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Synthesis of Finite-state and Definable Winning Strategies*

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ABSTRACT. Church's Problem asks for the construction of a procedure which, given a logical specification φ on sequence pairs, realizes for any input sequence I an output sequence O such that (I, O) satisfies φ. McNaughton reduced Church's Problem to a problem about two-player ω-games. Büchi and Landweber gave a solution for Monadic Second-Order Logic of Order (MLO) specifications in terms of finite-state strategies. We consider two natural generalizations of the Church problem to countable ordinals: the first deals with finite-state strategies; the second deals with MLO-definable strategies. We investigate games of arbitrary countable length and prove the computability of these generalizations of Church's problem.

1 Introduction

Two fundamental results of classical automata theory are decidability of the monadic second-order logic of order (MLO) over $\omega = (\mathbb{N}, <)$ and computability of the Church synthesis problem. These results have provided the underlying mathematical framework for the development of formalisms for the description of interactive systems and their desired properties, the algorithmic verification and the automatic synthesis of correct implementations from logical specifications, and advanced algorithmic techniques that are now embodied in industrial tools for verification and validation.

In order to prove decidability of the monadic theory of ω, Büchi introduced finite automata over ω-words. He provided a computable reduction from formulas to finite automata.

Büchi also introduced automata which "work" on words of any countable length (ordinal) and proved that the MLO-theory of any countable ordinal is decidable (see [BS73]).

What is known as the "Church synthesis problem" was first posed by Church in [Ch63] for the case of $(\omega, <)$. The Church problem is much more complex than the decidability problem for MLO. Church uses the language of automata theory. It was McNaughton [Mc65] who first observed that the Church problem can be equivalently phrased in game-theoretic language.

Let $\alpha > 0$ be an ordinal and let $\varphi(X_1, X_2)$ be a formula, where X_1 and X_2 are set (monadic predicate) variables. The *McNaughton game* $\mathcal{G}^{\alpha}_{\varphi}$ is defined as follows.

1. The game is played by two players, called Player I and Player II.
2. A *play* of the game has α rounds.
3. At round $\beta < \alpha$: first, Player I chooses $\pi_{X_1}(\beta) \in \{0, 1\}$; then, Player II chooses $\pi_{X_2}(\beta) \in \{0, 1\}$.

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4. By the end of the play two monadic predicates $\pi_{X_1}, \pi_{X_2} \subseteq \alpha$ have been constructed[†].
5. Then, Player I wins the play if $(\alpha, <) \models \varphi(\pi_{X_1}, \pi_{X_2})$; otherwise, Player II wins.

What we want to know is: Does either one of the players have a *winning strategy* in $\mathcal{G}^{\alpha}_{\varphi}$? If so, which one? That is, can Player I choose his moves so that, whatever way Player II responds we have $\varphi(\pi_{X_1}, \pi_{X_2})$? Or can Player II respond to Player I's moves in a way that ensures the opposite?

This leads to

Game version of the Church problem Let α be an ordinal. Given an MLO formula $\varphi(X_1, X_2)$, decide whether Player I has a winning strategy in $\mathcal{G}^{\alpha}_{\varphi}$.

In [BL69], Büchi and Landweber prove the computability of the Church problem in $\omega = (\mathbb{N}, <)$. Even more importantly, they show that in the case of ω we can restrict ourselves to MLO-*definable strategies*, or equivalently, to *finite-state strategies* (see Sect. 3 for the definitions of these strategies).

THEOREM 1.1 (BÜCHI-LANDWEBER, 1969) *Let $\varphi(X_1, X_2)$ be an MLO formula. Then:*

Determinacy *One of the players has a winning strategy in the game $\mathcal{G}^{\omega}_{\varphi}$.*

Decidability *It is decidable* which *of the players has a winning strategy.*

Definable strategy *The player who has a winning strategy, also has an MLO-*definable *winning strategy.*

Synthesis *We can compute a formula $\psi(X_1, X_2)$ that defines (in ω) a winning strategy for the winning player in $\mathcal{G}^{\omega}_{\varphi}$.*

After stating their main theorem, Büchi and Landweber write:

> "We hope to present elsewhere a corresponding extension of [our main theorem] from ω to any countable ordinal."

However, despite the fundamental role of the Church problem, no such extension is even mentioned in a later book by Büchi and Siefkes [BS73], which summarizes the theory of finite automata over words of countable ordinal length.

We proved in [RS08a, Rab09] that the Büchi-Landweber theorem extends fully to all ordinals $< \omega^{\omega}$ and its determinacy and decidability parts extend to all countable ordinals.

In [RS08], we provided a counter-example to a full extension of the Büchi-Landweber theorem to $\alpha \geq \omega^{\omega}$. For every ordinal $\alpha \geq \omega^{\omega}$ we constructed an MLO formula $\varphi_{\alpha}(X_1, X_2)$ such that Player I has a winning strategy in $\mathcal{G}^{\alpha}_{\varphi_\alpha}$; however, he has no MLO-definable winning strategy.

For $\alpha \leq \omega^{\omega}$, the set of MLO-definable in α strategies is the same as the set of finite-state strategies. However, for $\alpha > \omega^{\omega}$, the set of MLO-definable in α strategies properly contains the set of finite-state strategies. This leads to the following two synthesis problems for $\alpha \geq \omega^{\omega}$:

Synthesis Problems for α

Input: an MLO formula $\varphi(X_1, X_2)$.
Task1: Decide whether one of the players has a definable winning strategy in $\mathcal{G}^{\alpha}_{\varphi}$, and if so, construct ψ which defines his winning strategy.
Task2: Decide whether one of the players has a finite-state winning strategy in $\mathcal{G}^{\alpha}_{\varphi}$, and if so, construct such a strategy.

[†]We identify monadic predicates with their characteristic functions.

