Left-linear Bounded TRSs are Inverse Recognizability Preserving

Irène Durand and Marc Sylvestre

LaBRI, Université Bordeaux 1
351 Cours de la libération, F-33405 Talence cedex, France

Abstract

Bounded rewriting for linear term rewriting systems has been defined in (I. Durand, G. Sénizergues, M. Sylvestre. Termination of linear bounded term rewriting systems. Proceedings of the 21st International Conference on Rewriting Techniques and Applications) as a restriction of the usual notion of rewriting. We extend here this notion to the whole class of left-linear term rewriting systems, and we show that bounded rewriting is effectively inverse-recognizability preserving. The *bounded class* (*BO*) is, by definition, the set of left-linear systems for which every derivation can be replaced by a bottom-up derivation. The class *BO* contains (strictly) several classes of systems which were already known to be inverse-recognizability preserving: the left-linear growing systems, and the inverse right-linear finite-path overlapping systems.

1998 ACM Subject Classification Primary: F.4.2, Secondary: F.3.2, F.4.1

Keywords and phrases Term Rewriting, Preservation of Recognizability, Rewriting Strategies

Digital Object Identifier 10.4230/LIPIcs.RTA.2011.361

Category Regular Research Paper

1 Introduction

General framework. A term rewriting system (TRS) $\mathcal{R}$ is effectively recognizability preserving (respectively inverse recognizability preserving) if for every recognizable set of terms T the set $[T](\to_{\mathcal{R}}^*) = \{s \mid \exists t \in T, t \to_{\mathcal{R}}^* s\}$ (resp. $(\to_{\mathcal{R}}^*)[T] = \{s \mid \exists t \in T, s \to_{\mathcal{R}}^* t\}$) is recognizable and can be built. Many efforts have been made for finding subclasses of TRSs which are (inverse) recognizability preserving. The identification of such subclasses is important and has applications in equational reasoning, formal computation, automated deduction, and verification. For example, the reachability problem which is central in these areas, particularly in verification, is decidable for these classes of TRSs. The techniques used to prove reachability are often based on the computation of $[E](\to_{\mathcal{R}}^*)$ for some set E, and are coming from the Knuth and Bendix completion algorithm (see [14] for the seminal paper). An entire workshop is devoted to the reachability problem: the Workshop on Reachability Problem (RP). Each result of recognizability preservation yields also almost directly a new decidable call-by-need class [4] and decidability results on confluence (see [1] or [7] for a survey) and joinability. This notion has also been used to prove termination of systems for which none of the already known termination techniques work [10]. Different techniques for proving termination have been implemented in several softwares (Matchbox [24], AProVE [11], TORPA [25], CiME [3]). Consequently, the seek of a class which preserves the recognizability is well motivated. Many such classes have been defined by imposing syntactical restrictions on the rewrite rules (e.g. growing TRSs [16, 12] and finite-path overlapping TRSs [20, 21]). Another way is to use a *strategy*, i.e. restrictions on the derivations rather than on the rules, to ensure preservation of recognizability. Various such strategies where studied in [8, 17, 19, 5, 6]. In this paper,

22nd International Conference on Rewriting Techniques and Applications (RTA'11).
Editor: M. Schmidt-Schauß; pp. 361–376
Leibniz International Proceedings in Informatics
Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

we extend the bounded rewriting for linear TRSs to left-linear TRSs that may have non right-linear rules and we prove that this strategy is inverse recognizability preserving.

From linear TRS*s to left-linear* TRS*s.* Bounded rewriting for linear TRSs is essentially a new version of bottom-up rewriting [5] that is easier to define and has better properties. The reader may refer to [6] for more details on bounded rewriting for linear TRSs. Intuitively, for a linear TRS $\mathcal{R}$, a derivation is k-bounded ($\mathsf{lbo}(k)$) if when a rule is applied, the parts of the substitution located at a depth greater than k are not rewritten further in the derivation, i.e. do not match the left-handside of a rule applied further. A linear TRS $\mathcal{R}$ is $\mathsf{lbo}(k)$ if for any derivation $s \to_{\mathcal{R}}^{*} t$ there exists a $\mathsf{lbo}(k)$ derivation $s \;_{k}\!\!\to_{\mathcal{R}}^{*} t$. The class of linear $\mathsf{lbo}(k)$ TRSs is denoted by $\mathsf{LBO}(k)$. One of the goals of this paper is to drop the right-linear restriction and propose an extension of bounded rewriting to left-linear TRSs. This extension cannot be the trivial one: even if nothing in the definition of $\mathsf{LBO}(k)$ TRSs requires the linear condition, keeping this definition unchanged would define a class containing only linear TRSs (see example 4.14).

To solve this problem, we introduce a binary symbol E and a set $\mathcal{E}$ of three rewrite rules to handle this symbol: the introduction rule $x \to E(x, x)$, and two selection rules $E(x, y) \to x$ and $E(x, y) \to y$. Intuitively, E allows to store several descendants of the same initial subterm. Let $\mathcal{R}$ be a left-linear TRS over a signature $\mathcal{F}$. Roughly speaking, a derivation in $\mathcal{R} \cup \mathcal{E}$ is k-bounded if when a rule is applied, the parts of the substitution located at a depth greater than k (without taking the E into consideration) are not rewritten further in the derivation, i.e. do not match the left-handside of a rule of $\mathcal{R}$ applied further. A derivation in $s \to_{\mathcal{R}}^{*} t$ is k-bounded convertible ($\mathsf{boc}(k)$) if there exists a $\mathsf{bo}(k)$-derivation from s to t in $\mathcal{R} \cup \mathcal{E}$. Note that this definition does not constrain the application of the rules of $\mathcal{E}$. A TRS is $\mathsf{bo}(k)$ if every derivation is $\mathsf{boc}(k)$.

The class of $\mathsf{bo}(k)$ TRSs is denoted by $\mathsf{BO}(k)$. Let us see how we use the symbol E. Suppose that $\mathsf{f}(\mathsf{a}) \to_{\mathsf{f}(x)\to\mathsf{g}(x,x)} \mathsf{g}(\mathsf{a}, \mathsf{a}) \to_{\mathsf{a}\to\mathsf{b}} \mathsf{g}(\mathsf{a}, \mathsf{b})$. The symbol E is used to apply the rule $\mathsf{a} \to \mathsf{b}$ before the rule $\mathsf{f}(x) \to \mathsf{g}(x, x)$. First, we use E to create an envelop which contains a and b: $\mathsf{f}(\mathsf{a}) \to_{x\to E(x,x)} \mathsf{f}(E(\mathsf{a}, \mathsf{a})) \to_{\mathsf{a}\to\mathsf{b}} \mathsf{f}(E(\mathsf{a}, \mathsf{b}))$. Then we can apply the rule $\mathsf{f}(x) \to \mathsf{g}(x, x)$, and use the selections rules to obtain $\mathsf{g}(\mathsf{a}, \mathsf{b})$: $\mathsf{f}(E(\mathsf{a}, \mathsf{b})) \to \mathsf{g}(E(\mathsf{a}, \mathsf{b}), E(\mathsf{a}, \mathsf{b})) \to_{E(x,y)\to x} \mathsf{g}(\mathsf{a}, E(\mathsf{a}, \mathsf{b})) \to_{E(x,y)\to y} \mathsf{g}(\mathsf{a}, \mathsf{b})$. The introduction of the symbol E can be viewed as a counterpart of the construction of the powerset automaton in the extension of Jacquemard's saturation method [12] by Nagaya and Toyama [16] (this saturation method is used to prove that left-linear growing TRSs are inverse recognizability preserving).

Inverse recognizability preservation. In section 5, we prove that bounded rewriting for left-linear TRSs is effectively inverse recognizability preserving. This result is obtained by simulating $\mathsf{bo}(k)$-derivations by a ground tree transducer. The idea of simulating $\mathsf{bo}(k)$-derivations is similar to the idea developed in [5] where bottom-up(k) derivations are simulated using a ground TRS. This simulation yields directly to the inverse preservation result since GTTs are effectively inverse recognizability preserving.

Strongly bounded systems. In section 6, we introduce a subclass of $\mathsf{BO}(k)$ called the strongly bounded class ($\mathsf{SBO}(k)$). The membership problem for $\mathsf{SBO}(k)$ is decidable whereas the membership problem for $\mathsf{BO}(0)$ is undecidable. The class of strongly bounded TRSs contains inverse right-linear finite-path overlapping TRSs [22] and left-linear growing TRSs [16]. Note that a long version of this paper is available at: `http://hal.archives-ouvertes.fr/hal-00580528/fr/`.

2 Preliminaries

Given a set E, we denote by $\mathcal{P}(E)$ its powerset i.e. the set of all its subsets. Its cardinality is denoted by $\mathsf{Card}(E)$. A finite *word* over an alphabet A is a map $u : [0, \ell-1] \to A$, for some $\ell \in \mathbb{N}$. The integer ℓ is the *length* of the word u and is denoted by $|u|$. The set of words over A is denoted by A^* and endowed with the usual *concatenation* operation $u, v \in A^* \mapsto u \cdot v \in A^*$. The *empty* word is denoted by ε.

Assume that the set A is ordered. We denote by $\preceq_{\mathsf{Lex}_A}$ the lexicographic order on the set of words A^*. We may omit Lex_A when it is clear from the context. We assume the reader familiar with terms and automata (see e.g. [2] or [23] for an introduction). We call *signature* a set $\mathcal{F}$ of symbols with arity $ar : \mathcal{F} \to \mathbb{N}$. The subset of symbols with arity $m \in \mathbb{N}$ is denoted by $\mathcal{F}_m$.

As usual, a finite set $P \subseteq \mathbb{N}^*$ is called a *tree-domain* (or, *domain*, for short) iff for every $u \in \mathbb{N}^*, i \in \mathbb{N}$ $(u\cdot i \in P \Rightarrow u \in P)$ & $(u\cdot(i+1) \in P \Rightarrow u\cdot i \in P)$. We call $P' \subseteq P$ a *subdomain* of P iff, P' is a domain and, for every $u \in P, i \in \mathbb{N}$ $(u\cdot i \in P'$ & $u\cdot(i+1) \in P) \Rightarrow u\cdot(i+1) \in P'$.

A (first-order) *term* on a signature $\mathcal{F}$ is a partial map $t : \mathbb{N}^* \to \mathcal{F}$ whose domain is a non-empty tree-domain and which respects the arity assignment. We denote by $\mathcal{T}(\mathcal{F}, \mathcal{V})$ the set of first-order terms over the signature $\mathcal{F} \cup \mathcal{V}$, where $\mathcal{F}$ is a signature and $\mathcal{V}$ is a denumerable set of variables of arity 0.

The domain of t is also called its set of *positions* and denoted by $\mathcal{P}\mathsf{os}(t)$. The set of variables of t is denoted by $\mathcal{V}\mathsf{ar}(t)$. A variable x is said to occur at depth n in t if there exists a position $u \in \mathcal{P}\mathsf{os}(t)$ such that $t(u) = x$ and $|u| = n$. The root symbol of t is denoted by $\mathsf{root}(t)$. Given a set of symbols and variables $A \subseteq \mathcal{F} \cup \mathcal{V}$ and a term t, the set of positions $u \in \mathcal{P}\mathsf{os}(t)$ such that $t(u) \in A$ is denoted by $\mathcal{P}\mathsf{os}_A(t)$ and the set of position $u \in \mathcal{P}\mathsf{os}(t)$ such that $t(u) \notin A$ is denoted by $\mathcal{P}\mathsf{os}_{\setminus A}(t)$. Let X be either A or $\setminus A$ and $u \in \mathcal{P}\mathsf{os}_X(t)$. We denote by $\mathcal{P}\mathsf{os}_X{}^{\preceq u}(t)$ (respectively $\mathcal{P}\mathsf{os}_X{}^{\prec u}(t)$) the set of positions $v \in \mathcal{P}\mathsf{os}_X(t)$ such that $v \preceq u$ (resp. $v \prec u$) and by $\mathcal{P}\mathsf{os}_X{}^{\succeq u}(t)$ (respectively $\mathcal{P}\mathsf{os}_X{}^{\succ u}(t)$) the set of positions $v \in \mathcal{P}\mathsf{os}_X(t)$ such that $v \succeq u$ (resp. $v \succ u$). When $A = \{f\}$ for some $f \in \mathcal{F} \cup \mathcal{V}$ we may denote $\mathcal{P}\mathsf{os}_f(t)$ (respectively $\mathcal{P}\mathsf{os}_{\setminus f}(t)$) instead of $\mathcal{P}\mathsf{os}_{\{f\}}(t)$ (resp. $\mathcal{P}\mathsf{os}_{\setminus\{f\}}(t)$). A *substitution* σ is a mapping from $\mathcal{V}$ into $\mathcal{T}(\mathcal{F}, \mathcal{V})$. The substitution σ is naturally extended to a morphism $\sigma : \mathcal{T}(\mathcal{F}, \mathcal{V}) \to \mathcal{T}(\mathcal{F}, \mathcal{V})$, where $\sigma(f(t_1, .., t_n)) = f(\sigma(t_1), \ldots, \sigma(t_n))$, for each $f \in \mathcal{F}_n$, $t_i \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. Substitutions will often be used in postfix notation: $t\sigma$ is the result of applying σ to the term t. The *depth* of a term t is defined by $dpt(t) := \sup\{\mathsf{Card}(\mathcal{P}\mathsf{os}^{\preceq u}(t)) \mid u \in \mathcal{P}\mathsf{os}_{\setminus\mathcal{V}}(t)\}$. This definition is extended to substitutions $dpt(\sigma) := \max\{dpt(x\sigma) \mid x \in \mathcal{V}\}$. For a term t and a symbol $f \in \mathcal{F}$, we define $dpt_{\setminus f}(t)$ by: $dpt_{\setminus f}(t) := \sup\{\mathsf{Card}(\mathcal{P}\mathsf{os}_{\setminus f}{}^{\preceq u}(t)) \mid u \in \mathcal{P}\mathsf{os}_{\setminus\{f\}\cup\mathcal{V}}(t)\}$. This definition is extended to substitutions $dpt_{\setminus f}(\sigma) := \max\{dpt_{\setminus f}(x\sigma) \mid x \in \mathcal{V}\}$. The set of *leaves* of t is the set $\mathcal{P}\mathsf{os}_{\mathcal{V}\cup\mathcal{F}_0}(t)$ and is also denoted by $\mathcal{L}\mathsf{v}(t)$. For a variable $x \in \mathcal{V}\mathsf{ar}(t)$, the set of positions $\mathcal{P}\mathsf{os}_x(t)$ is also denoted by $\mathcal{P}\mathsf{os}(t, x)$. Let $w \in \mathcal{L}\mathsf{v}(t)$. The *branch* containing w is the set of positions u such that $u \preceq w$.

