Recent Hardness Results for Periodic Uni-processor Scheduling

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Abstract In the *synchronous periodic task model*, a set $\tau_1, \dots, \tau_n$ of tasks is given, each releasing jobs of running time c_i and relative deadline d_i at each integer multiple of the period p_i. It is a classical result that *Earliest Deadline First (EDF)* is an optimal preemptive uni-processor scheduling policy. For constrained deadlines, i.e. $d_i \leq p_i$, the EDF-schedule is feasible if and only if

$$\forall Q \geq 0 : \sum_{i=1}^{n} \left(\left\lfloor \frac{Q - d_i}{p_i} \right\rfloor + 1 \right) \cdot c_i \leq Q.$$

Though an enormous amount of literature deals with this topic, the complexity status of this test has remained unknown. We prove that testing EDF-schedulability of such a task system is (weakly) **coNP**-hard. This solves Problem 2 from the survey “Open Problems in Real-time Scheduling” by Baruah & Pruhs. The hardness result is achieved by applying recent results on inapproximability of Diophantine approximation.

1 Introduction

Nowadays more and more devices are controlled by embedded microprocessors, for example in power plants, car electronics, flight control systems, robotics and telecommunication systems, see Buttazzo [1] for an extensive introduction. Since many applications are safety critical, each task running on such a processor must produce the output not only correctly but also on time. Several tasks may run on the same processor and a *Real-time scheduling policy* decides which task should be active in which intervals, to guarantee that all deadlines are kept.

In the simple, but important *periodic task model* a set $\tau_1, \dots, \tau_n$ of tasks is given, where each τ_i is an infinite sequence of jobs, defined by an *execution time* $c_i \in \mathbb{Q}_+$, a *(relative) deadline* $d_i \in \mathbb{Q}_+$ and a *period* $p_i \in \mathbb{Q}_+$. We assume that the tasks are *synchronous*, i.e. there is a time, say 0, at which all tasks release a job simultaneously. In other words for each $i \in \{1, \dots, n\}$ and $z \in \mathbb{Z}_{\geq 0}$, a job of running time c_i and absolute deadline $z \cdot p_i + d_i$ is released at $z \cdot p_i$. Furthermore we assume *constrained-deadlines*, hence $d_i \leq p_i$ for each $i \in \{1, \dots, n\}$.

We consider *preemptive* uni-processor schedules, i.e. at any time a running job may be preempted and resumed later. As the name suggests, in the *Earliest Deadline First (EDF)* policy, at any time that job from the queue of released and not yet accomplished jobs is active, whose (absolute) deadline comes next. The EDF-schedule is provably optimal in this setting, meaning that if there is a schedule in which all jobs meet their deadlines, then the EDF-schedule is feasible as well (see Dertouzos [2]).

The main question of feasibility analysis however remains: Will each of the infinitely many jobs be finished in time? First observe, that

$$\left\lfloor \frac{Q - d_i}{p_i} \right\rfloor + 1$$

yields the number of jobs of τ_i that have both, their release time and deadline in the interval $[0, Q]$. Consequently the quantity

$$\mathtt{DBF}(\tau_i, Q) = \left(\left\lfloor \frac{Q - d_i}{p_i} \right\rfloor + 1 \right) \cdot c_i$$

gives the amount of running time that, regardless of the used scheduling policy, has to be spent on τ_i in this interval. More general, the *demand bound function*

$$\mathtt{DBF}(\mathcal{S}, Q) = \sum_{i=1}^{n} \left(\left\lfloor \frac{Q - d_i}{p_i} \right\rfloor + 1 \right) \cdot c_i$$

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gives the running time of *all* jobs, which have their release time and deadline in the interval $[0, Q]$. As a consequence, for feasibility it is necessary, that $\mathtt{DBF}(\mathcal{S}, Q) \leq Q$ for all $Q \geq 0$. Baruah et al. [3] showed that this condition is in fact sufficient, hence an *EDF-schedulability test* is a test which checks validity of the following formula

$$\forall Q \geq 0 : \sum_{i=1}^{n} \left(\left\lfloor \frac{Q - d_i}{p_i} \right\rfloor + 1 \right) \cdot c_i \leq Q,$$

see Figure 1 for an illustration.