The first task is the synthesis problem of definable strategy and it will be denoted by Dsynth(α); the second task is the synthesis problem of finite-state strategy and it will be denoted by Fsynth(α).

In [Rab09], we reduced the synthesis problem Dsynth(α) to Dsynth(ω^ω). However, the decidability of the latter remained open.

Two main contributions of this paper are: the computability of Dsynth(ω^ω) (and, as a consequence, the computability of Dsynth(α)), and the computability of Fsynth(α). Our results are stronger than the computability of Dsynth(α) and Fsynth(α). For every countable α we need *finite* amount of data (code of α) which determines its monadic theory (see Subsection 2.2). We prove that there is an algorithm that receives the code of an ordinal α and a formula φ and decides whether Player I has a definable or finite-state strategy in the McNaughton game $\mathcal{G}^\alpha_\varphi$.

Our proofs use both game theoretical techniques and the "composition method" developed by Feferman-Vaught, Shelah and others (see, e.g. [Sh75]).

The article is organized as follows. The next section recalls standard definitions about monadic logic of order, summarizes elements of the composition method and reviews known facts about the monadic theory of countable ordinals. In Sect. 3, we provide definitions of the finite-state and MLO-definable strategies and survey results about McNauughton games of countable length. In Section 4, we introduce special games on types and provide a reduction of these games to the McNaughton games. Section 5 contains the main results of the paper and outlines the proof of the computability of the synthesis problem for MLO-definable strategies. Finally, in Sect. 6, we discuss some open problems.

2 Preliminaries on Monadic Logic of Order

Notations and terminology We use n, k, l, m, p, q for natural numbers and $\alpha, \beta, \gamma, \delta$ for ordinals. We use $\mathbb{N}$ for the set of natural numbers and ω for the first infinite ordinal. We write $\alpha + \beta$, $\alpha\beta$, α^β for the sum, multiplication and exponentiation, respectively, of ordinals α and β. We use the expressions "*chain*" and "*linear order*" interchangeably. We use $\mathbb{P}(A)$ for the set of subsets of A.

2.1 The Monadic Logic of Order (MLO)

Syntax The syntax of the monadic second-order logic of order - MLO has in its vocabulary *individual* (first order) variables $t_1, t_2 \ldots$, monadic *second-order* variables $X_1, X_2 \ldots$ and one binary relation $<$ (the order).

Atomic formulas are of the form $X(t)$ and $t_1 < t_2$. Well-formed formulas of the monadic logic MLO are obtained from atomic formulas using Boolean connectives $\neg, \vee, \wedge, \rightarrow$, the first-order quantifiers $\exists t$ and $\forall t$, and the second-order quantifiers $\exists X$ and $\forall X$. The quantifier depth of a formula φ is denoted by $\mathrm{qd}(\varphi)$.

We use upper case letters X, Y, Z to denote second-order variables, and overlined letters $\bar{X}, \bar{Y}$ to denote finite tuples of variables.
Semantics A *structure* is a tuple $\mathcal{M} := (A^\mathcal{M}, <^\mathcal{M}, \bar{P}^\mathcal{M})$ where: $A^\mathcal{M}$ is a non-empty set, $<^\mathcal{M}$ is a binary relation on $A^\mathcal{M}$, and $\bar{P}^\mathcal{M} := \left(P_1^\mathcal{M}, \ldots, P_l^\mathcal{M}\right)$ is a *finite* tuple of subsets of $A^\mathcal{M}$.

If $\bar{P}^{\mathcal{M}}$ is a tuple of l sets, we call $\mathcal{M}$ an *l-structure*. If $<^{\mathcal{M}}$ linearly orders $A^{\mathcal{M}}$, we call $\mathcal{M}$ an *l-chain*.

Suppose $\mathcal{M}$ is an l-structure and φ a formula with free-variables among $X_1, \ldots, X_l$. We define the relation $\mathcal{M} \models \varphi$ (read: $\mathcal{M}$ *satisfies* φ) as usual, understanding that the second-order quantifiers range over *subsets* of $A^{\mathcal{M}}$.

Let $\mathcal{M}$ be an l-structure. The *monadic theory* of $\mathcal{M}$, $MTh(\mathcal{M})$, is the set of all formulas with free variables among $X_1, \ldots, X_l$ satisfied by $\mathcal{M}$.

¿From now on, we omit the superscript in '$<^{\mathcal{M}}$' and '$\bar{P}^{\mathcal{M}}$'. We often write $(A, <) \models \varphi(\bar{P})$ meaning $(A, <, \bar{P}) \models \varphi$.

2.2 The monadic theory of countable ordinals

Büchi (for instance [BS73]) has shown that there is a *finite* amount of data concerning any countable ordinal which determines its monadic theory:

THEOREM 2.1 *Let α be a countable ordinal. Write $\alpha = \omega^{\omega}\beta + \zeta$ where $\zeta < \omega^{\omega}$ (this can be done in a unique way). Then the monadic theory of $(\alpha, <)$ is determined by:*

1. *whether $\alpha < \omega^{\omega}$, and*
2. *ζ.*

We can associate with every countable α a finite *code* which holds the data required in the previous theorem. This is clear with respect to (1). As for (2), if $\zeta \neq 0$, write

$\zeta = \sum_{i \leq n} \omega^{n-i} \cdot a_{n-i}$, where $n, a_i \in \mathbb{N}$ for $i \leq n$ and $a_n \neq 0$

(this, too, can be done in a unique way), and let the sequence $(a_n, \ldots, a_0)$ encode ζ. The following is implicit in [BS73]:

THEOREM 2.2 (MONADIC DECIDABILITY THEOREM) *There is an algorithm that, given a sentence φ and the* code *of a countable ordinal α, determines whether $(\alpha, <) \models \varphi$.*

We conclude by a well-known Lemma which is easily derived from Büchi results [BS73], as well from the composition theorem (see Theorem 2.11).