Given a term t and $u \in \mathcal{P}\mathsf{os}(t)$ the *subterm of t at u* is denoted by t/u and defined by $\mathcal{P}\mathsf{os}(t/u) = \{w \mid uw \in \mathcal{P}\mathsf{os}(t)\}$ and $\forall w \in \mathcal{P}\mathsf{os}(t/u)$, $t/u(w) = t(uw)$.

A term that does not contain twice the same variable is called *linear*. Given a linear term $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, $x \in \mathcal{V}\mathsf{ar}(t)$, we denote by $\mathsf{pos}(t, x)$ the position of x in t.

A term containing no variable is called *ground*. The set of ground terms is abbreviated to $\mathcal{T}(\mathcal{F})$ or $\mathcal{T}$ whenever $\mathcal{F}$ is understood.

We denote by $C[t_1, \ldots, t_n]_{u_1,\ldots,u_n}$ the term obtained from $C[]_{u_1,\ldots,u_n}$ by replacing, for every $i \in \{1, \ldots, n\}$, the symbol $\Box$ at position u_i by the term t_i. Let t be a term, and $\{u_1, \ldots, u_n\} \subset \mathcal{P}\mathsf{os}(t)$ be a set of incomparable positions given in lexicographic order. We denote by $t[]_{u_1,\ldots,u_m}$ the context obtained from t by replacing each subterm t/u_i at a position u_i by a leaf labeled by $\Box$.

A *rewrite rule* over the signature $\mathcal{F}$ is a pair $l \to r$ of terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$.

We call l (resp. r) the *left-handside* (resp. *right-handside*) of the rule (*lhs* and *rhs* for short). A rule is *linear* if both its left and right-handsides are linear. A rule is *left-linear* if its left-handside is linear.

A *term rewriting system* (TRS for short) is a pair $(\mathcal{F}, \mathcal{R})$ where $\mathcal{F}$ is a signature and $\mathcal{R}$ a finite set of rewrite rules over the signature $\mathcal{F}$. When $\mathcal{F}$ is clear from the context or contains exactly the symbols of $\mathcal{R}$, we may omit $\mathcal{F}$ and write simply $\mathcal{R}$.

We denote by $\mathsf{LHS}(\mathcal{R})$ the set of lhs of $\mathcal{R}$, and by $\mathsf{RHS}(\mathcal{R})$ the set of rhs of $\mathcal{R}$.

Rewriting is defined as usual: for every $s, t \in \mathcal{T}(\mathcal{F})$, $s \to_{\mathcal{R}} t$ means that there exist a position $v \in \mathcal{P}\mathsf{os}(s)$, a rule $l \to r \in \mathcal{R}$, and a substitution σ such that $s = s[l\sigma]_v$ and $t = s[r\sigma]_v$.

We denote by $\to_{\mathcal{R}}^{+}$ the transitive closure of $\to$, by $\to_{\mathcal{R}}^{0,1}$ its reflexive closure, and by $\to_{\mathcal{R}}^{*}$ its reflexive and transitive closure. We may omit $\mathcal{R}$ when it is clear from the context. We say that there exists a derivation from s to t in $\mathcal{R}$ when $s \to_{\mathcal{R}}^{*} t$. The *length* of a derivation is the number of steps in this derivation. An n-step derivation from s to t is denoted by $s \to^{n} t$. More generally, the notation defined in [13] will be used in proofs.

A TRS is *linear* (resp. *left-linear*) if each of its rules is linear (resp. left-linear). A TRS $\mathcal{R}$ is *growing* [12] if every variable of a right-handside occurs at depth at most 1 in the corresponding left-handside.

We shall consider finite bottom-up term (tree) automata [2] (which we abbreviate to *f.t.a.*). An automaton $\mathcal{A}$ is given by a 4-tuple $(\mathcal{F}, \mathcal{Q}, \mathcal{Q}_f, \Gamma)$ where $\mathcal{F}$ is a signature, $\mathcal{Q}$ is a finite set of symbols of arity 0, called the set of states and such that $\mathcal{Q} \cap \mathcal{F}_0 = \emptyset$, $\mathcal{Q}_f \subseteq \mathcal{Q}$ is the set of final states, Γ is the set of transitions. A transition has either the form $q \to r$ for some $q, r \in \mathcal{Q}$, or $f(q_1, \ldots, q_m) \to q$ for some $m \geq 0, f \in \mathcal{F}_m, q_1, \ldots, q_m \in \mathcal{Q}$. Note that we can have rules of the form $c \to q$ with $c \in \mathcal{F}_0$, and $q \in \mathcal{Q}$. We shall also consider automaton on a denumerable signature. Such an automaton is given by a 4-tuple $(\mathcal{F}, \mathcal{Q}, \mathcal{Q}_f, \Gamma)$ where $\mathcal{F}$ is a denumerable signature, $\mathcal{Q}$ is a finite set of symbols of arity 0, $\mathcal{Q}_f \subseteq \mathcal{Q}$ is the set of final states, and Γ is an denumerable set of transitions.

The set of rules Γ can be viewed as a TRS over the signature $\mathcal{F} \cup \mathcal{Q}$. We then denote by $\to_{\mathcal{A}}$ the one-step rewriting relation generated by Γ. Given an automaton $\mathcal{A}$, the set of terms accepted by $\mathcal{A}$ is defined by: $\mathcal{L}(\mathcal{A}) := \{t \in \mathcal{T}(\mathcal{F}) \mid \exists q \in \mathcal{Q}_f, t \to_{\mathcal{A}}^{*} q\}$. A set of terms T is *recognizable* if there exists a term automaton $\mathcal{A}$ such that $T = \mathcal{L}(\mathcal{A})$. The automaton $\mathcal{A}$ is called *deterministic* if there is no rule of the form $q \to r$ for some $q, r \in \mathcal{Q}$ and if for every $t \in \mathcal{T}(\mathcal{F} \cup \mathcal{Q}), q, q' \in \mathcal{Q}$, $(t \to q \in \Gamma \;\&\; t \to q' \in \Gamma) \Rightarrow (q = q')$. The automaton $\mathcal{A}$ is called *complete* if for every $m \geq 0$, $f \in \mathcal{F}_m$ and m-tuple of states $(q_1, \ldots, q_m) \in \mathcal{Q}^m$, there exists $q \in \mathcal{Q}$ such that $f(q_1, \ldots, q_m) \to q \in \Gamma$.

Ground tree transducers have been introduced in [15]. A *ground tree transducer* (GTT) is a pair $V := (\mathcal{A}_1, \mathcal{A}_2)$ of f.t.a. automata over a signature $\mathcal{F}$. Let $\mathcal{A}_1 = (\mathcal{F}, \mathcal{Q}_1, \emptyset, \Gamma_1)$, $\mathcal{A}_2 = (\mathcal{F}, \mathcal{Q}_2, \emptyset, \Gamma_2)$. The relation recognized by V is the set $\mathcal{L}(V) := \{(t, t') \mid t, t' \in \mathcal{T}(\mathcal{F}), \exists s \in \mathcal{T}(\mathcal{F} \cup (\mathcal{Q}_1 \cap \mathcal{Q}_2)),\ t \to_{\mathcal{A}_1}^{*} s, t' \to_{\mathcal{A}_2}^{*} s\}$. A set $T \subseteq \mathcal{T}(\mathcal{F}) \times \mathcal{T}(\mathcal{F})$ is said to be *recognizable* by a GTT if there exists a GTT V such that $T = \mathcal{L}(V)$. The reflexive and transitive closure of the relation $\mathcal{L}(V)$ is recognizable by a GTT (see e.g. chapter 3.2 of [2]).

A *ground recognizable* TRS (GRS) $(\mathcal{F}, \mathcal{G})$ is a (possibly infinite) TRS of the form $\mathcal{G} = \{l \to r \mid i \in I, l \in R_i, r \in K_i\}$, where I is a finite set, R_i and K_i for all $i \in I$ are recognizable sets of terms over $\mathcal{F}$. One can easily check that the relation $\to_{\mathcal{G}}^{*}$ is recognizable by a GTT.

Given a TRS $\mathcal{R}$ and a set of terms T, we define $(\to_{\mathcal{R}}^{*})[T] := \{s \in \mathcal{T}(\mathcal{F}) \mid \exists t \in T, s \to_{\mathcal{R}}^{*} t\}$

and $[T](\to^*_{\mathcal{R}}) := \{s \in \mathcal{T}(\mathcal{F}) \mid \exists t \in T, t \to^*_{\mathcal{R}} s\}$. A TRS $\mathcal{R}$ is *effectively recognizability preserving* if for every recognizable T, $[T](\to^*_{\mathcal{R}})$ is recognizable and can be built. A TRS $\mathcal{R}$ is *effectively inverse recognizability preserving* if for every recognizable T, $(\to^*_{\mathcal{R}})[T]$ is recognizable and can be built.

We shall illustrate many of our definitions with the following left-linear TRS $(\mathcal{F}_1, \mathcal{R}_1)$ and the following complete deterministic automaton $\mathcal{A}_1$.

▶ **Example 2.1.** $\mathcal{F}_1 = \{\mathsf{a}, \mathsf{b}, \mathsf{f}(), \mathsf{h}(), \mathsf{g}(,), \mathsf{i}(,)\}$ is a signature, $\{x, y\}$ is a set of variables, $\mathcal{R}_1 = \{\mathsf{a} \to \mathsf{b}, \mathsf{f}(x) \to \mathsf{g}(x, x), \mathsf{h}(\mathsf{b}) \to \mathsf{b}, \mathsf{g}(\mathsf{h}(x), y) \to \mathsf{i}(x, y)\}$ is a set of rules, $\mathcal{A}_1 = (\mathcal{F}, \mathcal{Q}_{\mathcal{A}_1}, \{\mathsf{q}_f\}, \Gamma_{\mathcal{A}_1})$ with $\mathcal{Q}_{\mathcal{A}_1} = \{\mathsf{q}_f, \mathsf{q}_\mathsf{a}, \mathsf{q}_\mathsf{b}\, \mathsf{q}_\perp\}$, $\Gamma_{\mathcal{A}_1} = \{\mathsf{a} \to \mathsf{q}_\mathsf{a}, \mathsf{b} \to \mathsf{q}_\mathsf{b}, \mathsf{h}(\mathsf{q}_\mathsf{a}) \to \mathsf{q}_\mathsf{a}, \mathsf{h}(\mathsf{q}_\mathsf{b}) \to \mathsf{q}_\mathsf{b}, \mathsf{h}(\mathsf{q}_\perp) \to \mathsf{q}_\perp, \mathsf{i}(\mathsf{q}_\mathsf{a}, \mathsf{q}_\mathsf{b}) \to \mathsf{q}_f\} \cup \{\mathsf{f}(\mathsf{q}) \to \mathsf{q}_\perp \mid \mathsf{q} \in \mathcal{Q}_{\mathcal{A}_1}\} \cup \{\mathsf{g}(\mathsf{q}, \mathsf{q}') \to \mathsf{q}_\perp \mid \mathsf{q}, \mathsf{q}' \in \mathcal{Q}_{\mathcal{A}_1}\}$ $\cup \{\mathsf{i}(\mathsf{q}, \mathsf{q}') \to \mathsf{q}_\perp \mid \mathsf{q}, \mathsf{q}' \in \mathcal{Q}_{\mathcal{A}_1}, (\mathsf{q}, \mathsf{q}') \neq (\mathsf{q}_\mathsf{a}, \mathsf{q}_\mathsf{b})\}$ is a complete deterministic automaton. We have:
$\mathcal{L}(\mathcal{A}_1) = \{\mathsf{i}(t_1, t_2) \mid t_1 \in \{\mathsf{a}, \mathsf{h}(\mathsf{a}), \ldots, \mathsf{h}(\mathsf{h}(\ldots(\mathsf{a}))), \ldots\}, t_2 \in \{\mathsf{b}, \mathsf{h}(\mathsf{b}), \ldots, \mathsf{h}(\mathsf{h}(\ldots(\mathsf{b}))), \ldots\}\}$
and
$(\to^*_{\mathcal{R}_1})(\mathcal{L}(\mathcal{A}_1)) = \{\mathsf{i}(t_1, t_2), \mathsf{g}(\mathsf{h}(t_1), t_2), \mathsf{f}(\mathsf{h}(t_1)) \mid t_1 \in \{\mathsf{a}, \mathsf{h}(\mathsf{a}), \ldots, \mathsf{h}(\mathsf{h}(\ldots(\mathsf{a}))), \ldots\},$
$t_2 \in \{\mathsf{a}, \mathsf{h}(\mathsf{a}), \ldots, \mathsf{h}(\mathsf{h}(\ldots(\mathsf{a}))), \ldots\} \cup \{\mathsf{b}, \mathsf{h}(\mathsf{b}), \ldots, \mathsf{h}(\mathsf{h}(\ldots(\mathsf{b}))), \ldots\}\}$.

3 The TRS $\mathcal{E}$

From now on, until the end of this paper, we denote by $\mathcal{F}$ a finite signature, and by $\mathcal{R}$ a left-linear TRS over $\mathcal{F}$. Let $E \notin \mathcal{F}$ be a fresh symbol of arity 2.

▶ **Definition 3.1.** Let $x, y \in \mathcal{V}$. The left-linear TRS $\mathcal{E}$ is the TRS over $\mathcal{F} \cup \{E\}$ with the rules

$$x \to E(x, x) \quad (1) \qquad E(x, y) \to x \text{ and } E(x, y) \to y \quad (2)$$

Rule (1) is the *introduction rule*. Rules (2) are the *selection rules*. Note that the TRS $\mathcal{R} \cup \mathcal{E}$ is left-linear. This TRS will be used to define bounded rewriting (section 4) and has the following property.

