of difference hierarchy. (Recall that a countable ordinal γ is said to be even iff it can be written in the form $\gamma = \alpha + n$, where α is a limit ordinal and n is an even positive integer; otherwise the ordinal γ is said to be odd; notice that all limit ordinals are even ordinals.)

If $\eta < \omega_1$ and $(A_\theta)_{\theta<\eta}$ is an increasing sequence of subsets of some space X, then we set

$$D_\eta[(A_\theta)_{\theta<\eta}] := \{x \in X \mid \exists \theta < \eta \;\; x \in A_\theta \setminus \bigcup_{\theta'<\theta} A_{\theta'} \text{ and the parity of } \theta \text{ is opposite to that of } \eta\}.$$

If moreover $1 \leq \xi < \omega_1$, then we set :

$$D_\eta(\mathbf{\Sigma}^0_\xi) := \{D_\eta[(A_\theta)_{\theta<\eta}] \mid \text{ for each } \theta < \eta \;\; A_\theta \text{ is in the class } \mathbf{\Sigma}^0_\xi\}.$$

Recall that for each non null countable ordinal ξ, the sequence $(D_\eta(\mathbf{\Sigma}^0_\xi))_{\eta<\omega_1}$ is strictly increasing for the inclusion relation and that for each $\eta < \omega_1$ it holds that $D_\eta(\mathbf{\Sigma}^0_\xi) \subseteq \mathbf{\Delta}^0_{\xi+1}$. Moreover for each $\eta < \omega_1$ the class $D_\eta(\mathbf{\Sigma}^0_\xi)$ is a Wadge class and the class of $D_\eta(\mathbf{\Sigma}^0_\xi)$-complete subsets of 2^ω forms a single *non self-dual* Wadge degree.

Theorem 8.

1. *Let $1 \leq \xi < \omega_1$. Then there is $A \subseteq 2^{<\omega}$ such that A^ω is $\check{D}_2(\mathbf{\Sigma}^0_\xi)$-complete.*
2. *Let $3 \leq \xi < \omega_1$ and $1 \leq \theta < \omega_1$. Then there is $A \subseteq 2^{<\omega}$ such that A^ω is $\check{D}_{\omega^\theta}(\mathbf{\Sigma}^0_\xi)$-complete.*

Notice that for each ordinal ξ such that $3 \leq \xi < \omega_1$ we get uncountably many Wadge degrees of ω-powers of the same Borel rank $\xi + 1$. This confirms the great complexity of these ω-languages.
However the problem is still open to determine completely the Wadge hierarchy of ω-powers.

We now come to the effectiveness questions. It is natural to wonder whether the ω-powers obtained above are effective. For instance could they be obtained as ω-powers of recursive languages ?

In the paper [FL08] we prove effective versions of the results presented above. Using tools of effective descriptive set theory, such Kleene recursion Theorem and the notion of Borel codes, we first prove an effective version of Kuratowski's Theorem 6. Then we use it to prove the following effective version of Theorem 5, where Σ^0_ξ and Π^0_ξ denote classes of the hyperarithmetical hierarchy and ω_1^{CK} is the first non-recursive ordinal, usually called the Church-kleene ordinal.

Theorem 9. *Let ξ be a non-null ordinal smaller than ω_1^{CK}.*
(a) There is a recursive language $A \subseteq 2^{<\omega}$ such that $A^\omega \in \Sigma^0_\xi \setminus \mathbf{\Pi}^0_\xi$.
(b) There is a recursive language $A \subseteq 2^{<\omega}$ such that $A^\omega \in \Pi^0_\xi \setminus \mathbf{\Sigma}^0_\xi$.

Remark 10. *If $A \subseteq 2^{<\omega}$ is a recursive language, then the ω-power A^ω is an effective analytic set, i.e. a (lightface) Σ^1_1-set. And the supremum of the set of Borel ranks of Borel effective analytic sets is the ordinal γ^1_2. This ordinal is defined by Kechris, Marker, and Sami in [KMS89] and it is proved to be strictly greater than the ordinal δ^1_2 which is the first non Δ^1_2 ordinal. Thus the ordinal γ^1_2 is also strictly greater than the first non-recursive ordinal ω_1^{CK}. Thus Theorem 9 does not give the complete answer about the Borel hierarchy of ω-powers of recursive languages. Indeed there could exist some ω-powers of recursive languages of Borel ranks greater than ω_1^{CK}, but of course smaller than the ordinal γ^1_2.*

4 Concluding remarks

The question naturally arises to know what are all the possible infinite Borel ranks of ω-powers of finitary languages belonging to some natural class like the class of context free languages (respectively, languages accepted by stack automata, recursive languages, recursively enumerable languages, ...).
We know from [Fin06] that there are ω-languages accepted by Büchi 1-counter automata of every Borel rank (and even of every Wadge degree) of an effective analytic set. Every ω-language accepted by a Büchi 1-counter automaton can be written as a finite union $L = \bigcup_{1 \leq i \leq n} U_i^\frown V_i^\omega$, where for each integer i, U_i and V_i are finitary languages accepted by 1-counter automata. And the supremum of the set of Borel ranks of effective analytic sets is the ordinal γ^1_2. From these results it seems plausible that there exist some ω-powers of languages accepted by 1-counter automata which have Borel

ranks up to the ordinal γ_2^1, although these languages are located at the very low level in the complexity hierarchy of finitary languages.

Another interesting question would be to determine completely the Wadge hierarchy of ω-powers. A simpler open question is to determine the Wadge hierarchy of ω-powers of regular languages. The second author has given in [Lec05] a few Wadge degrees of ω-powers of regular languages. Notice however that even the question to determine the Wadge degrees of ω-powers of regular languages in the class $\mathbf{\Delta}_2^0$ is still open.

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