LEMMA 2.3 *For every n there is m computable from n such that for every MLO sentence φ of the quantifier depth at most n and every countable ordinals $\alpha > 0$ and β:*

$$\omega^m + \beta \models \varphi \text{ if and only if } \omega^m\alpha + \beta \models \varphi$$

2.3 Elements of the composition method

Our proofs make use of the technique known as the composition method developed by Feferman-Vaught and Shelah [FV59, Sh75]. To fix notations and to aid the reader unfamiliar with this technique, we briefly review the required definitions and results. A more detailed presentation can be found in [Th97] or [Gu85].

Let $n, l \in \mathbb{N}$. We denote by $\mathfrak{Form}^n_l$ the set of formulas with free variables among $X_1, \ldots, X_l$ and of quantifier depth $\leq n$.

DEFINITION 2.4 *Let $n, l \in \mathbb{N}$ and let $\mathcal{M}, \mathcal{N}$ be l-structures. The n-*theory *of $\mathcal{M}$ is*

$$Th^n(\mathcal{M}) := \{\varphi \in \mathfrak{Form}^n_l \mid \mathcal{M} \models \varphi\}.$$

*If $Th^n(\mathcal{M}) = Th^n(\mathcal{N})$, we say that $\mathcal{M}$ and $\mathcal{N}$ are n-*equivalent *and write $\mathcal{M} \equiv^n \mathcal{N}$.*

Clearly, $\equiv^n$ is an equivalence relation. For any $n \in \mathbb{N}$ and $l > 0$, the set $\mathfrak{Form}_l^n$ is infinite. However, it contains only finitely many semantically distinct formulas. So, there are finitely many $\equiv^n$-equivalence classes of l-structures. In fact, we can compute characteristic sentences for the $\equiv^n$-equivalence classes:

LEMMA 2.5 (HINTIKKA LEMMA) *For $n, l \in \mathbb{N}$, we can compute a* finite *set $Char_l^n \subseteq \mathfrak{Form}_l^n$ such that:*

1. For every $\equiv^n$-equivalence class A there is a unique *$\tau \in Char_l^n$ such that for every l-structure $\mathcal{M}$: $\mathcal{M} \in A$ iff $\mathcal{M} \models \tau$.*

2. Every MLO formula $\varphi(X_1, \ldots X_l)$ with $\mathrm{qd}(\varphi) \leq n$ is equivalent to a (finite) disjunction of characteristic formulas from $Char_l^n$. Moreover, there is an algorithm which for every formula $\varphi(X_1, \ldots X_l)$ computes a finite set $G_\varphi \subseteq Char_l^{\mathrm{qd}(\varphi)}$ of characteristic formulas, such that φ is equivalent to the disjunction of all the formulas in G.

Any member of $Char_l^n$ we call a (n, l)-*Hintikka formula* or (n, l)-characteristic formula. We use τ, τ_i, τ^j to range over the characteristic formulas and G, G_i, G' to range over sets of characteristic formulas. Usually, we do not distinguish between φ and the corresponding set G_φ of characteristic formulas.

DEFINITION 2.6 (n-TYPE) *For $n, l \in \mathbb{N}$ and an l-structure $\mathcal{M}$, we denote by $type_n(\mathcal{M})$ the unique member of $Char_l^n$ satisfied by $\mathcal{M}$ and call it the n-*type *of $\mathcal{M}$.*

Thus, $type_n(\mathcal{M})$ determines $Th^n(\mathcal{M})$ and, indeed, $Th^n(\mathcal{M})$ is computable from $type_n(\mathcal{M})$.

DEFINITION 2.7 (SUM OF CHAINS) *Let $l \in \mathbb{N}$, $\mathcal{I} := (I, <^{\mathcal{I}})$ a chain and $\mathfrak{S} := (\mathcal{M}_\alpha \mid \alpha \in I)$ a sequence of l-chains. Write $\mathcal{M}_\alpha := (A_\alpha, <^\alpha, P_1{}^\alpha, \ldots, P_l{}^\alpha)$ and assume that $A_\alpha \cap A_\beta = \emptyset$ whenever $\alpha \neq \beta$ are in I. The* ordered sum *of $\mathfrak{S}$ is the l-chain*

$$\sum_{\mathcal{I}} \mathfrak{S} := (\bigcup_{\alpha \in I} A_\alpha, <^{\mathcal{I},\mathfrak{S}}, \bigcup\nolimits_{\alpha \in I} P_1{}^\alpha, \ldots, \bigcup_{\alpha \in I} P_l{}^\alpha),$$

where: if $\alpha, \beta \in I$, $a \in A_\alpha$, $b \in A_\beta$, then $b <^{\mathcal{I},\mathfrak{S}} a$ iff $\beta <^{\mathcal{I}} \alpha$ or $\beta = \alpha$ and $b <^\alpha a$.

If the domains of the $\mathcal{M}_\alpha$'s are not disjoint, replace them with isomorphic l-chains that have disjoint domains, and proceed as before.

If $\mathcal{I} = (\{0, 1\}, <)$ and $\mathfrak{S} = (\mathcal{M}_0, \mathcal{M}_1)$, we denote $\sum_{\mathcal{I}} \mathfrak{S}$ by $\mathcal{M}_0 + \mathcal{M}_1$.

The next proposition states that taking ordered sums preserves $\equiv^n$-equivalence.

PROPOSITION 2.8 *Let $n, l \in \mathbb{N}$. Assume:*

1. *$(I, <^{\mathcal{I}})$ is a linear order,*
2. *$\left(\mathcal{M}_\alpha^0 \mid \alpha \in I\right)$ and $\left(\mathcal{M}_\alpha^1 \mid \alpha \in I\right)$ are sequences of l-chains, and*
3. *for every $\alpha \in I$, $\mathcal{M}_\alpha^0 \equiv^n \mathcal{M}_\alpha^1$.*

Then, $\sum_{\alpha \in I} \mathcal{M}_\alpha^0 \equiv^n \sum_{\alpha \in I} \mathcal{M}_\alpha^1$.