▶ **Proposition 3.2.** *Let $s, t \in \mathcal{T}(\mathcal{F})$. We have $s \to^*_{\mathcal{R}} t$ iff $s \to^*_{\mathcal{R} \cup \mathcal{E}} t$.*

4 Bounded Rewriting

Roughly speaking, a derivation in $\mathcal{R} \cup \mathcal{E}$ is k-bounded ($\mathsf{bo}(k)$) if when a rule is applied, the parts of the substitution located at a depth greater than k (without taking the E into consideration) do not match a left-handside of a rule of $\mathcal{R}$ applied further. To indicate which positions are allowed to be rewritten further, we are going to apply a marking process. A mark is an integer. A marked term $\overline{t}$ is just a term t where all the symbols are marked. To every derivation $s_0 \to_{\mathcal{R} \cup \mathcal{E}} \ldots \to_{\mathcal{R} \cup \mathcal{E}} s_n$, we associate a marked derivation $\overline{s_0} \circ\!\!\to_{\mathcal{R} \cup \mathcal{E}} \ldots \circ\!\!\to_{\mathcal{R} \cup \mathcal{E}} \overline{s_n}$ (i.e. a derivation where all terms are marked terms). This derivation starts on the term $\overline{s_0}$ which is obtained from s_0 by setting all the marks to 0. Now, if we consider a marked term $\overline{s_j}$ in this derivation, a mark i on a symbol f in s_j indicates that the maximal depth (again, without taking the E into consideration) at which the symbol appears in a substitution during the derivation $s_0 \to^*_{\mathcal{R} \cup \mathcal{E}} s_j$ is i. The derivation will be said $\mathsf{bo}(k)$ if the maximal mark that appears on a lhs in the marked derivation $\overline{s_0} \circ\!\!\to_{\mathcal{R} \cup \mathcal{E}} \ldots \circ\!\!\to_{\mathcal{R} \cup \mathcal{E}} \overline{s_n}$ is $\leq k$. Formal definitions are given in the next sections.

4.1 Marked Terms

We define the signature of marked symbols: $\mathcal{F}^{\mathbb{N}} := \{f^i \mid i \in \mathbb{N}, f \in \mathcal{F}\}$. The operation $\mathsf{m}()$ returns the mark of a marked symbol: for $f \in \mathcal{F}, i \in \mathbb{N}, \mathsf{m}(f^i) = i$. We extend this operation to the symbol E: $\mathsf{m}(E) = 0$, and to variables: $\forall x \in \mathcal{V}, \mathsf{m}(x) = 0$. We define $\mathcal{F}^{\leq k}$ by $\mathcal{F}^{\leq k} := \{f^i \mid i \in \{0, \ldots, k\}, f \in \mathcal{F}\}$ and by $\mathcal{F}^{\geq k}$ the signature $\mathcal{F}^{\geq k} := \{f^i \mid i \geq k, f \in \mathcal{F}\}$. Marked terms are elements of $\mathcal{T}_M(\mathcal{V}) := \mathcal{T}(\mathcal{F}^{\mathbb{N}} \cup \{E\}, \mathcal{V})$. The set of ground marked terms is denoted by $\mathcal{T}_M$. The operation m extends to marked terms: if $t \in \mathcal{V}$, $\mathsf{m}(t) = 0$, otherwise, $\mathsf{m}(t) = \mathsf{m}(\mathsf{root}(t))$. We use $\mathsf{mmax}(t)$ to denote the maximal mark on t. We denote by t^i the term obtained by setting all the marks in t at a position $u \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(t)$ to i. We extend this notation to sets of terms ($S^i := \{s^i | s \in S\}$), and to substitutions ($\sigma^i : x \to (x\sigma)^i$). For every $f \in \mathcal{F}$, we identify f^0 and f; it follows that $\mathcal{T}(\mathcal{F}) \subseteq \mathcal{T}(\mathcal{F}^{\mathbb{N}})$, and $\mathcal{T}(\mathcal{F} \cup \{E\}) \subseteq \mathcal{T}_M$. We usually denote by $\bar{t}$ (or $\widehat{t}$) a marked term such that $\bar{t}^0 = t$ (where $t \in \mathcal{T}(\mathcal{F}^{\mathbb{N}} \cup \{E\}, \mathcal{V})$). The same rule will apply to substitutions and contexts. For a set of terms $T \subseteq \mathcal{T}(\mathcal{F}, \mathcal{V})$, we denote by $T^{\mathbb{N}}$ the set of terms $\{\bar{t} \in \mathcal{T}_M \mid t \in T\}$.

▶ **Example 4.1.** $\mathsf{m}(\mathsf{f}^3(E(\mathsf{a}^4, \mathsf{b}^1))) = 3, \mathsf{m}(x) = 0, \mathsf{m}(E(\mathsf{a}^1, \mathsf{b}^2)) = 0, \mathsf{mmax}(\mathsf{f}^3(E(\mathsf{a}^4, \mathsf{b}^1))) = 4, \mathsf{mmax}(E(\mathsf{a}^1, \mathsf{b}^2)) = 2$, and if $\bar{t} = \mathsf{g}^3(\mathsf{a}^0, E(x, \mathsf{b}^2))$, then $\bar{t}^1 = \mathsf{g}^1(\mathsf{a}^1, E(x, \mathsf{b}^1))$.

From now on and until the end of section 5, let us fix, a language $T \subseteq \mathcal{T}(\mathcal{F})$ recognized by a complete deterministic automaton, $\mathcal{A} = (\mathcal{F}, \mathcal{Q}_{\mathcal{A}}, \mathcal{Q}_{f,\mathcal{A}}, \Gamma_{\mathcal{A}})$.

We start giving some technical definitions and lemmas.

The Automaton $\mathcal{A}_{\mathcal{P}}$

▶ **Definition 4.2.** We denote by $\overline{\mathcal{A}}$ the (infinite) automaton $\overline{\mathcal{A}} := (\mathcal{F}^{\mathbb{N}}, \mathcal{Q}_{\mathcal{A}}, \mathcal{Q}_{f,A}, \Gamma_{\overline{A}})$, with: $\Gamma_{\overline{A}} = \{f^i(q_1, \ldots, q_n) \to q \mid i \in \mathbb{N}, (f(q_1, \ldots, q_n) \to q) \in \Gamma_{\mathcal{A}}\}$.

Note that $\overline{\mathcal{A}}$ is deterministic and complete over $\mathcal{F}^{\mathbb{N}}$, and contains all the rules $c^i \to q$ for $i \in \mathbb{N}$, $c \in \mathcal{F}_0$, $(c \to q) \in \Gamma_{\mathcal{A}}$.

▶ **Lemma 4.3.** *Let* $\bar{t}, \widehat{t} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}}), q \in \mathcal{Q}_{\mathcal{A}}, m > 0$. *If* $\bar{t} \to^*_{\overline{\mathcal{A}}} q$ *then* $\widehat{t} \to^*_{\overline{\mathcal{A}}} q$.

▶ **Definition 4.4.** We define the (infinite) automaton $\mathcal{A}_{\mathcal{P}} := \{\mathcal{F}^{\mathbb{N}} \cup \{E\}, \mathcal{Q}_{\mathcal{P}}, \mathcal{Q}_{f,\mathcal{P}}, \Gamma_{\mathcal{P}}\}$ built from $\overline{\mathcal{A}}$, where $\mathcal{Q}_{\mathcal{P}} = \mathcal{P}(\mathcal{Q}_{\mathcal{A}})$, $\mathcal{Q}_{f,\mathcal{P}} = \{\{q\} \mid q \in \mathcal{Q}_{f,\mathcal{A}}\}$, $\Gamma_{\mathcal{P}} = \{E(S_1, S_2) \to S_1 \cup S_2 \mid S_1, S_2 \in \mathcal{Q}_{\mathcal{P}}\} \cup \{f^i(S_1, \ldots, S_n) \to S_{f^i(S_1,\ldots,S_n)} \mid i \in \mathbb{N}, f \in \mathcal{F}_n, S_1, \ldots, S_n \in \mathcal{Q}_{\mathcal{P}}\}$ with $S_{f^i(S_1,\ldots,S_n)} = \{q \in \mathcal{Q}_{\mathcal{A}} \mid \forall j \in \{1, \ldots, n\}, \exists s_j \in S_j \text{ s.t. } f^i(s_1, \ldots, s_n) \to q \in \Gamma_{\overline{\mathcal{A}}}\}$

Note that subsets rules are obtained like in a classical determinization procedure $\mathcal{A}_{\mathcal{P}}$ contains all the rules $c^i \to \{q\}$ for $c \in \mathcal{F}_0$, $i \in \mathbb{N}$, $c \to q \in \Gamma_{\mathcal{A}}$, and that $\mathcal{A}_{\mathcal{P}}$ is deterministic and complete over $\mathcal{F}^{\mathbb{N}} \cup \{E\}$. The language recognized by $\mathcal{A}_{\mathcal{P}}$ is $\mathcal{L}(\mathcal{A}_{\mathcal{P}}) = T^{\mathbb{N}}$. For every term $\bar{t} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}} \cup \{E\} \cup \mathcal{Q}_{\mathcal{P}})$, there is a unique state $Q \in \mathcal{Q}_{\mathcal{P}}$ such that $\bar{t} \to_{\mathcal{A}_{\mathcal{P}}} Q$. The state Q is the normal form associated to $\bar{t}$ and is denoted by $\mathsf{nf}_{\mathcal{A}_{\mathcal{P}}}(\bar{t})$. Since $\mathcal{A}_{\mathcal{P}}$ erases the marks $\mathsf{nf}_{\mathcal{A}_{\mathcal{P}}}(\bar{t}) = \mathsf{nf}_{\mathcal{A}_{\mathcal{P}}}(t)$. We extend the operation m to $\mathcal{T}(\mathcal{F}^{\mathbb{N}} \cup \mathcal{Q}_{\mathcal{P}} \cup \{E\}, \mathcal{V})$ by setting $\mathsf{m}(S) = 0$. For a term $t \in \mathcal{T}(\mathcal{F}^{\mathbb{N}} \cup \mathcal{Q}_{\mathcal{P}} \cup \{E\}, \mathcal{V})$, and an integer i we denote by t^i the term obtained by setting all the marks in t at a position $u \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(t)$ to i, and we usually denote by $\bar{t}$ or $\widehat{t}$ a term such that $\bar{t}^0 = t$ (where $t \in \mathcal{T}(\mathcal{F} \cup \mathcal{Q}_{\mathcal{P}} \cup \{E\}, \mathcal{V})$).

▶ **Example 4.5.** Let us consider the automaton $\mathcal{A}_1$ from example 2.1. The following rules belong to the set of rules of $\mathcal{A}_{1\mathcal{P}}$: $E(\{\mathsf{q_a}\}, \{\mathsf{q_b}\}) \to \{\mathsf{q_a}, \mathsf{q_b}\}, \mathsf{a}^3 \to \{\mathsf{q_a}\}, \mathsf{h}^2(\{\mathsf{q_a}, \mathsf{q}_\perp\}) \to \{\mathsf{q_a}, \mathsf{q}_\perp\}, \mathsf{i}^1(\{\mathsf{q_a}, \mathsf{q_b}\}, \{\mathsf{q_b}\}) \to \{\mathsf{q}_\perp, \mathsf{q}_f\}$,

$E(\{\mathsf{q_a}, \mathsf{q}_f\}, \{\mathsf{q_b}, \mathsf{q_a}\}) \to \{\mathsf{q_a}, \mathsf{q_b}, \mathsf{q}_f\}$, $\mathsf{g}^4(\{\mathsf{q_a}\}, \{\mathsf{q_b}\}) \to \{\mathsf{q}_\perp\}$.

▶ **Definition 4.6.** We denote by $\mathcal{A}^+_{\mathcal{P}}$ the automaton $(\mathcal{F}^{\mathbb{N}} \cup \{E\}, \mathcal{Q}_{\mathcal{P}}, \mathcal{Q}_{f,\mathcal{P}}, \Gamma^+_{\mathcal{P}})$, where $\Gamma^+_{\mathcal{P}} = \Gamma_{\mathcal{P}} \cup \{S \to S' \mid S \in \mathcal{Q}_{\mathcal{P}}, S' \subset S\}$.

The language recognized by the automaton $\mathcal{A}^+_{\mathcal{P}}$ is $\mathcal{L}(\mathcal{A}^+_{\mathcal{P}}) = (\to^*_{\{E(x,y)\to x, E(x,y)\to y\}})[T^{\mathbb{N}}]$.

▶ **Definition 4.7.** For an automaton $\mathcal{C} = (\mathcal{F}^{\mathbb{N}} \cup \{E\}, \mathcal{Q}_{\mathcal{C}}, \mathcal{Q}_{\mathcal{C},f}, \Gamma_{\mathcal{C}})$ and for every $n \in \mathbb{N}$, we denote by $\mathcal{C}^{\leq n}$ (respectively $\mathcal{C}^{\geq n}$) the (finite) automaton $\mathcal{C}^{\leq n} := (\mathcal{F}^{\leq n} \cup \{E\}, \mathcal{Q}_{\mathcal{C}}, \mathcal{Q}_{\mathcal{C},f}, {\Gamma_{\mathcal{C}}}^{\leq n})$ (resp. $\mathcal{C}^{\geq n} := (\mathcal{F}^{\geq n} \cup \{E\}, \mathcal{Q}_{\mathcal{C}}, \mathcal{Q}_{\mathcal{C},f}, {\Gamma_{\mathcal{C}}}^{\geq n})$) where ${\Gamma_{\mathcal{C}}}^{\leq n} := \{l \to r \in \Gamma_{\mathcal{C}} \mid l, r \in \mathcal{T}(\mathcal{F}^{\leq n} \cup \{E\} \cup \mathcal{Q}_{\mathcal{C}})\}$ (resp. ${\Gamma_{\mathcal{C}}}^{\geq n} := \{l \to r \in \Gamma_{\mathcal{C}} \mid l, r \in \mathcal{T}(\mathcal{F}^{\geq n} \cup \{E\} \cup \mathcal{Q}_{\mathcal{C}})\}$).