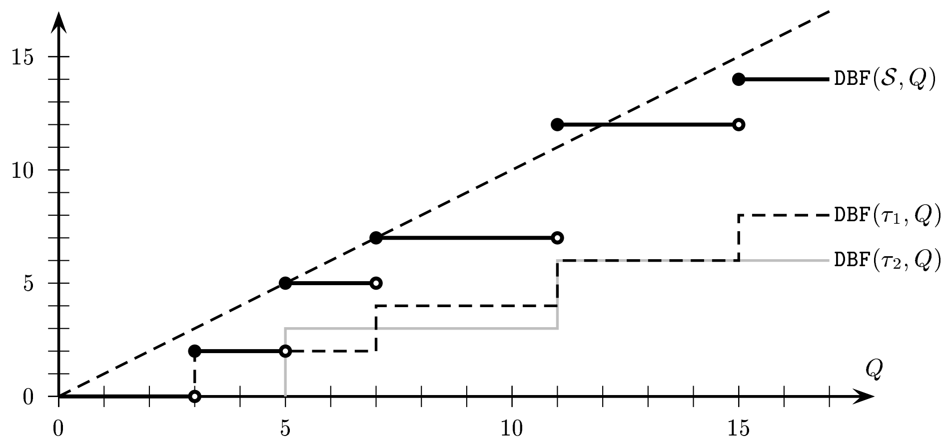

Figure 1. Constrained deadline task system $\mathcal{S} = \{\tau_1, \tau_2\}$ with $\tau_1 = (2, 3, 4)$, $\tau_2 = (3, 5, 6)$, using notation $\tau_i = (c_i, d_i, p_i)$. One has $\mathtt{DBF}(\mathcal{S}, Q) > Q$ for $Q = 11$, thus $\mathcal{S}$ is not EDF-schedulable.

Much effort has been spent on developing sufficient polynomial or exact pseudo-polynomial time tests for EDF-schedulability of periodic tasks, see [4,5,3,6,7]. But none of the algorithms suggested in these papers was able to decide EDF-schedulability on a unit speed processor correctly and in polynomial time for all instances. The question whether EDF-schedulability can be decided in polynomial time is stated as a major open problem in the survey of Baruah & Pruhs [8] on open problems in Real-time scheduling. We settle the complexity status of testing EDF-schedulability by proving the following theorem.

Theorem 1. *Given a set $\mathcal{S} = \{\tau_1, \ldots, \tau_n\}$ of synchronous, periodic, constrained-deadline tasks defined by rational numbers $0 \leq c_i \leq d_i \leq p_i$, it is (weakly)* **coNP***-hard to decide, whether $\mathcal{S}$ is EDF-schedulable, i.e. testing the condition*

$$\forall Q \geq 0 : \sum_{i=1}^{n} \left(\left\lfloor \frac{Q - d_i}{p_i} \right\rfloor + 1 \right) \cdot c_i \leq Q,$$

is (weakly) **coNP***-hard. This holds even if $d_i = p_i$ for $i = 1, \ldots, n-1$.*

This, together with the result in [3] implies the following corollary.

Corollary 1. *Given a set $\mathcal{S} = \{\tau_1, \ldots, \tau_n\}$ of sporadic tasks with worst-case execution time c_i, relative deadline d_i and minimum inter-arrival time p_i it is (weakly) **coNP**-hard to determine, whether the EDF-schedule of $\mathcal{S}$ is feasible.*

Related work

One approach to obtain algorithms to test EDF-feasibility lies in bounding the interval, in which the demand bound function has to be evaluated. Let $u = \sum_{i=1}^{n} \frac{c_i}{p_i}$ be the *utilization* of a task system. Given that $\mathcal{S}$ is not EDF-schedulable, the smallest $Q > 0$, certifying the infeasibility must have

$$Q < \frac{u}{1-u} \max_{i=1,\ldots,n} \{p_i - d_i\},$$

see e.g. [9,10]. This admits a pseudo-polynomial time algorithm for the feasibility test, if the utilization of $\mathcal{S}$ is bounded by $1-\varepsilon$ for some constant $\varepsilon > 0$.