This allows us to define the sum of formulas in $Char_l^n$ with respect to any linear order.

DEFINITION 2.9 *Let $n, l \in \mathbb{N}$, $\mathcal{I} := (I, <^{\mathcal{I}})$ a chain, $\mathfrak{H} := (\tau_\alpha \mid \alpha \in I)$ a sequence of (n, l)-Hintikka formulas. The* ordered sum *of $\mathfrak{H}$, (notations $\sum_{\mathcal{I}} \mathfrak{H}$ or $\sum_{\alpha \in \mathcal{I}} \tau_\alpha$), is an* element *$\tau$ of $Char_l^n$ such that:*

if $\mathfrak{S} := (\mathcal{M}_\alpha \mid \alpha \in I)$ is a sequence of l-chains and $type_n(\mathcal{M}_\alpha) = \tau_\alpha$ for $\alpha \in I$, then

$$type_n(\sum_{\mathcal{I}} \mathfrak{S}) = \tau.$$

If $\mathcal{I} = (\{0, 1\}, <)$ and $\mathfrak{H} = (\tau_0, \tau_1)$, we denote $\sum_{\alpha \in \mathcal{I}} \tau_\alpha$ by $\tau_0 + \tau_1$.

The next Lemma states that the sum of two types is computable.

LEMMA 2.10 (ADDITION LEMMA) *The function which maps the pairs of characteristic formulas to their sum is recursive. Formally, $\lambda n, l \in \mathbb{N}.\lambda \tau_0, \tau_1 \in Char^n_l.\tau_0 + \tau_1$ is recursive.*

The following fundamental result of Shelah can be found in [Sh75]:

THEOREM 2.11 (COMPOSITION THEOREM) *Let $\varphi(X_1, \ldots, X_l)$ be a formula, let $n = \text{qd}(\varphi)$ and let $\{\tau_1, \ldots, \tau_m\} = Char^n_l$. Then, there is a formula $\psi(Y_1, \ldots, Y_m)$ such that for every chain $\mathcal{I} = (I, <)$ and sequence $(\mathcal{M}_\alpha \mid \alpha \in I)$ of l-chains the following holds:*

$$\sum_{\alpha \in I} \mathcal{M}_\alpha \models \varphi \quad \textit{iff} \quad \mathcal{I} \models \psi(Q_1, \ldots, Q_m), \textit{ where}$$

$Q_j = \{\alpha \in I \mid M_\alpha \models \tau_j\}$. Moreover, ψ is computable from φ.

3 Finite-state and MLO-definable strategies

In the McNaughton game $\mathcal{G}^\alpha_\varphi$, at round $\beta < \alpha$, Player I has access only to $\pi_{X_2} \cap [0, \beta)$ and Player II has access only to $\pi_{X_1} \cap [0, \beta]$. Therefore, the following formalizes well the notion of a strategy in this game:

DEFINITION 3.1 (CAUSAL OPERATOR) *Let α be an ordinal, $F : \mathbb{P}(\alpha) \to \mathbb{P}(\alpha)$ maps the subsets of α into the subsets of α. We call F* causal *(resp.* strongly causal*) iff for all $P, P' \subseteq \alpha$ and $\beta < \alpha$: if $P \cap [0, \beta] = P' \cap [0, \beta]$ (resp. $P \cap [0, \beta) = P' \cap [0, \beta)$), then*
$F(P) \cap [0, \beta] = F(P') \cap [0, \beta]$.
That is, if P and P' agree up to and including *(resp.* up to*) β, then so do $F(P)$ and $F(P')$.*

So, a winning strategy for Player I is a strongly causal $F : \mathbb{P}(\alpha) \to \mathbb{P}(\alpha)$ such that for every $P \subseteq \alpha$, $(\alpha, <) \models \varphi(F(P), P)$; a winning strategy for Player II is a causal $F : \mathbb{P}(\alpha) \to \mathbb{P}(\alpha)$ such that for every $P \subseteq \alpha$, $(\alpha, <) \models \neg\varphi(P, F(P))$.

Let $\psi(X_1, X_2)$ be a formula where X_2 is declared as the "domain" variable and X_1 as the "range" variables. Let $\mathcal{M} := (A, <)$ be a chain and let $F : \mathbb{P}(A) \to \mathbb{P}(A)$ be an operator. We say that ψ *defines* F in $\mathcal{M}$ if $\mathcal{M} \models \psi(P_1, P_2)$ iff $P_1 = F(P_2)$.

It is easy to formalize in MLO that ψ defines in $\mathcal{M}$ a causal or strongly causal operator. Hence, for every ψ there are sentences I-Player-strategy$_\psi$ and II-Player-strategy$_\psi$ such that $\alpha \models$ I-Player-strategy$_\psi$ iff ψ defines (in α) a strategy for Player I, and $\alpha \models$ II-Player-strategy$_\psi$ iff ψ defines (in α) a strategy for Player II. A play $\rho := (\rho_{X_1}(0), \rho_{X_2}(0)) \ldots (\rho_{X_1}(\beta), \rho_{X_2}(\beta)) \ldots$ is consistent with the strategy defined in α by ψ if $\alpha \models \psi(\rho_{X_1}, \rho_{X_2})$. A Player I's strategy defined by ψ is winning in $\mathcal{G}^\alpha_\varphi$ if $\alpha \models \forall X_1 X_2 \psi(X_1, X_2) \to \varphi(X_1, X_2)$. Hence, the monadic theory of an ordinal α "knows" which formulas defines in α a strategy and which definable strategies are winning in $\mathcal{G}^\alpha_\varphi$.