The automaton ${\mathcal{A}^+_{\mathcal{P}}}^{\geq k+1}$ will be used to define the top of a marked term, i.e. the top part of the term that could be used in a k-bounded derivation (see definition 5.3 given further). The automaton ${\mathcal{A}^+_{\mathcal{P}}}^{\leq k}$ will be a part of the GRS $\mathcal{G}$ used to simulate k-bounded derivations (see definition 5.14).

▶ **Definition 4.8.** For all linear terms $\overline{t} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}} \cup \{E\} \cup \mathcal{Q}_{\mathcal{P}}, \mathcal{V})$, for all $n \in \mathbb{N}$, we define $\overline{t} \odot n$ as the unique marked term such that $(\overline{t} \odot n)^0 = t$, and, $\forall u \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(t)$, $\mathsf{m}(\overline{t} \odot n/u) = \max(\mathsf{m}(\overline{t}/u), \mathsf{Card}(\mathcal{P}\mathsf{os}_{\setminus E}{}^{\prec u}(t)) + n)$

4.2 Marked Rewriting

From now on and until the end of this paper, let us fix an integer $k > 0$. We introduce here the rewrite relation $\circ\!\!\longrightarrow$ between marked terms.

▶ **Definition 4.9** (Marked rewriting step)**.** A ground marked term $\overline{s} \in \mathcal{T}_M$ rewrites to a ground marked term $\overline{t} \in \mathcal{T}_M$ in $\mathcal{R} \cup \mathcal{E}$ if there exist a rule $l \to r \in \mathcal{R} \cup \mathcal{E}$, a position $v \in \mathcal{P}\mathsf{os}(s)$, a marked term $\overline{l}$, and a marked substitution $\overline{\sigma}$ such that : $\overline{s} = \overline{s}[\overline{l}\overline{\sigma}]_v, \overline{t} = \overline{s}[r(\overline{\sigma} \odot j)]_v$, where: $j = 0$, if $l \to r \in \mathcal{E}$, and $j = 1$, if $l \to r \in \mathcal{R}$. We then just write $\overline{s} \circ\!\!\longrightarrow_{\mathcal{R}\cup\mathcal{E}} \overline{t}$.

We may omit $\mathcal{R} \cup \mathcal{E}$ when it is clear from the context. We use two different marking ($j = 0$ or $j = 1$) depending on the rule applied only to properly extend the notion of weakly bottom-up for linear TRSs (defined in [5]) to left-linear TRSs (see section 6). This notion is helpful to prove that several already known classes of TRSs belong to the class of bounded TRSs. Let us give some properties of marked derivations.

Associated Marked Derivation

Every derivation
$d: \ s_0 = s_0[l_0\sigma_0]_{v_0} \to_{\mathcal{R}\cup\mathcal{E}} s_0[r_0\sigma_0]_{v_0} = s_1 \to_{\mathcal{R}\cup\mathcal{E}} \dots \to_{\mathcal{R}\cup\mathcal{E}} s_{n-1}[r_{n-1}\sigma_{n-1}]_{v_{n-1}} = s_n,$
is mapped to a marked derivation $\overline{d}$ called the *marked derivation associated* to d
$\overline{d}: \ \overline{s_0} = \overline{s_0}[\overline{l_0}\overline{\sigma_0}]_{v_0} \circ\!\!\longrightarrow_{\mathcal{R}\cup\mathcal{E}} \overline{s_0}[r_0(\overline{\sigma_0} \odot i_0)]_{v_0} = \overline{s_1} \circ\!\!\longrightarrow_{\mathcal{R}\cup\mathcal{E}}$
$\dots \circ\!\!\longrightarrow_{\mathcal{R}\cup\mathcal{E}} \overline{s_{n-1}}[r_{n-1}(\overline{\sigma_{n-1}} \odot i_{n-1})]_{v_{n-1}} = \overline{s_n}$
where $\overline{s_0} = s_0$. Note that this map is unique since the position v_j, the rule (l_j, r_j), and $\overline{s_j}$ completely determine $\overline{s}_{j+1}$.

4.3 Bounded Derivations and Bounded TRSs

▶ **Definition 4.10** (Bounded derivations)**.** A marked rewriting step $\overline{s} = \overline{s}[\overline{l}\overline{\sigma}]_v \circ\!\!\longrightarrow_{\mathcal{R}\cup\mathcal{E}} \overline{t} = \overline{s}[r(\overline{\sigma} \odot j)]_v$ is k*-bounded* ($\mathsf{bo}(k)$) if $l \to r \in \mathcal{E}$ or if $l \to r \in \mathcal{R}$ and the following assertion holds: $(l \notin \mathcal{V} \Rightarrow \mathsf{mmax}(\overline{l}) \leq k)$ and $(l \in \mathcal{V} \Rightarrow \sup(\{\mathsf{m}(\overline{s}/u) \mid u \prec v\}) \leq k)$.
A marked derivation in $\mathcal{R} \cup \mathcal{E}$ is $\mathsf{bo}(k)$ if all its rewriting steps are $\mathsf{bo}(k)$. A derivation in $\mathcal{R} \cup \mathcal{E}$ is $\mathsf{bo}(k)$ if the associated marked derivation is $\mathsf{bo}(k)$. A derivation $s \to^*_{\mathcal{R}} t, s, t \in \mathcal{T}(\mathcal{F})$

is *k-bounded convertible* ($\mathsf{boc}(k)$) if there exists a $\mathsf{bo}(k)$-derivation $s \to^*_{\mathcal{R}\cup\mathcal{E}} t$ in $\mathcal{R}\cup\mathcal{E}$. The left-linear TRS $\mathcal{R}$ is k-bounded if every derivation in $\mathcal{R}$ is $\mathsf{boc}(k)$. We denote by $\mathsf{BO}(k)$ the class of k-bounded TRS and by BO the class $\bigcup_{k\in\mathbb{N}} \mathsf{BO}(k)$.

▶ **Example 4.11.** Let $\mathcal{R}_1$ be the TRS of example 2.1, and let $d : s_0 = \mathsf{f}(\mathsf{h}(\mathsf{a})) \to_{\mathsf{f}(x)\to\mathsf{g}(x,x)} s_1 = \mathsf{g}(\mathsf{h}(\mathsf{a}),\mathsf{h}(\mathsf{a})) \to_{\mathsf{a}\to\mathsf{b}} s_2 = \mathsf{g}(\mathsf{h}(\mathsf{a}),\mathsf{h}(\mathsf{b})) \to_{\mathsf{g}(\mathsf{h}(x),y)\to\mathsf{i}(x,y)} s_3 = \mathsf{i}(\mathsf{a},\mathsf{h}(\mathsf{b})) \to_{\mathsf{h}(\mathsf{b})\to\mathsf{b}} s_4 = \mathsf{i}(\mathsf{a},\mathsf{b})$ be a derivation in $\mathcal{R}_1$. Let us prove that this derivation is $\mathsf{boc}(1)$, i.e. that there is a derivation d' in $\mathcal{R}_1\cup\mathcal{E}$ which is $\mathsf{bo}(1)$.

- Let us take $d' = d$. We are going to prove that this derivation is $\mathsf{bo}(2)$ but not $\mathsf{bo}(1)$, i.e. that in the associated marked derivation, the maximal mark that appears on a lhs is 2 By definition, $\overline{d}$ starts on the term $\mathsf{f}^0(\mathsf{h}^0(\mathsf{a}^0))$. To build the associated marked derivation, we just apply the marking process exposed in section 4.2. We obtain the following derivation $\overline{d} : \overline{s_0} = \mathsf{f}^0(\mathsf{h}^0(\mathsf{a}^0)) \circ\!\!\to \overline{s_1} = \mathsf{g}^0(\mathsf{h}^1(\mathsf{a}^2),\mathsf{h}^1(\mathsf{a}^2)) \circ\!\!\to \overline{s_2} = \mathsf{g}^0(\mathsf{h}^1(\mathsf{a}^2),\mathsf{h}^1(\mathsf{b}^0)) \circ\!\!\to \overline{s_3} = \mathsf{i}^0(\mathsf{a}^2,\mathsf{h}^1(\mathsf{b}^2)) \circ\!\!\to \overline{s_4} = \mathsf{i}^0(\mathsf{a}^2,\mathsf{b}^0)$. We now look at the marks that appear on a lhs during this derivation. The lhs are f^0, a^2, $\mathsf{g}^0(\mathsf{h}^1(x),y)$, and $\mathsf{h}^1(\mathsf{b}^2)$, and the maximal mark that appears on the lhs is 2. Thus, we have proved that d is $\mathsf{boc}(2)$, but we want to prove that d is $\mathsf{boc}(1)$.
- To obtain a derivation d' which is $\mathsf{bo}(1)$, we apply the rules going from the bottom to the top. We apply the rules in this order: $\mathsf{a}\to\mathsf{b}$, $\mathsf{h}(\mathsf{b})\to\mathsf{b}$, then $\mathsf{f}(x)\to\mathsf{g}(x,x)$ and, to finish $\mathsf{g}(\mathsf{h}(x),y)\to\mathsf{i}(x,y)$. Since the rule $\mathsf{f}(x)\to\mathsf{g}(x,x)$ is not linear and duplicates the variable x we need to use the symbol E to apply the rules in the correct order. We introduce an E just above a and then apply the rule $\mathsf{a}\to\mathsf{b}$:
$\mathsf{f}^0(\mathsf{h}^0(\mathsf{a}^0)) \to_{x\to E(x,x)} \mathsf{f}^0(\mathsf{h}^0(E(\mathsf{a}^0,\mathsf{a}^0))) \to_{\mathsf{a}\to\mathsf{b}} \mathsf{f}^0(\mathsf{h}^0(E(\mathsf{a}^0,\mathsf{b}^0)))$.
Now, we introduce a second symbol E above the symbol h and get ride of the first one with selection rules, and then apply the rule $\mathsf{h}(\mathsf{b})\to\mathsf{b}$:
$\mathsf{f}^0(\mathsf{h}^0(E(\mathsf{a}^0,\mathsf{b}^0))) \to_{x\to E(x,x)} \mathsf{f}^0(E(\mathsf{h}^0(E(\mathsf{a}^1,\mathsf{b}^1)),\mathsf{h}^0(E(\mathsf{a}^1,\mathsf{b}^1)))) \to_{E(x,y)\to x}$
$\mathsf{f}^0(E(\mathsf{h}^0(\mathsf{a}^1),\mathsf{h}(E(\mathsf{a}^1,\mathsf{b}^1)))) \to_{E(x,y)\to y} \mathsf{f}^0(E(\mathsf{h}^0(\mathsf{a}^1),\mathsf{h}^0(\mathsf{b}^1))) \to_{\mathsf{h}(\mathsf{b})\to\mathsf{b}} \mathsf{f}^0(E(\mathsf{h}^0(\mathsf{a}^1),\mathsf{b}^0))$.
Hence, we apply the rule $\mathsf{f}(x)\to\mathsf{g}(x,x)$:
$\mathsf{f}^0(E(\mathsf{h}^0(\mathsf{a}^1),\mathsf{b}^0)) \to_{\mathsf{f}(x)\to\mathsf{g}(x,x)} \mathsf{g}^0(E(\mathsf{h}^1(\mathsf{a}^2),\mathsf{b}^1),E(\mathsf{h}^1(\mathsf{a}^2),\mathsf{b}^1))$, then select the needed copies:
$\mathsf{g}^0(E(\mathsf{h}^1(\mathsf{a}^2),\mathsf{b}^1),E(\mathsf{h}^1(\mathsf{a}^2),\mathsf{b}^1)) \to_{E(x,y)\to x} \mathsf{g}^0(\mathsf{h}^1(\mathsf{a}^2),E(\mathsf{h}^1(\mathsf{a}^2),\mathsf{b}^1))$
$\to_{E(x,y)\to y} \mathsf{g}^0(\mathsf{h}^1(\mathsf{a}^2),\mathsf{b}^1)$,
and apply the last rule:
$\mathsf{g}^0(\mathsf{h}^1(\mathsf{a}^2),\mathsf{b}^1) \to_{\mathsf{g}(\mathsf{h}(x),y)\to\mathsf{i}(x,y)} \mathsf{i}^0(\mathsf{a}^2,\mathsf{b}^1)$.
The maximal mark that appears on a lhs is 1. So, d' is $\mathsf{bo}(1)$ and d is $\mathsf{boc}(1)$.

Let us introduce a convenient notation.

▶ **Definition 4.12.** The binary relation ${}_k\!\circ\!\!\to_{\mathcal{R}\cup\mathcal{E}}$ over $\mathcal{T}(\mathcal{F}^{\mathbb{N}}\cup\{E\})$ is defined by $\overline{s}\ {}_k\!\circ\!\!\to_{\mathcal{R}\cup\mathcal{E}}\ \overline{t}$ if there is a $\mathsf{bo}(k)$ marked rewriting step in $\mathcal{R}\cup\mathcal{E}$ between $\overline{s}$ and $\overline{t}$. The binary relation ${}_k\!\circ\!\!\to^*_{\mathcal{R}\cup\mathcal{E}}$ over $\mathcal{T}(\mathcal{F}^{\mathbb{N}}\cup\{E\})$ is defined by $\overline{s}\ {}_k\!\circ\!\!\to^*_{\mathcal{R}\cup\mathcal{E}}\ \overline{t}$ if there is a $\mathsf{bo}(k)$ marked derivation from $\overline{s}$ to $\overline{t}$. The binary relation ${}_k\!\to^*_{\mathcal{R}}$ over $\mathcal{T}(\mathcal{F})$ is defined by $s\ {}_k\!\to^*_{\mathcal{R}}\ t$ if there is a $\mathsf{boc}(k)$ derivation in $\mathcal{R}$ from s to t.

Since the composition of two $\mathsf{bo}(k)$ marked derivations is a $\mathsf{bo}(k)$ marked derivation, ${}_k\!\circ\!\!\to^*_{\mathcal{R}\cup\mathcal{E}}$ is the transitive and reflexive closure of ${}_k\!\circ\!\!\to_{\mathcal{R}\cup\mathcal{E}}$. Note that the composition of two $\mathsf{boc}(k)$-derivation is not always a $\mathsf{boc}(k)$-derivation.