Albers & Slomka [11] gave an FPTAS for approximating the speed of a processor, needed to make the EDF-schedule of $\mathcal{S}$ feasible. Their algorithm is also interpreted as follows. It either asserts that the tasks are feasible, or it asserts that the tasks are infeasible on a processor of speed $1-\varepsilon$. A similar result was also provided in the setting of fixed priority scheduling [12]. See [1] for more details on fixed priority scheduling policies and [6,4,7,13] for further approaches to feasibility analyzes of EDF-schedules. Recently, Bonifaci et al. [14] extended the result of Albers & Slomka to the case of multiprocessor scheduling with migration. The algorithm asserts that a set of tasks is feasible on m speed-$(2-1/m+\varepsilon)$ machines or infeasible on m speed-1 machines.

In a popular special case, the tasks have *implicit-deadlines*, i.e. $d_i = p_i$ for all i. In that case the condition $\texttt{DBF}(S,Q) \leq Q$ has only to evaluated at $Q = \mathrm{scm}(p_1,\ldots,p_n)$ and the set is EDF-schedulable if and only if the utilization is bounded by 1, see Liu & Layland [15]. In other words, the EDF-schedulability in this special case is decidable in polynomial time. If the tasks may be asynchronous, i.e. each task has on offset a_i, such that jobs are released at $z \cdot p_i + a_i$, then testing the feasibility is strongly **coNP**-hard [16]. This even holds if the utilization of the system is bounded from above by an arbitrarily small constant.

In the *sporadic* task model neither release times nor running times are predetermined. There, c_i denotes the *worst-case execution time* and p_i denotes the *minimum inter-arrival time*. But the worst-case is attained in a *synchronous arrival sequence*, that is when all tasks release jobs at time 0, all jobs fully use the worst-case execution time c_i and jobs arrive as early as permissible, see Baruah, Mok & Rosier [3]. In other words, the sporadic task system is EDF-schedulable if and only if this is true for the corresponding synchronous periodic task system.

2 Diophantine approximation

The EDF-schedulability test contains only one single unknown variable Q. This is unusual for **NP**/**coNP**-hard problems and helps us to narrow down the search for **NP**/**coNP**-hard remote relatives. The relative that we found helpful for problems in Real-time scheduling is *Diophantine approximation*, a problem in the field of algorithmic number theory (see e.g. [17]). Roughly speaking, there the objective is to replace a number or a vector, by another number or vector which is very close to the original, but less complex in terms of fractionality.

More precisely, a sequence $\alpha_1,\ldots,\alpha_n$ of rational numbers together with a bound $N \in \mathbb{N}$ and an error bound $\varepsilon \in \mathbb{Q}_+$ is given. One has to decide whether

$$\exists Q \in \{1,\ldots,N\} : \max_{i=1,\ldots,n} |\lfloor Q\alpha_i \rceil - Q\alpha_i| \leq \varepsilon, \tag{1}$$

where $\lfloor x \rceil$ is the integer closest to $x \in \mathbb{R}$. In a seminal work, Lagarias [18] has shown, that testing (1) is **NP**-hard. This was later extended by Rössner & Seifert [19] and Chen & Meng [20] to inapproximability results. In [21], the authors of this paper applied these results to show that response-time computation of tasks in a *Rate-monotonic schedule* is **NP**-hard, where tasks with smaller period always preempt that of larger period.

The EDF-schedulability test uses a rounding operation, where one replaces a rational by the closest integer which is equal or smaller, i.e, one *rounds down*. In Diophantine approximation, one rounds up or down to the nearest integer. The variant of Diophantine approximation, where one has to round up is called *directed Diophantine approximation (DDA)*. Recently the authors of this paper provided the following hardness result for directed Diophantine approximation.