A formula $\psi(\bar{X}, t)$ with at most one free individual variable t is (syntactically) *bounded* if all its first-order quantifiers are of the form $\exists^{<t} y \ldots$ (short for $\exists y(y < t \wedge \ldots)$ and $\forall^{<t} y \ldots$ (short for $\forall y(y < t \to \ldots)$).

If $\psi(X_1, X_2, t)$ is syntactically bounded and does not contain the atomic formulas $X_1(t)$ and $X_2(t)$, then $\forall t\big(X_1(t) \leftrightarrow \psi\big)$ defines in every ordinal a strategy for Player I (a strongly causal operator); ψ is said to be an *explicit definition* of this strategy. Similarly, if $\psi(X_1, X_2, t)$ is syntactically bounded and does not contain the atomic formula $X_2(t)$, then $\forall t\big(X_2(t) \leftrightarrow \psi\big)$ defines in every ordinal a strategy for Player II (a causal operator).

The strategies explicitly defined by the bounded formulas can be computed by finite-state transducers. A finite state transducer consists of a finite set Q - memory states, an initial state q_{init}, next-state functions $\text{next}_1 : Q \to Q$ and $\text{next}_2 : Q \times \{0,1\} \to Q$, a limit transition function $\Delta : \mathbb{P}(Q) \to Q$, and an output function $\text{out} : Q \to \{0,1\}$.

During a play, according to a transducer, at round β, Player I first updates the state according to next_1 or Δ, outputs value according to out, and then after a move of Player II updates the state. Formally, a play $\rho := (\rho_{X_1}(0), \rho_{X_2}(0)) \dots (\rho_{X_1}(\beta), \rho_{X_2}(\beta)) \dots$ is consistent with such a strategy if there are $q_0, q'_0 \dots q_\beta q'_\beta \dots$ such that $q_0 = q_{init}$

1. If $\beta = \beta' + 1$ is a successor ordinal, then $q_\beta = \text{next}_1(q'_{\beta'})$
2. If β is a limit ordinal then $q_\beta = \Delta(L)$, where $L := \{q \in Q \mid q$ appears cofinally often in $q_0, q'_0 \dots q_\gamma q'_\gamma \dots (\gamma < \beta)\}$.
3. $\rho_{X_1}(\beta) = \text{out}(q_\beta)$.
4. $q'_\beta = \text{next}_2(q_\beta, \rho_{X_2}(\beta))$

It is clear that a transducer defines a strategy *st* for Player I. Moreover, *st* is definable by a transducer iff it is explicitly definable by a bounded a formula.

Every ordinal $\alpha < \omega^\omega$ is MLO-definable. It is not difficult to show that a strategy is MLO-definable in $\alpha < \omega^\omega$ iff it is finite-state strategy (equivalently is explicitly defined by a bounded formula). If for a countable ordinal α every cofinal interval (β, α) is isomorphic to α, then a strategy is finite-state iff it is MLO-definable in α. However, the set of MLO-definable strategies is larger than the set of finite-state strategies; e.g., if $n > 0$ and φ expresses "X_1 contains exactly the last element", then Player I has a definable winning strategy in $\mathcal{G}_\varphi^{\omega^\omega+n}$, but he has no finite-state winning strategy in this game.

We recall below results from [Rab09, RS08] about McNauughton games over ordinals, and results from [CH08] about reachability and safety games of length ω^ω.

THEOREM 3.2 *Let α be a* countable *ordinal, $\varphi(X_1, X_2)$ a formula.*

Determinacy *One of the players has a winning strategy in the game $\mathcal{G}_\varphi^\alpha$.*

MLO characterization of the winner *There is a sentence $win(\varphi)$ such that for every countable ordinal α: Player I wins $\mathcal{G}_\varphi^\alpha$ if and only if $\alpha \models win(\varphi)$. Furthermore, $win(\varphi)$ is computable from φ.*

Decidability *There is an algorithm that given α and φ decides* which *of the players has a winning strategy in $\mathcal{G}_\varphi^\alpha$.*

No definable winning strategy *For every $\alpha \geq \omega^\omega$, there is a formula φ such that no player has a definable winning strategy in $\mathcal{G}_\varphi^\alpha$.*

Finite-state winning strategy *If $\alpha < \omega^\omega$, then the player who has a winning strategy, also has a* finite-state *winning strategy.*

Synthesis *If $\alpha < \omega^\omega$, then we can compute a finite-state winning strategy for the winning player in $\mathcal{G}_\varphi^\alpha$.*

Hence, the Büchi-Landweber theorem extends fully to the ordinals less than ω^ω, and its determinacy and decidability parts extends to all countable ordinals.

REMARK 3.3 *1. In this paper, whenever we say that an algorithm is "given an ordinal..." or "returns an ordinal...", we mean the* code *of the ordinal. In particular, this holds for the decidability and synthesis parts of Theorem 3.2.*

2. Sometimes, like in the MLO characterization part of Theorem 3.2, we state our result only for Player I. However, in all these cases there is a duality between the players, and similar assertions hold for Player II. For every φ we can construct ψ such that Player I has a definable (respectively,

finite-state) winning strategy st in $\mathcal{G}_{\psi}^{\alpha}$ iff Player II has a definable (respectively, finite-state) winning strategy in $\mathcal{G}_{\varphi}^{\alpha}$. Moreover, this strategy is computable from (the description of) st.
3. To simplify notations, games and the Church problem were previously defined for formulas with two free variables X_1 and X_2. It is easy to generalize all definitions and results to formulas $\psi(X_1, \dots, X_m, Y_1, \dots Y_n)$ with many variables. In this generalization at round β, Player I chooses values for $X_1(\beta), \dots, X_m(\beta)$, then Player II replies by choosing values for $Y_1(\beta), \dots, Y_n(\beta)$. Note that, strictly speaking, the input to the Church problem is not only a formula, but a formula plus a partition of its free variables to Player I's variables and Player II's variables.
In [CH08] reachability games of ordinal length over finite graphs were considered. The next theorem reformulates results from [CH08] in logical terms.