Let us recall the notion of linear k-bounded rewriting defined in [6] which will be denoted here lbo to avoid confusion.

▶ **Definition 4.13.** Let $\mathcal{R}$ be a linear TRS. A marked rewriting step $\overline{s} = \overline{s}[\overline{l}\overline{\sigma}]_v \circ\!\!\longrightarrow_{\mathcal{R}} \overline{t} = \overline{s}[r(\overline{\sigma} \odot 1)]_v$ is *linear k-bounded* ($\mathsf{lbo}(k)$) if the following assertion holds

$$(l \notin \mathcal{V} \Rightarrow \mathsf{mmax}(\overline{l}) \leq k), \text{ and } (l \in \mathcal{V} \Rightarrow \sup(\{\mathsf{m}(\overline{s}/u) \mid u \prec v\}) \leq k) \tag{3}$$

A marked derivation is $\mathsf{lbo}(k)$ if all its rewriting steps are $\mathsf{bo}(k)$. A derivation in $\mathcal{R}$ is $\mathsf{lbo}(k)$ if the associated marked derivation is $\mathsf{lbo}(k)$. The TRS $\mathcal{R}$ is linear k-bounded if every derivation $s \rightarrow^*_{\mathcal{R}} t$ can be replaced by a $\mathsf{lbo}(k)$ derivation from s to t. We denote by $\mathsf{LBO}(k)$ the class of linear k-bounded TRSs and by LBO the class $\bigcup_{k\in\mathbb{N}} \mathsf{LBO}(k)$.

By definition, $\mathsf{LBO}(k) \subseteq \mathsf{BO}(k)$. Moreover, one can easily check that for every linear TRS $\mathcal{R}$, $\mathcal{R} \in \mathsf{LBO}(k)$ iff $\mathcal{R} \in \mathsf{BO}(k)$. Since the $\mathsf{LBO}(0)$ membership problem is undecidable, the $\mathsf{BO}(0)$ membership problem is undecidable too. Note that in the definition of an $\mathsf{lbo}(k)$-derivation, nothing requires the linear condition. But if we consider $\mathsf{lbo}(k)$-derivations for left-linear TRSs, then the class $\mathsf{LBO}(k)$ does not contain left-linear TRSs with non right-linear rules. This is illustrated in the following example.

▶ **Example 4.14.** Let $\mathcal{R}_2 = \{\mathsf{f}(x) \rightarrow \mathsf{g}(x,x), \mathsf{a} \rightarrow \mathsf{b}\}$ and let $k \in \mathbb{N}$. There is a $\mathsf{bo}(0)$-derivation $\mathsf{f}(\mathsf{f}(\ldots(\mathsf{f}(\mathsf{a}))\ldots)) \rightarrow_{\mathcal{E}} \mathsf{f}(\mathsf{f}(\ldots(\mathsf{f}(E(\mathsf{a},\mathsf{a}))\ldots))) \rightarrow_{\mathsf{a}\rightarrow\mathsf{b}} \mathsf{f}(\mathsf{f}(\ldots(\mathsf{f}(E(\mathsf{a},\mathsf{b}))\ldots)))$ $\rightarrow_{\mathsf{f}(x)\rightarrow\mathsf{g}(x,x)} \mathsf{g}(\mathsf{f}(\ldots(\mathsf{f}(E(\mathsf{a},\mathsf{b}))\ldots)),\mathsf{f}(\ldots(\mathsf{f}(E(\mathsf{a},\mathsf{b}))\ldots))) \rightarrow_{E(x,y)\rightarrow x}$ $\mathsf{g}(\mathsf{f}(\ldots(\mathsf{f}(\mathsf{a})\ldots)),\mathsf{f}(\ldots(\mathsf{f}(E(\mathsf{a},\mathsf{b}))\ldots))) \rightarrow_{E(x,y)\rightarrow y} \mathsf{g}(\mathsf{f}(\ldots(\mathsf{f}(\mathsf{a}))\ldots),\mathsf{f}(\ldots(\mathsf{f}(\mathsf{b}))\ldots))$ but there is no $\mathsf{bo}(k)$-derivation from $\mathsf{f}(\mathsf{f}(\ldots(\mathsf{f}(\mathsf{a}))\ldots))$ to $\mathsf{g}(\mathsf{f}(\ldots(\mathsf{f}(\mathsf{a}))\ldots),\mathsf{f}(\ldots(\mathsf{f}(\mathsf{b}))\ldots))$ which does not use the rules of $\mathcal{E}$. Note that the TRS $\mathcal{R}_2$ is $\mathsf{bo}(0)$ since every derivation in $\mathcal{R}_2$ is $\mathsf{boc}(0)$.

Well-marked Derivation

Terms that appear on a marked derivation starting on a term $s \in \mathcal{T}(\mathcal{F})$ have a special form and are said to be *well-marked*.

▶ **Definition 4.15** (well-marked). A term $\overline{s} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}} \cup \{E\} \cup \mathcal{Q}_{\mathcal{P}}, \mathcal{V})$ is *well-marked* for k if these two assertions holds

1. for all $w \in \mathcal{P}\mathsf{os}_{\mathcal{V}}(s)$, for all $v \preceq w$, $\mathsf{m}(\overline{s}/v) \leq k$,
2. for all $w \in \mathcal{L}\mathsf{v}(s) \backslash \mathcal{P}\mathsf{os}_{\mathcal{V}}(s)$, one of these two assertions holds
   - **a.** for all $v \preceq w$, $\mathsf{m}(\overline{s}/v) \leq k$,
   - **b.** there exists $u \in \mathcal{P}\mathsf{os}_{\mathcal{F}}{}^{\preceq w}(\overline{s})$ such that:
     - for all $v \prec u$, $\mathsf{m}(\overline{s}/v) \leq k$,
     - for all $v \in \mathcal{P}\mathsf{os}_{\mathcal{F}}{}^{\preceq w}(s)$ such that $v \succeq u$, $\mathsf{m}(\overline{s}/v) = k + 1 + \mathsf{Card}(\mathcal{P}\mathsf{os}_{\backslash E}{}^{\prec v}(s)) - \mathsf{Card}(\mathcal{P}\mathsf{os}_{\backslash E}{}^{\prec u}(s))$.

A marked derivation is *well-marked* if every term in the derivation is well-marked.

So, a term is well-marked if for every $w \in \mathcal{L}\mathsf{v}(t)$, the sequence of marks on the symbols of $\mathcal{F}$ that appear on the branch containing w has the form: $m_0, m_1, \ldots, m_n, k+1, k+2, \ldots, k+l$ with $m_i \leq k$ in case $2b$. is satisfied and $m_0, m_1, \ldots, m_n$ with $m_i \leq k$ in case 1. or $2a$. is satisfied. Note that an unmarked term is well-marked, and that condition $2a$. is equivalent to for all $v \prec u$,$\mathsf{m}(\overline{s}/v) \leq k$, $\mathsf{m}(\overline{s}/u) = k+1$, and for all $v \in \mathcal{P}\mathsf{os}_{\mathcal{F}}{}^{\preceq w}(s)$ such that $v \succeq u$, $\mathsf{m}(\overline{s}/v) - \mathsf{m}(\overline{s}/u) = \mathsf{Card}(\mathcal{P}\mathsf{os}_{\backslash E}{}^{\prec v}(s)) - \mathsf{Card}(\mathcal{P}\mathsf{os}_{\backslash E}{}^{\prec u}(s))$.

▶ **Example 4.16.** Let $k = 3$ and let $\mathcal{R}_1$ and $\mathcal{A}_1$ be the TRS and the automaton from example 2.1. The terms $\mathsf{f}^1(E(\mathsf{f}^2(\mathsf{a}^2),x))$, $\mathsf{f}^0(E(\mathsf{f}^3(\mathsf{a}^4),x))$, $\mathsf{f}^0(\mathsf{f}^3(E(\mathsf{f}^4(\mathsf{a}^5),\mathsf{f}^3(\mathsf{a}^3))))$, $\mathsf{f}^2(E(\mathsf{f}^0(\mathsf{a}^4),x))$, $\mathsf{f}^4(\mathsf{f}^5(E(\mathsf{f}^6(\{\mathsf{q_a},\mathsf{q}_\perp\}),\mathsf{f}^6(\mathsf{b}^7))))$ are well-marked. The terms $\mathsf{f}^4(E(\mathsf{f}^5(\mathsf{a}^6),x))$, $\mathsf{f}^2(\mathsf{f}^3(E(\mathsf{f}^4(\mathsf{a}^4),\mathsf{f}^3(\mathsf{a}^4))))$, and $\mathsf{f}^2(\mathsf{f}^3(E(\mathsf{f}^4(\mathsf{a}^6),\mathsf{f}^3(\{\mathsf{q_b},\mathsf{q}_f\}))))$ are not well-marked since neither 1 or 2 hold.

▶ **Lemma 4.17.** *A* bo(k)*-derivation is well-marked iff it is starting on a well-marked term.*

5 Main Result

The main theorem of this section (and of the paper) is the following.

▶ **Theorem 5.1.** *Let $\mathcal{R}$ be a left-linear rewriting* TRS *over a signature $\mathcal{F}$. Let T be some recognizable subset of $\mathcal{T}(\mathcal{F})$ and let $k > 0$. Then, the set $(\,_k\!\!\to_{\mathcal{R}}^*)[T]$ is recognizable too and can be built.*

To obtain this result, we simulate bo(k)-derivations using a GTT. The construction of the proof can be divided into three steps:

- First, we define the top part $\mathsf{Top}(\bar{t})$ of a well-marked term $\bar{t}$ which is the part of $\bar{t}$ that could be rewritten using a rule of $\mathcal{R}$ in a bo(k)-derivation.
- Then we define a GRS $\mathcal{G}$ which has the following properties:
  - If $\bar{s} \to_{\mathcal{G}}^* \bar{t}$, then there exists $\overline{t'}$ such that $\bar{s}\ {}_k\!\!\circ\!\!\to_{\mathcal{R}}^* \overline{t'} \to_{\mathcal{A}_{\mathcal{P}}+}^* \bar{t}$ (lifting rewriting with $\mathcal{G}$ to $\mathcal{R}$).
  - If $\bar{s}\ {}_k\!\!\circ\!\!\to_{\mathcal{R}}^* \bar{t}$ then $\mathsf{Top}(\bar{s}) \to_{\mathcal{G}}^* \mathsf{Top}(\bar{t})$ (projecting rewriting with $\mathcal{R}$ to $\mathcal{G}$).
- From these two properties of $\mathcal{G}$ and using some technical lemmas, we obtain the simulation lemma 5.22. The relation $\to_{\mathcal{G}}^*$ is recognizable by a GTT, and since GTTs are effectively inverse recognizability preserving, we obtain theorem 5.1.

Top of a Marked Term

By definition of a bo(k)-derivation, a symbol in a term $\bar{t}$ can match a *lhs* of a rule of $\mathcal{R}$ only if the mark of this symbol is smaller or equal to k. This leads us to define the top part of a well-marked term $\bar{t}$ which is (intuitively) obtained by replacing all the useless subterms $\bar{t}/u$ by their normal form $\mathsf{nf}_{\mathcal{A}_{\mathcal{P}}}(\bar{t}/u)$.

▶ **Definition 5.2.** Let $\bar{t} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}} \cup \{E\} \cup \mathcal{Q}_{\mathcal{P}}, \mathcal{V})$ be well-marked. We define $\mathsf{Topd}(\bar{t})$ the top domain of $\bar{t}$ as: $u \in \mathsf{Topd}(\bar{t})$ iff $u \in \mathcal{P}os(t)$ and $\forall v \prec u, \mathsf{m}(\bar{t}/v) \leq k$.

▶ **Definition 5.3.** Let $\bar{t} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}} \cup \{E\} \cup \mathcal{Q}_{\mathcal{P}}, \mathcal{V})$ be well-marked. We denote by $\mathsf{Top}(\bar{t})$ the unique term such that

- $\mathcal{P}os(\mathsf{Top}(\bar{t})) = \mathsf{Topd}(\bar{t})$,
- $\bar{t} \to_{\mathcal{A}_{\mathcal{P}}^{\geq k+1}}^* \mathsf{Topd}(\bar{t})$,
- for all $\overline{t'}$ such that $\mathcal{P}os(\overline{t'}) = \mathsf{Topd}(\bar{t})$ and $\bar{t} \to_{\mathcal{A}_{\mathcal{P}}^{\geq k+1}}^* \overline{t'}$, we have $\overline{t'} \to_{\mathcal{A}_{\mathcal{P}}^{\geq k+1}}^* \mathsf{Top}(\bar{t})$.