Theorem 2 (Hardness of $\mathbf{DDA}_\rho$ [22]). *There exists a constant $c > 0$, such that the following Directed Diophantine Approximation problem (DDA_ρ) with gap parameter $\rho = \lfloor n^{c/\log\log n} \rfloor$ is* **NP***-hard: Given numbers $\alpha_1,\ldots,\alpha_n \in \mathbb{Q}$, a bound $N \in \mathbb{N}$ and an error bound $\varepsilon \in \mathbb{Q}_+$ as input, distinguish the following cases*

- YES : $\exists Q \in \{\lceil N/2 \rceil,\ldots,N\} : \max_{i=1,\ldots,n}(\lceil Q\alpha_i \rceil - Q\alpha_i) \leq \varepsilon$
- NO : $\nexists Q \in \{1,\ldots,\rho \cdot N\} : \max_{i=1,\ldots,n}(\lceil Q\alpha_i \rceil - Q\alpha_i) \leq 2^n \cdot \varepsilon$

Note that the union of the YES and NO cases does not represent all possible inputs. But there is a polynomial time reduction, taking the input of an **NP**-complete problem, say a SAT clause C, and yielding a DDA_ρ instance respecting the YES-case if C is satisfiable and the NO-case otherwise. See, e.g., [23,24] for more details on gap reductions.

Despite of some similarities between DDA_ρ and EDF-schedulability, we still observe crucial differences:

1. DDA_ρ contains a ceiling instead of a floor operation.
2. The number Q is restricted to be integer.
3. The approximation error is measured with $\|\cdot\|_\infty$-norm instead of $\|\cdot\|_1$-norm.
4. For DDA_ρ, one has a bound N on the number Q.

We can easily eliminate the first difference by observing that $\lceil Q\alpha_i \rceil - Q\alpha_i = Q \cdot (-\alpha_i) - \lfloor Q(-\alpha_i) \rfloor$. Consequently replacing the numbers by their negatives, we obtain a DDA_ρ problem with a floor operation. By adding a sufficiently large integer z and using $Q(\alpha_i + z) - \lfloor Q(\alpha_i + z) \rfloor = Q\alpha_i - \lfloor Q\alpha_i \rfloor$ for $Q \in \mathbb{N}$ we may then make the α_i's positive. We conclude that given $\alpha_1, \ldots, \alpha_n \in \mathbb{Q}_+$, $N \in \mathbb{N}$ and $\varepsilon \in \mathbb{Q}_+$, it is **NP**-hard to distinguish

- YES : $\exists Q \in \{\lceil N/2 \rceil, \ldots, N\} : \max_{i=1,\ldots,n}(Q\alpha_i - \lfloor Q\alpha_i \rfloor) \leq \varepsilon$
- NO : $\nexists Q \in \{1, \ldots, \rho \cdot N\} : \max_{i=1,\ldots,n}(Q\alpha_i - \lfloor Q\alpha_i \rfloor) \leq 2^n \cdot \varepsilon$

for $\rho = \lfloor n^{c/\log\log n} \rfloor$. In a next step, we introduce a variant of directed Diophantine approximation which incorporates differences (2) & (3). We use the notation $[\alpha, \beta]$ to denote the set of real numbers $[\alpha, \beta] = \{x \in \mathbb{R} : \alpha \leq x \leq \beta\}$.

Theorem 3 (Hardness of DDA^*_ρ). *There exists a constant $c > 0$, such that the following DDA^*_ρ problem with gap parameter $\rho = \lfloor n^{c/\log\log n} \rfloor$ is **NP**-hard: Given numbers $\alpha_1, \ldots, \alpha_n \in \mathbb{Q}_+$, weights $w_1, \ldots, w_n \in \mathbb{Q}_+$, a bound $N \in \mathbb{N}$ and an error bound $\varepsilon \in \mathbb{Q}_+$, distinguish*

- YES : $\exists Q \in [\lceil N/2 \rceil, N] : \sum_{i=0}^n w_i(Q\alpha_i - \lfloor Q\alpha_i \rfloor) \leq \varepsilon$
- NO : $\nexists Q \in [1, \rho \cdot N] : \sum_{i=0}^n w_i(Q\alpha_i - \lfloor Q\alpha_i \rfloor) \leq \rho \cdot \varepsilon$