Let $\vartheta(X_1, X_2)$ be a formula. Let $\vartheta^{<t}$ be the relativization of ϑ to the interval $[0, t)$, i.e., obtained from $\vartheta(X_1, X_2)$ by changing the first-order quantifiers $\exists y$ and $\forall y$ to $\exists^{<t} y$ and $\forall^{<t} y$. A *reachability* formula is a formula of the form $\exists t \vartheta^{<t}$. A *safety* formula is a formula of the form $\forall t \vartheta^{<t}$.

THEOREM 3.4 *Let φ be a reachability or safety formula. Then*
Finite-state strategy *The player who has a winning strategy in $\mathcal{G}_{\varphi}^{\omega^\omega}$ also has a finite-state winning strategy.*
Synthesis *We can compute a finite-state winning strategy for the winning player in $\mathcal{G}_{\varphi}^{\omega^\omega}$.*

4 Special Games on Types

In this section we introduce special games on types. These games play an important role in our proof that Dsynth(ω^ω) is computable. We reduce special games to safety games and derive that a winning player in these games has a definable winning strategy.

DEFINITION 4.1 (RESIDUAL) *Let $k \in \mathbb{N}$, $G \subseteq Char_2^k$ and $\tau \in Char_2^k$. Define $res_\tau(G)$ as $res_\tau(G) := \{\tau' \in Char_2^k \mid \tau + \tau' \in G\}$.*
Let F assign to every $\tau \in G$ a non-empty subset of $\mathcal{P}(res_\tau(G)) \setminus \{\emptyset\}$. The ω^ω-game on types, $Game(F, G)$, is defined as follows. There are ω^ω rounds.
Round 0: Player I sets $G_0 := G$. Player II chooses $\tau_0 \in G_0$.
Round α (for $\alpha > 0$): Let $\tau_{<\alpha} := \sum_{\beta \in \alpha} \tau_\beta$. If $\tau_{<\alpha} \notin G$, then Player II wins. Otherwise, Player I chooses $G_\alpha \in F(\tau_{<\alpha})$ and then Player II chooses $\tau_\alpha \in G_\alpha$.
Winning Conditions: Player I wins a play $G_0\tau_0 \dots G_\beta\tau_\beta \dots$ if $\sum_{\beta \in \alpha} \tau_\beta \in G$ for every $\alpha \leq \omega^\omega$.
The proof of the next proposition is based on a reduction of special games to safety games.
PROPOSITION 4.2 *There is an algorithm that given a game $Game(F, G)$, decides whether Player I has a winning strategy. Furthermore, if such a strategy exists, then there is definable winning strategy, and we can compute a formula $\psi(\bar{X}, \bar{Y})$ that defines in ω^ω a winning strategy for Player I.*
Since a strategy is definable in ω^ω iff it is finite-state, we can replace "definable" by "finite-state" in the above Proposition.

5 Main Results

In the next lemma and throughout this paper we often use $G \subseteq Char_2^k$ for φ defined as $\vee_{\tau \in G} \tau$. In particular, we use $\mathcal{G}_G^\alpha$, for the McNaughton game $\mathcal{G}_\varphi^\alpha$, and $win(G)$ for $win(\varphi)$, where $win(\varphi)$ is the sentence from Theorem 3.2.

LEMMA 5.1 (MAIN) *Let $G \subseteq Char_2^k$. The following are equivalent:*

1. *Player I has a definable winning strategy in $\mathcal{G}_G^{\omega^\omega}$.*
2. *There is $G' \subseteq G$ and a special game $Game(F, G')$ such that*
   (a) $\omega^\omega \models win(G')$.
   (b) $\omega^\omega \models win(G_1)$ *for every $\tau \in G'$ and $G_1 \in F(\tau)$.*
   (c) *Player I has a winning strategy in $Game(F, G')$.*

The implication (2)⇒(1) will be proved in Subsection 5.1. The implication (1)⇒(2) will be proved in Subsection 5.2. As a consequence, we obtain the computability of Dsynth(ω^ω).

THEOREM 5.2 (COMPUTABILITY OF DSYNTH(ω^ω)) *There is an algorithm that given a formula $\varphi(X_1, X_2)$ decides whether Player I has a definable winning strategy in the game $\mathcal{G}_\varphi^{\omega^\omega}$. Furthermore, if such a strategy exists we can compute a formula $\psi(X_1, X_2)$ that defines (in ω^ω) a winning strategy for Player I.*

PROOF. Since condition (2) of Lemma 5.1 is decidable, we obtain the decidability part of the theorem. The "furthermore part" of the theorem can be extracted from our proof of Lemma 5.1. ∎

In [Rab09] we provided reduction from Dsynth(α) to Dsynth(ω^ω). As a consequence of Theorem 5.2 and results in [Rab09] we obtain:

THEOREM 5.3 (COMPUTABILITY OF DSYNTH(α)) *1. There is an algorithm that given a formula $\varphi(X_1, X_2)$ computes a sentence $Dwin_\varphi$ such that for every countable ordinal $\alpha \geq \omega^\omega$: Player I has a definable (in α) winning strategy in $\mathcal{G}_\varphi^\alpha$ iff $\alpha \models Dwin_\varphi$.*
2. There is an algorithm that given a formula $\varphi(X_1, X_2)$ and the code of an ordinal α decides whether Player I has a definable winning strategy in $\mathcal{G}_\varphi^\alpha$, and if so, computes a formula ψ_α which defines in α such a strategy.

The next theorem states that the synthesis problem for finite-state strategies is computable. Its proof refines the proof of Theorem 5.3 and will be presented in the full paper.