▶ **Example 5.4.** Let $k = 3$ and let $\mathcal{R}_1$ and $\mathcal{A}_1$ be the TRS and the automaton from example 2.1. Let $\overline{t_0} = \mathsf{f}^0(E(\{\mathsf{q_a}\}, \mathsf{g}^0(\mathsf{a}^0, \mathsf{b}^0)))$, $\overline{t_1} = \mathsf{f}^2(E(\{\mathsf{q_a}\}, \mathsf{g}^0(\mathsf{a}^3, \mathsf{b}^4)))$, $\overline{t_2} = \mathsf{f}^2(E(\{\mathsf{q_a}\}, \mathsf{g}^3(\mathsf{a}^4, \mathsf{b}^4)))$, $\overline{t_3} = \mathsf{f}^2(E(\{\mathsf{q_a}\}, \mathsf{g}^4(\mathsf{a}^5, \mathsf{b}^5)))$, $\overline{t_4} = \mathsf{f}^4(E(\{\mathsf{q_a}\}, \mathsf{g}^5(\mathsf{a}^6, \mathsf{b}^6)))$. Note that these terms are well-marked. We have $\mathsf{Topd}(\overline{t_0}) = \mathsf{Topd}(\overline{t_1}) = \mathsf{Topd}(\overline{t_2}) = \mathcal{P}os(t_0)$, $\mathsf{Topd}(\overline{t_3}) = \{\epsilon, 0, 00, 01\}$, $\mathsf{Topd}(\overline{t_4}) = \{\epsilon\}$ and $\overline{t_0} \to_{\mathcal{A}_{1\mathcal{P}}^{\geq 4}}^0 \mathsf{Top}(\overline{t_0}) = \overline{t_0}$, $\overline{t_1} \to_{\mathcal{A}_{1\mathcal{P}}^{\geq 4}} \mathsf{f}^2(E(\{\mathsf{q_a}\}, \mathsf{g}^0(\mathsf{a}^3, \{\mathsf{q_b}\}))) = \mathsf{Top}(\overline{t_1})$,
$\overline{t_2} \to_{\mathcal{A}_{1\mathcal{P}}^{\geq 4}} \mathsf{f}^2(E(\{\mathsf{q_a}\}, \mathsf{g}^3(\mathsf{a}^4, \{\mathsf{q_b}\}))) \to_{\mathcal{A}_{1\mathcal{P}}^{\geq 4}} \mathsf{f}^2(E(\{\mathsf{q_a}\}, \mathsf{g}^3(\{\mathsf{q_a}\}, \{\mathsf{q_b}\}))) = \mathsf{Top}(\overline{t_2})$,
$\overline{t_3} \to_{\mathcal{A}_{1\mathcal{P}}^{\geq 4}} \mathsf{f}^2(E(\{\mathsf{q_a}\}, \mathsf{g}^4(\mathsf{a}^5, \{\mathsf{q_b}\}))) \to_{\mathcal{A}_{1\mathcal{P}}^{\geq 4}} \mathsf{f}^2(E(\{\mathsf{q_a}\}, \mathsf{g}^4(\{\mathsf{q_a}\}, \{\mathsf{q_b}\}))) \to_{\mathcal{A}_{1\mathcal{P}}^{\geq 4}}$
$\mathsf{f}^2(E(\{\mathsf{q_a}\}, \{\mathsf{q_\perp}\})) = \mathsf{Top}(\overline{t_3})$, $\overline{t_4} \to_{\mathcal{A}_{1\mathcal{P}}^{\geq 4}} \mathsf{f}^4(E(\{\mathsf{q_a}\}, \mathsf{g}^5(\mathsf{a}^6, \{\mathsf{q_b}\}))) \to_{\mathcal{A}_{1\mathcal{P}}^{\geq 4}}$
$\mathsf{f}^4(E(\{\mathsf{q_a}\}, \mathsf{g}^5(\{\mathsf{q_a}\}, \{\mathsf{q_b}\}))) \to_{\mathcal{A}_{1\mathcal{P}}^{\geq 4}} \mathsf{f}^4(E(\{\mathsf{q_a}\}, \{\mathsf{q_\perp}\})) \to_{\mathcal{A}_{1\mathcal{P}}^{\geq 4}} \mathsf{f}^4(\{\mathsf{q_a}, \mathsf{q_\perp}\})$
$\to_{\mathcal{A}_{1\mathcal{P}}^{\geq 4}} \{\mathsf{q_\perp}\} = \mathsf{Top}(\overline{t_4})$

5.1 Definition of the GRS $\mathcal{G}$ Used for the Simulation

Comb Associated to a Term and the Set $\mathcal{C}_{\leq n}$

Before giving the definition of $\mathcal{G}$ we need to introduce some notations.

▶ **Definition 5.5.** We define the binary relation $\sqsubset$ over $\mathcal{T}(\mathcal{F}^{\mathbb{N}} \cup \mathcal{Q}_{\mathcal{P}} \cup \{E\})$ by $s \sqsubset t$ if $\mathsf{root}(s) \in \mathcal{F}^{\mathbb{N}} \cup \mathcal{Q}_{\mathcal{P}}$, $\mathsf{root}(t) = E$, and there exists $u \in \{w \in \mathcal{P}\mathsf{os}_{\setminus E}(t) \mid \mathcal{P}\mathsf{os}^{\prec w}_{\setminus E}(t) = \emptyset\}$ such that $t/u = s$. We define $s \not\sqsubset t$ by $s \not\sqsubset t$ if $s \sqsubset t$ does not hold.

Note that for all t, $t \not\sqsubset t$.

▶ **Definition 5.6.** Let $t \in \mathcal{T}(\mathcal{F}^{\mathbb{N}} \cup \mathcal{Q}_{\mathcal{P}} \cup \{E\})$. Let B be the TRS $B := \{E(x, E(y, z)) - > E(E(x, y), z)\}$. We denote by $\ll t \gg$ the normal form associated to t: $\ll t \gg = \mathsf{nf}_B(t)$.

Note that B can be easily shown to be terminating and confluent.

▶ **Definition 5.7.** Let $t \in \mathcal{T}(\mathcal{F}^{\mathbb{N}} \cup \mathcal{Q}_{\mathcal{P}} \cup \{E\})$. Let D be the (infinite) ground TRS
$D := \{E(E(x\sigma, y\sigma), z\sigma) \to E(x\sigma, y\sigma) \mid \sigma : \mathcal{V} \to \mathcal{T}(\mathcal{F}^{\mathbb{N}} \cup \mathcal{Q}_{\mathcal{P}} \cup \{E\})\ \&\ z\sigma \sqsubset E(x\sigma, y\sigma)\}$
$\cup \{E(E(x\sigma, y\sigma), z\sigma) \to E(x\sigma, z\sigma) \mid \sigma : \mathcal{V} \to \mathcal{T}(\mathcal{F}^{\mathbb{N}} \cup \mathcal{Q}_{\mathcal{P}} \cup \{E\})\ \&\ x\sigma = y\sigma\}$.
The *comb* associated to t is denoted by $\lfloor t \rfloor$ and is defined by $\lfloor t \rfloor := \mathsf{nf}_D(\ll t \gg)$. We extend this definition to marked substitutions ($\lfloor \sigma \rfloor : x \mapsto \lfloor x\sigma \rfloor$).

Note that D can be shown to be terminating, and that there is a unique normal form associated to $\ll t \gg$.

▶ **Example 5.8.** Let $t_0 = E(a^0, a^0)$, $t_1 = E(a^1, E(a^2, b^1))$, $t_2 = E(a^1, E(a^1, b^1))$ and $t_3 = f(E(E(b^1, a^2), E(b^2, a^2)))$. We have $\ll t_0 \gg = t_0$, $\ll t_1 \gg = E(E(a^1, a^2), b^1)$, $\ll t_2 \gg = E(E(a^1, a^1), b^1)$, $\ll t_3 \gg = f(E(E(E(b^1, a^2), b^2), a^2))$, $\lfloor t_0 \rfloor = t_0$, $\lfloor t_1 \rfloor = \ll t_1 \gg$, $\ll t_2 \gg = E(a^1, b^1)$, and $\lfloor t_3 \rfloor = f(E(E(b^1, a^2), b^2))$.

▶ **Definition 5.9.** Let $n \in \mathbb{N}$. Let $A = \{\}$. We denote by $\mathcal{C}_{\leq n}$ the (finite) set of combs: $\mathcal{C}_{\leq n} := \{\lfloor t \rfloor \mid t \in \mathcal{T}(\mathcal{F}^{\leq k} \cup \mathcal{Q}_{\mathcal{P}} \cup \{E\})\ \&\ dpt_{\setminus E}(t) \leq n\}$.

Note that for a comb t, the set of term $\{s \mid \lfloor s \rfloor = t\}$ is recognizable. The next lemma is used to prove the projecting lemma 5.18.

▶ **Proposition 5.10** (Comb form proposition). *Let* $\overline{s}, \overline{t} \in \mathcal{T}(\mathcal{F}^{\leq k} \cup \mathcal{Q}_{\mathcal{P}} \cup \{E\})$, $\overline{u} \in \mathcal{C}_{\leq k+2}$, $n \in \mathbb{N}$. *If* $\lfloor \overline{s} \odot n \rfloor = \overline{u}$ *and* $\lfloor \overline{t} \odot n \rfloor = \overline{u}$, *then* $\overline{s} \odot n \circ\!\!\longrightarrow^*_{\mathcal{E}} \overline{t} \odot n$.

▶ **Definition 5.11.** Let $\overline{t} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}} \cup \mathcal{Q}_{\mathcal{P}} \cup \{E\}, \mathcal{V})$, $\mathcal{V}\mathsf{ar}(t) = \{x_1, \ldots, x_n\}$. For all $1 \leq i \leq n$, let $j_i = \mathsf{Card}(\mathcal{P}\mathsf{os}(t, x_i))$, and let $\mathcal{P}\mathsf{os}(t, x_i) = \{v_{1,1}, \ldots, v_{1,j_i}\}$ where the $v_{p,q}$ are given in lexicographic order. We define $\mathsf{lin}(\overline{t})$ as the term
$\mathsf{lin}(\overline{t}) := \overline{t}[x_{1,1}, \ldots, x_{1,j_1}, \ldots, x_{n,1}, \ldots, x_{n,j_n}]_{v_{1,1}, \ldots, v_{1,j_1}, \ldots v_{n,1}, \ldots v_{n,j_n}}$,
where the $x_{i,j}$ are distinct variables.

Each time we use the notation $\mathsf{lin}(\overline{t})$, we implicitly suppose that if $\mathcal{V}\mathsf{ar}(t) = \{x_1, \ldots, x_n\}$, then the variables in $\mathcal{V}\mathsf{ar}(\mathsf{lin}(t))$ are denoted $x_{i,j}$ as in definition 5.11.

Overview of the Simulation

Let us give an overview of the proof of the projecting and lifting lemmas used to simulate $\mathsf{bo}(k)$-derivations by a GTT (lemmas 5.18 and 5.15).
For every rule $l \to r \in \mathcal{R} \cup \mathcal{E}$, every term $\overline{l} \in \mathcal{T}(\mathcal{F}^{\leq k} \cup \mathcal{Q}_{\mathcal{P}} \cup \{E\})$ and every substitution $\overline{\tau} : \mathcal{V} \to \mathcal{C}_{\leq k+2}$ such that $\overline{\tau} \odot a : \mathcal{V} \to \mathcal{C}_{\leq k+2}$ (where $a = 1$ if $l \to r \in \mathcal{R}$, and $a = 0$ otherwise),

we build a GRS $\mathcal{G}_{\bar{l},\bar{r},\overline{\tau}} = (L, R)$, where L is the recognizable set containing all the terms $\bar{l}\overline{\sigma}$ such that $\lfloor\overline{\sigma}\rfloor = \overline{\tau}$ and R is the recognizable set containing all the terms $\mathsf{lin}(r)(\overline{\sigma} \odot a)$ such that for all $x_{i,j} \in \mathcal{V}\mathsf{ar}(\mathsf{lin}(r))$, the associated comb of $(x_{i,j}\overline{\sigma} \odot a)$ is $\lfloor x_i\overline{\tau} \odot a\rfloor$. The GRS $\mathcal{G}$ over $\mathcal{F}^{\leq k} \cup \mathcal{Q}_{\mathcal{P}} \cup \{E\}$ is hence defined as the union of all the GRS $\mathcal{G}_{\bar{l},r,\overline{\tau}}$ and $\mathcal{A}_{\mathcal{P}}^{+\leq k}$.
Now, let us see how the simulation works. The simulation is based on the projecting lemma 5.18 and the lifting lemma 5.15. Let us start with the projecting lemma. Let $l \to r \in \mathcal{R}$, and let us suppose that we want to simulate a rewriting step $\overline{s} = \overline{s}[\overline{l\sigma}]_v \ {}_k\!\circ\!\!\to_{\mathcal{R}\cup\mathcal{E}} \overline{t} = \overline{s}[\overline{r\sigma} \odot a]_v$ using the GRS $\mathcal{G}$.

The projecting lemma 5.18 claims that we can rewrite the useful top part of $\overline{s}$ to the useful top part of $\overline{t}$, i.e. that $\mathsf{Top}(\overline{s}) = \mathsf{Top}(\overline{s})[\bar{l}\mathsf{Top}(\overline{\sigma})]_v \to^*_{\mathcal{G}} \mathsf{Top}(\overline{t}) = \mathsf{Top}(\overline{s})[r\mathsf{Top}(\overline{\sigma} \odot a)]$. We obtain this derivation in two steps:

- First, we cut the useless part of $\mathsf{Top}(\overline{s})$ using $\mathcal{A}_{\mathcal{P}}{}^{\leq k}$, i.e. the parts of $x\overline{\sigma}$ that are marked by an integer greater than k in $x\overline{\sigma} \odot a$. Let us denote by $\overline{\sigma'}$ the substitution obtained after this step (i.e. the unique substitution such that $\overline{\sigma'} \odot a = \mathsf{Top}(\overline{\sigma} \odot a)$).
- Then, since $l\lfloor\overline{\sigma'}\rfloor \to r\lfloor\overline{\sigma'} \odot a\rfloor \in \mathcal{G}(\bar{l}, r, \lfloor\overline{\sigma'}\rfloor)$, we obtain the required derivation $\mathsf{Top}(\overline{s}) = \overline{s}[\bar{l}\overline{\sigma}]_v \to_{\mathcal{A}_{\mathcal{P}}^{\leq k}} \mathsf{Top}(\overline{s})[\overline{l\sigma'}]_v \to_{\mathcal{G}(\bar{l},r,\lfloor\overline{\sigma'}\rfloor)} \mathsf{Top}(\overline{s})[r(\overline{\sigma'} \odot a)]_v = \mathsf{Top}(\overline{s})[r\mathsf{Top}(\overline{\sigma} \odot a)]_v = \mathsf{Top}(\overline{t})$.

Now, let us see how the lifting lemma works. Let $\overline{s} = \overline{s}[l\overline{\sigma}] \to_{\mathcal{G}(\bar{l},r,\overline{\tau})} \overline{t} = \overline{s}[\mathsf{lin}(r)\overline{\sigma'}]$. We want to prove that there exists a term $\overline{s'}$ and a derivation $\overline{s} \ {}_k\!\circ\!\!\to_{\mathcal{R}\cup\mathcal{E}} \overline{s'} \to^*_{\mathcal{A}_{\mathcal{P}}} \overline{t}$. First, we apply the rule $l \to r$. We obtain a derivation $\overline{s} = \overline{s}[\bar{l}\overline{\sigma}]_v \ {}_k\!\circ\!\!\to_{\mathcal{R}} \overline{s}[r(\overline{\sigma} \odot a)]$. We then use the comb form proposition 5.10, and some other technical lemmas to obtain a term $\overline{s'}$ and a derivation $\overline{s} \ {}_k\!\circ\!\!\to_{\mathcal{R}} \overline{s}[r(\overline{\sigma} \odot a)] \ {}_k\!\circ\!\!\to^*_{\mathcal{E}} \overline{s'} \to_{\mathcal{A}_{\mathcal{P}}} \overline{t}$.