Proof. We reduce DDA_ρ to DDA^*_ρ. Let $(\alpha_1, \ldots, \alpha_n; N; \varepsilon)$ be the given DDA_ρ instance (with rounding down and $\alpha_i > 0$ for all i). Since the α_i's are rational numbers, we can write them as $\alpha_i = \frac{a_i}{b_i}$ with pairwise co-prime integers $a_i, b_i \in \mathbb{N}$. Our DDA^*_ρ instance consists of the same numbers $\alpha_1, \ldots, \alpha_n$, equipped with unit weights $w_1 = \cdots = w_n = 1$. Furthermore we choose the same bound N, but a different error bound $\varepsilon' = n \cdot \varepsilon$ and we add one more number $\alpha_0 = 1$ with a very high weight of $w_0 = 2 \cdot \max\{a_i : i = 1, \ldots, n\} \cdot \varepsilon \cdot \rho \cdot n$. Intuitively the weight w_0 is large enough, such that any reasonable DDA^*_ρ solution Q of this instance must be an integer. It suffices to show the following implications:

- YES : $\exists Q \in \{\lceil N/2 \rceil, \ldots, N\} : \max_{i=1,\ldots,n}(Q\alpha_i - \lfloor Q\alpha_i \rfloor) \leq \varepsilon$
  $\Rightarrow \exists Q \in [\lceil N/2 \rceil, N] : \sum_{i=0}^n w_i(Q\alpha_i - \lfloor Q\alpha_i \rfloor) \leq \varepsilon'$
- NO : $\nexists Q \in \{1, \ldots, \rho \cdot N\} : \max_{i=1,\ldots,n}(Q\alpha_i - \lfloor Q\alpha_i \rfloor) \leq 2^n \cdot \varepsilon$
  $\Rightarrow \nexists Q \in [1, \rho \cdot N] : \sum_{i=0}^n w_i(Q\alpha_i - \lfloor Q\alpha_i \rfloor) \leq \rho \cdot \varepsilon'$

<u>YES-case:</u> Clearly YES instances for DDA_ρ are mapped to YES instances of DDA^*_ρ by simply using the same solution Q. This is the case since given a $Q \in \{\lceil N/2 \rceil, \ldots, N\}$ that matches the conditions of the YES case for DDA_ρ, one has

$$\sum_{i=0}^n w_i(Q\alpha_i - \lfloor Q\alpha_i \rfloor) = w_0 \cdot \underbrace{(Q - \lfloor Q \rfloor)}_{=0} + \sum_{i=1}^n 1 \cdot \underbrace{(Q\alpha_i - \lfloor Q\alpha_i \rfloor)}_{\leq \varepsilon} \leq n \cdot \varepsilon = \varepsilon'.$$

<u>NO-case:</u> Now suppose that we have a $Q \in [1, \rho \cdot N]$ with $\sum_{i=0}^n w_i(Q\alpha_i - \lfloor Q\alpha_i \rfloor) \leq \rho \cdot \varepsilon' = \rho \cdot n \cdot \varepsilon$. Decrease Q continuously until $Q\alpha_j \in \mathbb{Z}$ for at least one $j \in \{0, \ldots, n\}$. This can only decrease the approximation error since $\lfloor Q\alpha_i \rfloor$ remains invariant. Furthermore Q will never be decreased below 1 since $\alpha_0 = 1$. If Q is then an integer, we are done since

$$\max_{i=1,\ldots,n}(Q\alpha_i - \lfloor Q\alpha_i \rfloor) \leq \sum_{i=0}^n w_i(Q\alpha_i - \lfloor Q\alpha_i \rfloor) \leq \rho \cdot n \cdot \varepsilon \leq 2^n \varepsilon$$

for n large enough. Now suppose that Q is not integer. Then we may write $Q\alpha_j = Q\frac{a_j}{b_j} =: z \in \mathbb{Z}$, thus $Q = \frac{zb_j}{a_j} \in \mathbb{Z}\frac{1}{a_j}$. We write $Q = \frac{y}{a_j}$ where y is integer but not a multiple of a_j (since $Q \notin \mathbb{Z}$). Hence

$$Q - \lfloor Q \rfloor = \frac{y}{a_j} - \frac{\lfloor Q \rfloor a_j}{a_j} = \underbrace{(y - \lfloor Q \rfloor a_j)}_{\geq 1} \cdot \frac{1}{a_j} \geq \frac{1}{a_j}$$

where we use that $y - \lfloor Q \rfloor a_j$ is a non-negative integer but $y - \lfloor Q \rfloor a_j \neq 0$. We obtain

$$\sum_{i=0}^{n} w_i (Q\alpha_i - \lfloor Q\alpha_i \rfloor) \geq w_0 \cdot (Q - \lfloor Q \rfloor) \geq w_0 \cdot \frac{1}{a_j} > \rho \cdot n \cdot \varepsilon$$

by the choice of w_0. This contradiction yields that $Q \in \mathbb{N}$ and the claim follows.