THEOREM 5.4 (COMPUTABILITY OF FSYNTH(α)) *1. There is an algorithm that given a formula $\varphi(X_1, X_2)$ computes a sentence $Fswin_\varphi$ such that for every countable ordinal $\alpha \geq \omega^\omega$: Player I has a finite-state winning strategy in $\mathcal{G}_\varphi^\alpha$ if and only if $\alpha \models Fswin_\varphi$.*
2. There is an algorithm that given a formula $\varphi(X_1, X_2)$ and a code of α decides whether Player I has a finite-state winning strategy in $\mathcal{G}_\varphi^\alpha$, and if so, computes such a strategy.

5.1 Implication (2)⇒ (1) of Lemma 5.1

Terminology. (k-type of a play) For a (partial) play $\pi := (\pi_{X_1}(0), \pi_{X_2}(0)) \ldots (\pi_{X_1}(\beta), \pi_{X_2}(\beta)) \ldots$ ($\beta \in \alpha$) its k-type is defined as the k-type of the chain $(\alpha, <, \pi_{X_1}, \pi_{X_2})$.

Let n be an upper bound on the quantifier depth of $win(H)$ for $H \subseteq Char_2^k$, where $win(H)$ is as in Theorem 3.2. By Lemma 2.3, we can compute m such that no sentence of the quantifier depth $\leq n$ can distinguish between multiples of ω^m.

¿From condition 2(a), and our choice of m, it follows that $\omega^m \models win(G')$ and therefore, by the synthesis part of Theorem 3.2, Player I has a definable winning strategy in $\mathcal{G}_{G'}^{\omega^m}$. We fix such a strategy $st_{G'}$. Similarly, condition 2(b) implies that for every $\tau \in G'$ and $G_1 \in F(\tau)$ Player I has a definable winning strategy in $\mathcal{G}_{G_1}^{\omega^m}$, we denote such a strategy by st_{G_1}. Condition 2(c) implies that Player I has a definable winning strategy st_F in $Game(F, G')$.

We organize our description of a winning strategy for $\mathcal{G}_G^{\omega^\omega}$ in sessions; each session is played for ω^m rounds. Each session "corresponds" to one round in $Game(F, G')$.

We show that this strategy wins G' on every multiple of ω^m.

Session 0: Play first ω^m rounds according to a definable winning strategy for $G_0 := G'$. Set τ_0 to be the k-type of the play during this session. Note that $\tau_0 \in G_0$ and this session corresponds to the play $\pi_0 := G_0\tau_0$ consistent with st_F in the game $Game(F, G')$.

Session α (for $\alpha > 0$): Let $\pi := G_0\tau_0, \dots, G_\beta\tau_\beta \dots$ (for $\beta < \alpha$) be the play of $Game(F, G')$ which corresponds to the previous sessions of the play.

Let G_α be defined as the response of st_F after π. Play the next ω^m rounds according to the winning strategy st_{G_α} in $\mathcal{G}_{G_\alpha}^{\omega^m}$.

Set τ_α to be the k-type of the play during this session. Note that $\tau_\alpha \in G_\alpha$ and the play $\pi G_\alpha \tau_\alpha$ is a play according to st_F.

It is clear that the above strategy is winning in $\mathcal{G}_{G'}^{\omega^\omega}$ and hence in $\mathcal{G}_G^{\omega^\omega}$.

It is easy to see that the above description of the strategy can be formalized in MLO.

5.2 Implication (1)⇒(2) of Lemma 5.1

DEFINITION 5.5 *Let $G \subseteq Char_2^k$. We say that a strategy* realizes *G on α if it wins $\mathcal{G}_G^\alpha$ and there is no $G_1 \subsetneq G$ such that it wins $\mathcal{G}_{G_1}^\alpha$.*

Note that for each k and a strategy st, the set $G \subseteq Char_2^k$ realized by st on α is unique. For every ψ and $G \subseteq Char_2^k$, there is a sentence $\text{Realize}(\psi, G)$ such that for every α: ψ defines in α a strategy which realizes G iff $\alpha \models \text{Realize}(\psi, G)$.

Assume that st defines a strategy and the quantifier depth of st is s. For $\tau \in Char_2^s$, let $st_\tau := \{\tau' \in Char_2^s \mid \tau + \tau' \to st\}$ be the residual of st wrt τ.

LEMMA 5.6 *Assume that st defines in ω^ω a strategy, its quantifier depth is s, and $\tau \in Char_2^s$.*

1. *If $st \wedge \tau$ is satisfiable, then st_τ defines in ω^ω a strategy.*
2. *If $\mathcal{M}_0 + \mathcal{M}_1 \models st$ and $type_s(\mathcal{M}_0) = \tau$ then $\mathcal{M}_1 \models st_\tau$.*
3. *If $\mathcal{M}_0 \models st$ and $type_s(\mathcal{M}_0) = \tau$ and $\mathcal{M}_1 \models st_\tau$, then $\mathcal{M}_0 + \mathcal{M}_1 \models st$.*
4. *If $\tau_\beta = type_s(\mathcal{M}_\beta)$, and $\mathcal{M}_0 \models st$ and $\mathcal{M}_\beta \models st_{\Sigma_{\gamma\in\beta}\tau_\gamma}$ for every $\beta \in (0, \alpha)$, then $\Sigma_{\beta\in[0,\alpha)}\mathcal{M}_\beta \models st$.*

For $k \in \mathbb{N}$ and a strategy st, we denote by $R(k, st)$ the subset of $Char_2^k$ realized by st on ω^ω. Define $F_{st}^k : R(k, st) \to \mathcal{P}(\mathcal{P}(Char_2^k)) \setminus \{\emptyset\}$ as follows:

$$F_{st}^k(\tau) := \{R(k, st_\delta) \mid \delta \in Char_2^s \text{ and } \delta \wedge st \wedge \tau \text{ is satisfiable on } \omega^\omega\}$$

The implication (1) ⇒ (2) of Lemma 5.1 immediately follows from the next lemma and the observation that st_δ wins $\mathcal{G}_{R(k,st_\delta)}^{\omega^\omega}$.