The GRS $\mathcal{G}$

For every linear term $\overline{t} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}} \cup \mathcal{Q}_{\mathcal{P}} \cup \{E\}, \mathcal{V})$, and every substitution $\overline{\sigma} : \mathcal{V} \to \mathcal{C}_{\leq k+2}$, the set $\{\overline{t}\,\overline{\sigma'} \mid \overline{\sigma'} : \mathcal{V} \to \mathcal{T}(\mathcal{F}^k \cup \mathcal{Q}_{\mathcal{P}} \cup \{E\}), \forall x \in \mathcal{V}, \lfloor x\overline{\sigma'}\rfloor = x\overline{\sigma}\}$ is recognizable.

▶ **Definition 5.12.** We denote by Λ_a the set of substitutions $\overline{\tau} : \mathcal{V} \to \mathcal{C}_{\leq k+2}$ such that $\lfloor\overline{\tau} \odot a\rfloor : \mathcal{V} \to \mathcal{C}_{\leq k+2}$.

▶ **Definition 5.13.** Let $l \to r \in \mathcal{R} \cup \mathcal{E}$, $\bar{l} \in \mathcal{T}(\mathcal{F}^{\leq k} \cup \mathcal{Q}_{\mathcal{P}} \cup \{E\}, \mathcal{V})$. Let $a = 0$ if $l \to r \in \mathcal{E}$, and let $a = 1$ if $l \to r \in \mathcal{R}$. Let $\mathcal{V}\mathsf{ar}(l) = \{x_1, \ldots, x_n\}$ and $\mathcal{V}\mathsf{ar}(r) = \{x_1, \ldots, x_m\}$. Let $\tau \in \Lambda_a$, and let
$L = \{\bar{l}\overline{\sigma} | \overline{\sigma} : \mathcal{V} \to \mathcal{T}(\mathcal{F}^{\leq k} \cup \mathcal{Q}_{\mathcal{P}} \cup \{E\}), \forall i \in \{1, \ldots, n\}, \lfloor x_i\overline{\sigma}\rfloor = x_i\overline{\tau}\}$, $R = \{\mathsf{lin}(r)(\overline{\sigma} \odot a) | \overline{\sigma} : \mathcal{V} \to \mathcal{T}(\mathcal{F}^{\leq k} \cup \mathcal{Q}_{\mathcal{P}} \cup \{E\}), \forall i \in \{1, \ldots, m\}, \forall j \in \{1, \ldots, \mathsf{Card}(\mathcal{P}\mathsf{os}(r, x_i))\}, \lfloor x_{i,j}\overline{\sigma} \odot a\rfloor = \lfloor x_i\overline{\tau} \odot a\rfloor\}$. Note that L and R are two recognizable sets of ground terms. We define the $\mathcal{G}(\bar{l}, r, \overline{\tau})$ over $\mathcal{F}^{\leq k} \cup \mathcal{Q}_{\mathcal{P}} \cup \{E\}$ by: $\mathcal{G}(\bar{l}, r, \overline{\tau}) := \{l \to r \mid l \in L, r \in R\}$.

▶ **Definition 5.14.** Let $\mathcal{G}_{\mathcal{R}} = \{\mathcal{G}(\bar{l}, r, \overline{\tau}) \mid l \to r \in \mathcal{R}, \bar{l} \in \mathcal{T}(\mathcal{F}^{\leq k} \cup \{E\}, \mathcal{V}), \overline{\tau} \in \Lambda_1\}$, $\mathcal{G}_{\mathcal{E}} := \{\mathcal{G}(l, r, \overline{\tau}) \mid l \to r \in \mathcal{E}, \overline{\tau} \in \Lambda_0\}$. We define $\mathcal{G}$ as the GRS over $\mathcal{F}^{\leq k} \cup \{E\} \cup \mathcal{Q}_{\mathcal{P}}$ $\mathcal{G} := \mathcal{G}_{\mathcal{R}} \cup \mathcal{G}_{\mathcal{E}} \cup \Gamma_{\mathcal{P}(\mathcal{A})^{+\leq k}}$.

The transitive and reflexive closure of a GRS is recognizable by a GTT. The GTT recognizing $\to^*_{\mathcal{G}}$ will be used to simulate $\mathsf{bo}(k)$-derivations in $\mathcal{R} \cup \mathcal{E}$ (see lemma 5.22).

Lifting Lemma

The lifting lemma simulates a derivation $\overline{s} \to^*_{\mathcal{A}^+_{\mathcal{P}}} \overline{s'} \to_{\mathcal{G}} \overline{t}$ by a $\mathsf{bo}(k)$-derivation in $\mathcal{R} \cup \mathcal{E}$ followed by a derivation in $\mathcal{A}_{\mathcal{P}}{}^+$. The proof can be found in the long version of this article.

▶ **Lemma 5.15** (lifting lemma). *Let* $l \to r \in \mathcal{R} \cup \mathcal{E}$*,* $\bar{l} \in \mathcal{T}(\mathcal{F}^{\leq k} \cup \mathcal{Q}_{\mathcal{P}} \cup \{E\})$*, let* $a = 0$ *if* $l \to r \in \mathcal{E}$ *and* $a = 1$ *if* $l \to r \in \mathcal{R}$*, and let* $\overline{\sigma} \in \Lambda_a$ *be a substitution. Let* $\overline{s} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}} \cup \{E\})$*,* $\overline{s'}, \overline{t} \in \mathcal{T}(\mathcal{F}^{\leq k} \cup \mathcal{Q}_{\mathcal{P}} \cup \{E\})$ *be such that* $\overline{s} \to^*_{\mathcal{A}^+_{\mathcal{P}}} \overline{s'} \to_{\mathcal{G}(\bar{l},r,\overline{\sigma})} \overline{t}$*. There exists* $\overline{t'} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}}, \cup\{E\})$ *such that* $\overline{s}\ {}_k\!\circ\!\!\to^+_{\mathcal{R}\cup\mathcal{E}} \overline{t'} \to^*_{\mathcal{A}^+_{\mathcal{P}}} \overline{t}$*.*

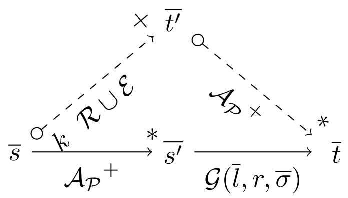

Figure 1 Lifting One Step.

▶ **Example 5.16.** Let $\mathcal{R}_1$ and $\mathcal{A}_1$ be the TRS and the automaton of example 2.1 and let $k = 1$. Let $\overline{s} = \mathsf{f}^0(E(\mathsf{h}^0(\mathsf{a}^1), E(\mathsf{a}^0, \mathsf{a}^1))) \to_{\mathcal{A}_{\mathcal{P}}+} \overline{s'} = \mathsf{f}^0(E(\mathsf{h}^0(\{\mathsf{q_a}\}), E(\mathsf{a}^0, \mathsf{a}^1)))$, let $x\overline{\sigma} = E(\mathsf{h}^0(\{\mathsf{q_a}\}), E(\mathsf{a}^0, \mathsf{a}^1))$, and let $\overline{\tau} = \lfloor \overline{\sigma} \rfloor$. Let $\overline{s'} \to_{\mathcal{G}(\mathsf{f}^0(x), \mathsf{g}^0(x,x), \overline{\tau})} \overline{t} = \mathsf{g}^0(E(E(\mathsf{h}^1(\{\mathsf{q_a}\}), \mathsf{h}^1(\{\mathsf{q_a}\})), \mathsf{a}^1), E(\mathsf{h}^1(\{\mathsf{q_a}\}), E(\mathsf{a}^1, \mathsf{a}^1)))$ (this step holds since $\lfloor E(E(\mathsf{h}^1(\{\mathsf{q_a}\}), \mathsf{h}^1(\{\mathsf{q_a}\})), \mathsf{a}^1)\rfloor = \lfloor E(\mathsf{h}^1(\{\mathsf{q_a}\}), E(\mathsf{a}^1, \mathsf{a}^1))\rfloor = \lfloor x\overline{\tau} \odot a \rfloor$ $= E(\mathsf{h}^1(\{\mathsf{q_a}\}), \mathsf{a}^1))$. We want to find a term $\overline{t'}$ such that $\overline{s}\ {}_1\!\circ\!\!\to^*_{\mathcal{R}\cup\mathcal{E}} \overline{t'} \to^*_{\mathcal{A}^+_{\mathcal{P}}} \overline{t}$. First, we apply the rule $\mathsf{f}(x) \to \mathsf{g}(x, x)$ which gives the $\mathsf{bo}(1)$-step $\overline{s'}\ {}_1\!\circ\!\!\to_{\mathcal{R}} \overline{t'} = \mathsf{g}^0(E(\mathsf{h}^1(\mathsf{a}^2), E(\mathsf{a}^1, \mathsf{a}^1)), E(\mathsf{h}^1(\mathsf{a}^2), E(\mathsf{a}^1, \mathsf{a}^1)))$. Then, since $\lfloor E(\mathsf{h}^1(\mathsf{a}^2), E(\mathsf{a}^1, \mathsf{a}^1))\rfloor = x\lfloor \overline{\tau} \odot 1 \rfloor$, using the comb form proposition 5.10 we obtain a derivation $\overline{t'}\ {}_1\!\circ\!\!\to^*_{\mathcal{E}} \mathsf{g}^0(E(E(\mathsf{h}^1(\mathsf{a}^2), \mathsf{h}^1(\mathsf{a}^2)), \mathsf{a}^1), E(\mathsf{h}^1(\mathsf{a}^2), E(\mathsf{a}^1, \mathsf{a}^1))) \to^*_{\mathcal{A}_{\mathcal{P}}+} \overline{t}$.

▶ **Corollary 5.17** (lifting n-steps). *Let* $\overline{s} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}} \cup \mathcal{Q}_{\mathcal{P}} \cup \{E\})$*,* $\overline{s'}, \overline{t} \in \mathcal{T}(\mathcal{F}^{\leq k} \cup \mathcal{Q}_{\mathcal{P}} \cup \{E\})$ *be such that* $\overline{s} \to^*_{\mathcal{G}} \overline{t}$*. There exists* $\overline{t'} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}} \cup \mathcal{Q}_{\mathcal{P}} \cup \{E\})$ *such that* $\overline{s}\ {}_k\!\circ\!\!\to^*_{\mathcal{R}\cup\mathcal{E}} \overline{t'} \to^*_{\mathcal{A}^+_{\mathcal{P}}} \overline{t}$*.*

Projecting Lemma

The projecting lemma simulates one $\mathsf{bo}(k)$-step $\overline{s}\ {}_k\!\circ\!\!\to_{\mathcal{R}\cup\mathcal{E}} \overline{t}$ by a derivation in $\mathcal{G}$ from $\mathsf{Top}(\overline{s})$ to $\mathsf{Top}(\overline{t})$. The full proof is given in the long version of this paper.

▶ **Lemma 5.18** (Projecting one step). *Let* $\overline{s}$*,* $\overline{t} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}} \cup \mathcal{Q}_{\mathcal{P}} \cup \{E\})$*, be such that* $\overline{s}$ *is well-marked and* $\overline{s} = \overline{s}[\bar{l}\overline{\sigma}]_v\ {}_k\!\circ\!\!\to_{\mathcal{R}\cup\mathcal{E}} \overline{t} = \overline{s}[r(\overline{\sigma} \odot j)]_v$*.*

1. *If* $\forall u \prec v$*,* $\mathsf{m}(\overline{s}/u) \leq k$ *then there exist a term* $\overline{s'} \in \mathcal{T}(\mathcal{F}^{\leq k} \cup \mathcal{Q}_{\mathcal{P}} \cup \{E\})$*, a substitution* $\overline{\sigma'} : \mathcal{V} \to \mathcal{C}_{\leq k+2}$ *such that* $\mathsf{Top}(\overline{s}) \to^*_{\mathcal{A}_{\mathcal{P}}{}^{\leq k}} \overline{s'} \to_{\mathcal{G}(\bar{l},r,\overline{\sigma'})} \mathsf{Top}(\overline{t})$*,*
2. *otherwise,* $\mathsf{Top}(\overline{s}) \to^*_{\mathcal{A}^+_{\mathcal{P}}{}^{\leq k}} \mathsf{Top}(\overline{t})$*.*

▶ **Example 5.19.** Let us consider the TRS $\mathcal{R}_1$, and the automaton $\mathcal{A}_1$ of example 2.1, and $k = 1$. We have the following derivation between these two well-marked terms $\overline{s} = \mathsf{f}^1(E(\mathsf{a}^0, E(\mathsf{a}^1, E(\mathsf{b}^1, \mathsf{h}^0(\mathsf{h}^0(\mathsf{a}^1)))))) \to_{\mathsf{f}(x)\to\mathsf{g}(x,x)} \overline{t} =$ $\mathsf{g}(E(\mathsf{a}^1, E(\mathsf{a}^1, E(\mathsf{b}^1, \mathsf{h}^1(\mathsf{h}^2(\mathsf{a}^3))))), E(\mathsf{a}^1, E(\mathsf{a}^1, E(\mathsf{b}^1, \mathsf{h}^1(\mathsf{h}^2(\mathsf{a}^3))))))$.
Let $x\overline{\sigma} = E(\mathsf{a}^0, E(\mathsf{a}^1, E(\mathsf{b}^1, \mathsf{h}^0(\mathsf{h}^0((\mathsf{a}^1))))))$. We have $\overline{s} = \mathsf{f}^1(x\overline{\sigma})$, $\overline{t} = \mathsf{g}(x\overline{\sigma} \odot 1, x\overline{\sigma} \odot 1)$, $\mathsf{Top}(\overline{s}) = \overline{s}$ and $\mathsf{Top}(\overline{t}) = \mathsf{g}(E(\mathsf{a}^1, E(\mathsf{a}^1, E(\mathsf{b}^1, \mathsf{h}^1(\{\mathsf{q}_\perp\})))), E(\mathsf{a}^1, E(\mathsf{a}^1, E(\mathsf{b}^1, \mathsf{h}^1(\{\mathsf{q}_\perp\})))))$. First, we cut the "useless" part of $\overline{s}$ using $\mathcal{A}_{1\mathcal{P}}{}^{\leq 1}$ i.e. the part of $x\overline{\sigma}$ that is marked by an integer greater that 1 in $x\overline{\sigma} \odot 1 = E(\mathsf{a}^1, E(\mathsf{a}^1, E(\mathsf{b}^1, \mathsf{h}^1(\mathsf{h}^2(\mathsf{a}^3)))))$. We obtain the following