3 Hardness of EDF-schedulability

In this section we will see that the **NP**-hard problem DDA_ρ^* is close enough to the EDF-schedulability condition to admit a direct reduction. To achieve this, YES (NO, resp.) instances for DDA_ρ^* are mapped to NO (YES, resp.) instances of EDF-schedulability. Intuitively this is done as follows: Suppose we are given a DDA_ρ^* instance $(\alpha_1, \ldots, \alpha_n; w_1, \ldots, w_n; N; \varepsilon)$. The first idea is to create implicit-deadline tasks $\tau_1, \ldots, \tau_n$ with $p_i = d_i = \frac{1}{\alpha_i}$. Then we have

$$\left\lfloor \frac{Q - d_i}{p_i} \right\rfloor + 1 = \lfloor Q\alpha_i \rfloor$$

hence a Q that maximizes $\texttt{DBF}(\mathcal{S}, Q)/Q$, minimizes the approximation error. On the other hand we need to forbid Q with $Q \gg N$. a common multiple of all p_i's. For this purpose we add a special task τ_0 which has a deadline of $N/2$ and a sufficiently large period (we may imagine $p_0 = \infty$). Then the quantity $\texttt{DBF}(\tau_0, Q)/Q$ contributes significantly to $\texttt{DBF}(\mathcal{S}, Q)/Q$ only if Q is of order N.

Theorem 4. *Given an instance of* DDA_ρ^* *consisting of rational numbers* $\alpha_1, \ldots, \alpha_n \in \mathbb{Q}_+$*, weights* $w_1, \ldots, w_n \in \mathbb{Q}_+$*, a bound* $N \in \mathbb{N}_{\geq 2}$ *and an error bound* $\varepsilon > 0$*, we can find in polynomial time a constrained-deadline task system* $\mathcal{S}$ *consisting of* $n+1$ *tasks such that*

- YES*:* $\exists Q \in [\lceil N/2 \rceil, N] : \sum_{i=1}^{n} w_i (Q\alpha_i - \lfloor Q\alpha_i \rfloor) \leq \varepsilon \Rightarrow \mathcal{S}$ *not EDF-schedulable*
- NO*:* $\nexists Q \in [\lceil N/2 \rceil, 3N] : \sum_{i=1}^{n} w_i (Q\alpha_i - \lfloor Q\alpha_i \rfloor) \leq 3\varepsilon \Rightarrow \mathcal{S}$ *EDF-schedulable*

Furthermore n *tasks in* $\mathcal{S}$ *have implicit-deadlines.*

Proof. A set of tasks is EDF-schedulable on a processor of speed $\beta > 0$ if and only if the tasks with running times scaled by $\frac{1}{\beta}$ are feasible on a unit speed processor. Thus we may assume to have an oracle for the test

$$\forall Q \geq 0 : \sum_{i=1}^{n} \left(\left\lfloor \frac{Q - d_i}{p_i} \right\rfloor + 1 \right) \cdot c_i \leq \beta \cdot Q$$

Let $N \in \mathbb{N}, \alpha_1, \ldots, \alpha_n, w_1, \ldots, w_n \in \mathbb{Q}_+, \varepsilon > 0$ be the DDA_ρ^* instance. We choose a constrained-deadline task system $\mathcal{S}$ consisting of $n+1$ tasks

$$\tau_i = (c_i, d_i, p_i) = \left(w_i, \frac{1}{\alpha_i}, \frac{1}{\alpha_i} \right) \quad \forall i = 1, \ldots, n$$
$$\tau_0 = (c_0, d_0, p_0) = (3\varepsilon, \lceil N/2 \rceil, 12N)$$

and processor speed

$$\beta = \frac{\varepsilon}{N} + \sum_{i=1}^{n} w_i \alpha_i = \frac{\varepsilon}{N} + u(\{\tau_1, \ldots, \tau_n\})$$

which just slightly exceeds the utilization.