LEMMA 5.7 *Assume that st defines in ω^ω a strategy for Player I, and the quantifier depth of st is s. Then for every $k \le s$, Player I has a winning strategy in $Game(F_{st}^k, R(k, st))$.*

PROOF. Let m be defined from $n := s + 2$ as in Lemma 2.3. In particular, st realizes $R(k, st)$ on every multiple of ω^m. Note that for $\delta \in Char_2^s$, the quantifier depth of st_δ is s. Therefore, st_δ realizes $R(k, st_\delta)$ on every multiple of ω^m.

We will describe a strategy for Player I and show that it is winning in $Game(F_{st}^k, R(k, st))$. Each round in this game corresponds to ω^m rounds in $\mathcal{G}_{R(k,st)}^{\omega^\omega}$. A play according to this strategy corresponds to a play according to the strategy st in $\mathcal{G}_{R(k,st)}^{\omega^\omega}$.

In addition to the description of the strategy we are going to define for each round α: $\delta_\alpha, v_\alpha \in Char_2^s$, and a play $\mathcal{M}_\alpha$ of length ω^m.

Round 0 Play $G_0 := R(k, st)$. Assume that Player II has replied by $\tau_0 \in G_0$ in round 0. Choose $v_1 = \delta_0 \in Char_2^s$ consistent with $\tau_0 \wedge st$ on ω^ω. Choose 2-chain $\mathcal{M}_0 := (\omega^m, X_1, X_2)$ such that $\mathcal{M}_0 \models \tau_0 \wedge \delta_0$. The structure $\mathcal{M}_0$ is a (partial) play, according to the strategy st.

Round α (for $\alpha > 0$) Assume that $\pi_{<\alpha} = G_0\tau_0 \dots G_\beta\tau_\beta \dots$ is the (partial) play up to round α and we have chosen $\delta_\beta \in Char_2^s$ at round $\beta < \alpha$.
Set $v_\alpha := \sum_{\beta\in\alpha} \delta_\beta$.
Play $G_\alpha := R(k, st_{v_\alpha})$.
Assume that Player II replies by $\tau_\alpha \in G_\alpha$ at this round.
Choose $\delta_\alpha \in Char_2^s$ to be consistent with $\tau_\alpha \wedge st_{v_\alpha}$.
Choose 2-chain $\mathcal{M}_\alpha := (\omega^m, X_1, X_2)$ such that $\mathcal{M}_\alpha \models \tau_\alpha \wedge \delta_\alpha \wedge st_{v_\alpha}$.

By the induction on α, using Lemma 5.6 and our choice of m, one can show that for every play $G_0\tau_0 \dots G_\beta\tau_\beta \dots$ which is consistent with the described strategy the following invariants hold:

1. $\delta_\alpha \wedge \tau_\alpha$ are satisfiable on ω^ω and therefore on ω^m.
2. $(\sum_{\beta\in\alpha} \delta_\beta) \wedge (\sum_{\beta\in\alpha} \tau_\beta)$ are satisfiable on ω^ω, and therefore on $\omega^m\alpha$.
3. $type_s(\sum_{\beta\in\alpha} \mathcal{M}_\beta) = \sum_{\beta\in\alpha} \delta_\beta = v_\alpha$
4. $\sum_{\beta\in\alpha} \mathcal{M}_\beta \models st$, i.e., the play $\sum_{\beta\in\alpha} \mathcal{M}_\beta$ is consistent with st.
5. $\sum_{\beta\in\alpha} \tau_\beta \in R(k, st)$.
6. $G_\alpha \in F_{st}^k(\sum_{\beta\in\alpha} \tau_\beta)$.

¿From (5)-(6) it follows that the described strategy is a winning strategy in $Game(F_{st}^k, R(k, st))$. ∎

6 Open Problems and Further Directions

The Büchi-Landweber theorem (Theorem 1.1) states that for the ω-games with MLO winning conditions, the player who has a winning strategy also has an MLO-definable winning strategy. In [RT07], we considered fragments of MLO logics. We proved that the Büchi-Landweber theorem fully extends to the first-order fragment of MLO (FOMLO) for ω-games; i.e., for every winning conditions $\varphi(X_1, X_2) \in$ FOMLO, the player who has a winning strategy in $\mathcal{G}_\varphi^\omega$, also has a FOMLO-definable winning strategy. We also proved that the theorem extends fully to the FOMLO extended by modular counting quantifiers.

In [RS08], we proved that for every ordinal $\alpha \geq \omega^\omega$ there is a FOMLO formula $\varphi_\alpha(X_1, X_2)$ such that Player I has a winning strategy in $\mathcal{G}_{\varphi_\alpha}^\alpha$; however, he has no MLO-definable winning strategy.

We plan to consider several fragments of MLO including FOMLO, FOMLO extended by the modular counting quantifiers and FOMLO extended by the quantifications over the finite sets (WMLO). For each of the above fragments $\mathcal{L}$ we address the problem of deciding for a formula $\varphi \in \mathcal{L}$ and an ordinal α, whether one of the player has $\mathcal{L}$-definable winning strategy in $\mathcal{G}_\varphi^\alpha$.

We reduced the synthesis problems to the satisfiability problem for MLO which has non-elementary complexity. We plan to analyze the complexity of the synthesis problems when winning conditions are described by automata which have the same expressive power as MLO or by temporal logic formulas which have the same expressive power as FOMLO. For the winning conditions expressed in these formalisms we hope to prove that the synthesis problems have a reasonable complexity.

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