Yes-case: Suppose that we have a $Q \in [\lceil N/2 \rceil, N]$ with $\sum_{i=1}^n w_i(Q\alpha_i - \lfloor Q\alpha_i \rfloor) \leq \varepsilon$. Then

$$\begin{aligned}
\mathtt{DBF}(\{\tau_0,\dots,\tau_n\},Q) &= \mathtt{DBF}(\tau_0,Q) + \sum_{i=1}^n \left(\left\lfloor \frac{Q-d_i}{p_i} \right\rfloor + 1 \right) c_i \\
&= 3\varepsilon + \sum_{i=1}^n \lfloor Q\alpha_i \rfloor w_i \\
&\overset{(*)}{\geq} 3\varepsilon + \left(\left(\sum_{i=1}^n Q\alpha_i w_i \right) - \varepsilon \right) \\
&= 2\varepsilon + Q \sum_{i=1}^n \alpha_i w_i \\
&\overset{(**)}{>} Q \cdot \underbrace{\left(\frac{\varepsilon}{N} + \sum_{i=1}^n \alpha_i w_i \right)}_{=\beta} \\
&= \beta Q
\end{aligned}$$

Here we use $\sum_{i=1}^n w_i(Q\alpha_i - \lfloor Q\alpha_i \rfloor) \leq \varepsilon$ in $(*)$ and $Q \leq N < 2N$ in $(**)$. Thus the task system $\mathcal{S}$ is not EDF-schedulable (on a processor of speed β).

No-case: Next we assume that $\mathcal{S}$ is not EDF-schedulable. Then there is a $Q > 0$ such that $\mathtt{DBF}(\{\tau_0,\dots,\tau_n\},Q) > \beta Q$. We need to show that $Q \in [\lceil N/2 \rceil, 3N]$ and $\sum_{i=1}^n w_i(Q\alpha_i - \lfloor Q\alpha_i \rfloor) \leq 3\varepsilon$.

Observe that using the definition of β and $\lfloor Q\alpha_i \rfloor \leq Q\alpha_i$, one has

$$\begin{aligned}
\mathtt{DBF}(\tau_0,Q) &= \mathtt{DBF}(\mathcal{S},Q) - \mathtt{DBF}(\{\tau_1,\dots,\tau_n\},Q) \\
&> \beta Q - \sum_{i=1}^n \lfloor Q\alpha_i \rfloor w_i \\
&\geq \beta Q - Q \sum_{i=1}^n \alpha_i w_i \\
&= \beta Q - Q \underbrace{\left(\frac{\varepsilon}{N} + \sum_{i=1}^n \alpha_i w_i \right)}_{=\beta} + Q\frac{\varepsilon}{N} \\
&= Q\frac{\varepsilon}{N}
\end{aligned}$$

Since τ_0 has its first deadline at $d_0 = \lceil N/2 \rceil$ and $\mathtt{DBF}(\tau_0,Q) > 0$ we must have $Q \geq \lceil N/2 \rceil$. Suppose for contradiction that already the second deadline of τ_0 occurred before Q, i.e. $Q \geq p_0 = 12N$. Then

$$\mathtt{DBF}(\tau_0,Q) \leq c_0 \cdot \left\lceil \frac{Q}{p_0} \right\rceil \leq 2 \cdot 3\varepsilon \cdot \frac{Q}{12N} < Q\frac{\varepsilon}{N},$$

leading to a contradiction. Hence, till time Q exactly one deadline of τ_0 has passed, thus $\mathtt{DBF}(\tau_0,Q) = 3\varepsilon$. But we already inferred the bound $\mathtt{DBF}(\tau_0,Q) > Q\frac{\varepsilon}{N}$, thus even $Q < 3N$. Finally

$$\sum_{i=1}^n w_i(Q\alpha_i - \lfloor Q\alpha_i \rfloor) = Q \underbrace{\sum_{i=1}^n \alpha_i w_i}_{<\beta} - (\mathtt{DBF}(\mathcal{S},Q) - \mathtt{DBF}(\tau_0,Q)) \leq \underbrace{Q\beta - \mathtt{DBF}(\mathcal{S},Q)}_{<0} + 3\varepsilon \leq 3\varepsilon$$

and the claim follows.

Theorem 1 follows by combining Theorem 3 and 4, with $\rho = 4$.

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