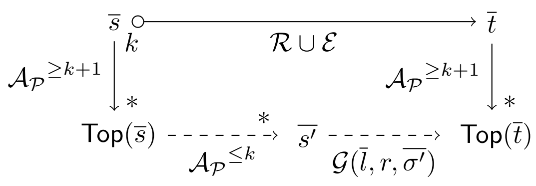

Figure 2 Projecting One Step, case 1.

derivation $\mathsf{Top}(\overline{s}) \to^*_{\mathcal{A}^{\leq k}_{1\mathcal{P}}} \mathsf{f}^1(E(\mathsf{a}^0, E(\mathsf{a}^1, E(\mathsf{b}^1, \mathsf{h}^0(\{\mathsf{q}_\perp\})))))$. We are now ready to apply the step of the GRS that simulates the rule $l \to r$. Let $x\overline{\sigma'} = E(\mathsf{a}^0, E(\mathsf{a}^1, E(\mathsf{b}^1, \mathsf{h}^1(\{\mathsf{q}_\perp\}))))$. Let $\overline{\tau} = \lfloor \overline{\sigma'} \rfloor$ and $\overline{s'} = \mathsf{f}(x\overline{\sigma'})$. The comb associated to $x\overline{\sigma'}$ is $x\overline{\tau} = E(E(E(\mathsf{a}^0, \mathsf{a}^1), \mathsf{b}^1), \mathsf{h}^1(\{\mathsf{q}_\perp\}))$. Moreover, the comb associated to $x\overline{\sigma'} \odot 1 = E(\mathsf{a}^1, E(\mathsf{a}^1, E(\mathsf{b}^1, \mathsf{h}^1(\{\mathsf{q}_\perp\}))))$ is $\lfloor x\overline{\tau} \odot 1 \rfloor = E(E(\mathsf{a}^1, \mathsf{b}^1), \mathsf{h}^1(\{\mathsf{q}_\perp\}))$. By definition, it means that $\overline{l\sigma'} \to r(\overline{\sigma'} \odot 1) \in \mathcal{G}(\overline{l}, r, \overline{\tau})$. Hence, we obtain the derivation $\mathsf{Top}(\overline{s}) \to_{\mathcal{A}^{\leq 1}_{1\mathcal{P}}} \overline{s'} = \mathsf{f}^1(x\overline{\sigma'}) \to_{\mathcal{G}(\overline{l}, r, \lfloor x\overline{\sigma'} \rfloor)} \mathsf{g}(x\overline{\sigma'} \odot 1, x\overline{\sigma'} \odot 1) = \mathsf{Top}(\overline{t})$.

▶ **Corollary 5.20** (Projecting n -steps). *Let $\overline{s}$, $\overline{t} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}} \cup \mathcal{Q_P} \cup \{E\})$, be such that $\overline{s}$ is well-marked and $\overline{s}\ {}_k\!\circ\!\!\to^*_{\mathcal{R} \cup \mathcal{E}} \overline{t}$. We have $\mathsf{Top}(\overline{s}) \to^*_{\mathcal{G}} \mathsf{Top}(\overline{t})$.*

▶ **Lemma 5.21.** *Let $s \in \mathcal{T}(\mathcal{F})$, $q \in \mathcal{Q_A}$. We have $\exists \overline{t} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}})$, $s\ {}_k\!\circ\!\!\to^*_{\mathcal{R} \cup \mathcal{E}} \overline{t} \to^*_{\overline{\mathcal{A}}} q$ iff $s \to^*_{\mathcal{G}} \{q\}$.*

Inverse Recognizability Preservation

▶ **Lemma 5.22** (simulation lemma). *We have $(\to^*_{\mathcal{G}})[\mathcal{Q}_{f,\mathcal{P}}] \cap \mathcal{T}(\mathcal{F}) = ({}_k\!\!\to^*_{\mathcal{R}})[T]$.*

Proof. Let $s \in ({}_k\!\!\to^*_{\mathcal{R}})[T]$. By definition, there exist $t \in \mathcal{T}(\mathcal{F})$ and $q \in \mathcal{Q}_{f,\mathcal{A}}$ such that $s\ {}_k\!\!\to^*_{\mathcal{R}} t \to^*_{\mathcal{A}} q$. By definition of a $\mathsf{boc}(k)$ derivation, there exists a marked term $\overline{t}$ such that $s\ {}_k\!\circ\!\!\to^*_{\mathcal{R} \cup \mathcal{E}} \overline{t}$. By lemma 4.3, since $t \to^*_{\mathcal{A}} q$, we have $\overline{t} \to^*_{\overline{\mathcal{A}}} q$. By lemma 5.21, $s \to^*_{\mathcal{G}} \{q\}$, and since $\{q\} \in \mathcal{Q}_{f,\mathcal{P}}$, we have $s \in (\to^*_{\mathcal{G}})[\mathcal{Q}_{f,\mathcal{P}}]$. Hence, $({}_k\!\!\to^*_{\mathcal{R}})[T] \subseteq (\to^*_{\mathcal{G}})[\mathcal{Q}_{f,\mathcal{P}}] \cap \mathcal{T}(\mathcal{F})$.
Let $s \in (\to^*_{\mathcal{G}})[\mathcal{Q}_{f,\mathcal{P}}] \cap \mathcal{T}(\mathcal{F})$. There exists $q \in \mathcal{Q}_{f,\mathcal{A}}$ such that $s \to^*_{\mathcal{G}} \{q\}$. By lemma 5.21, there exists $\overline{t} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}})$ such that $s\ {}_k\!\circ\!\!\to^*_{\mathcal{R} \cup \mathcal{E}} \overline{t} \to^*_{\overline{\mathcal{A}}} q$. By proposition 3.2, $s \to^*_{\mathcal{R}} t$, and since there exists a $\mathsf{bo}(k)$ marked derivation from s to $\overline{t}$, $s\ {}_k\!\!\to^*_{\mathcal{R}} t$. By lemma 4.3, since $\overline{t} \to^*_{\overline{\mathcal{A}}} q$, we have $t \to^*_{\mathcal{A}} q$. So, $t \in T$, and $s \in ({}_k\!\!\to^*_{\mathcal{R}})[T]$. ◀

We are now ready to prove theorem 5.1.

▶ **Theorem 5.1.** *Let $\mathcal{R}$ be some (finite) left-linear TRS over a signature $\mathcal{F}$. Let T be some recognizable subset of $\mathcal{T}(\mathcal{F})$ and let $k > 0$. Then, the set $({}_k\!\!\to^*_{\mathcal{R}})[T]$ is recognizable too.*

By lemma 5.22, $(\to^*_{\mathcal{G}})[\mathcal{Q}_{f,\mathcal{P}}] \cap \mathcal{T}(\mathcal{F}) = ({}_k\!\!\to^*_{\mathcal{R}})[T]$. The relation $\to^*_{\mathcal{G}}$ is recognizable by a GTT, and since GTTs are inverse recognizability preserving (see e.g. chapter 3.2 of [2]), $(\to^*_{\mathcal{G}})[\mathcal{Q}_{f,\mathcal{P}}] \cap \mathcal{T}(\mathcal{F})$ is recognizable, and thus $({}_k\!\!\to^*_{\mathcal{R}})[T]$ is recognizable. To effectively build the set $({}_k\!\!\to^*_{\mathcal{R}})[T]$, we need to construct the automaton $\mathcal{A}^{\leq k}_{\mathcal{P}}$ and the GTT $(\to^*_{\mathcal{G}})$. Since GTTs are effectively inverse recognizability preserving, the result holds.

▶ **Corollary 5.23.** *Every* $\mathsf{BO}(k)$ TRS *are effectively inverse recognizability preserving.*

6 Strongly Bounded TRSs

We introduce here strongly bounded TRSs. The reader may refer to the long version of the article for more details.

▶ **Definition 6.1.** A marked step $\overline{s} = \overline{s}[\overline{l}\overline{\sigma}]_v \circ\!\!\to_{\mathcal{R}} \overline{t} = \overline{s}[r(\overline{\sigma} \odot j)]_v$ is *weakly bottom-up* (wbu for short) if $l \to r \in \mathcal{E}$ or if $l \to r \in \mathcal{R}$ and the following assertion holds: $(l \notin \mathcal{V} \Rightarrow \mathsf{m}(\overline{l}) = 0)$ and $(l \in \mathcal{V} \Rightarrow \sup(\{\mathsf{m}(\overline{s}/u) \mid u \prec v) = 0)$. A marked derivation is wbu if all its rewriting steps are wbu. A derivation $s \to^*_{\mathcal{R}\cup\mathcal{E}} t$ is wbu if the associated marked derivation is wbu. A derivation $s \to^*_{\mathcal{R}} t$ is *weakly bottom-up convertible* (wbuc for short) if there exists a wbu derivation $s \to^*_{\mathcal{R}\cup\mathcal{E}} t$. Let $k \in \mathbb{N}$. A TRS is strongly k-bounded ($\mathsf{SBO}(k)$ for short) if every wbu derivation starting on a term $s \in \mathcal{T}(\mathcal{F})$ is $\mathsf{bo}(k)$. We denote by $\mathsf{SBO}(k)$ the class of $\mathsf{SBO}(k)$ TRSs. Finally, the class of strongly bounded TRS SBO is defined by: $\mathsf{SBO} = \bigcup_{k\in\mathbb{N}} \mathsf{SBO}(k)$.

Note that every marked derivation in $\mathcal{E}$ is wbu. Roughly speaking, a wbu derivation is a derivation in which the rules of $\mathcal{R}$ are applied going from the bottom to the top. Moreover, every derivation is wbuc and $\mathsf{SBO}(k) \subset \mathsf{BO}(k)$. The class SBO contains inverse right-linear finite-path overlapping TRSs [20], and left-linear growing TRSs [16]. Moreover, the membership problem for the class of $\mathsf{SBO}(k)$ TRSs such that $\mathsf{LHS}(\mathcal{R}) \cap \mathcal{V} = \emptyset$ is decidable, whereas the membership problem for $\mathsf{BO}(0)$ is undecidable, as shown in [5].

7 Perspectives

Here are some natural perspectives of development for this work.

- The method developed here also might be used for testing some termination properties and might lead to a proof of the decidability of the termination of left-linear growing TRSs as conjectured in [16].
- A dual notion of *top-down* rewriting should be defined (at least for linear *TRS*s). The class would presumably extend the class of layered transducing TRSs defined in [18].
- The *TRS*s considered in [9] and the *TRS*s considered here might be treated in a unifed manner for the linear case and if so, might be extended to the left-linear case.

Some work in these directions has been undertaken by the authors.

Acknowledgment. We thank the anonymous referees for their useful comments, which improved the presentation of our results.

References

1 B. Buchberger. Basic features and development of the critical-pair/completion procedure. In *Proceedings of the 1st International Conference on Rewriting Techniques and Applications*, pages 1–45, 1985.

2 H. Comon, M. Dauchet, R. Gilleron, C. Löding, F. Jacquemard, D. Lugiez, S. Tison, and M. Tommasi. Tree automata techniques and applications. Available at: `http://www.grappa.univ-lille3.fr/tata`, 2007. release October, 12th 2007.

3 Evelyne Contejean, Claude Marché, and Xavier Urbain. CiME, 2004. Available at `http://cime.lri.fr/`.

4 I. Durand and A. Middeldorp. Decidable call-by-need computations in term rewriting. *Inf. Comput.*, 196(2):95–126, 2005.

5 I. Durand and G. Sénizergues. Bottom-up rewriting for words and terms, March 2009. Available at: `http://arXiv.org/abs/0903.2554`.

6 I. Durand, G. Sénizergues, and M. Sylvestre. Termination of linear bounded term rewriting systems. In Christopher Lynch, editor, *Proceedings of the 21st International Conference on Rewriting Techniques and Applications*, pages 341–356. LIPIcs, July 2010.

7 Guillaume Feuillade, Thomas Genet, and Valérie Viet Triem Tong. Reachability Analysis over Term Rewriting Systems. Research Report RR-4970, INRIA, 2003.

8 Z. Fülöp, E. Jurvanen, M. Steinby, and S. Vágvölgyi. On one-pass term rewriting. In *MFCS*, pages 248–256, 1998.

9 A. Geser, D. Hofbauer, and J. Waldmann. Match-bounded string rewriting systems. *Journal Applicable Algebra in Engineering, Communication and Computing*, 15(3-4):149–171, November, 2004.

10 A. Geser, D. Hofbauer, J. Waldmann, and H. Zantema. On tree automata that certify termination of left-linear term rewriting systems. *Inform. and Comput.*, 205(4):512–534, 2007.

11 J. Giesl, R. Thiemann, P. Schneider-Kamp, and S. Falke. Automated Termination Proofs with AProVE (system description). In *Proceedings of the 15th International Conference on Rewriting Techniques and Applications*, pages 210–220, 2004.

12 F. Jacquemard. Decidable approximations of term rewriting systems. In *Proceedings of the 7th International Conference on Rewriting Techniques and Applications*, volume 1103 of *LNCS*, pages 362–376, 1996.

13 J.W. Klop. Term rewriting systems. In *Handbook of Logic in Computer Science, Vol. 2*, pages 1–116. Oxford University Press, 1992.

14 D. Knuth and P. Bendix. Simple word problems in universal algebras. In Leech, editor, *Computational problems in abstract algebra*, pages 263–297. Pergamon Press, 1970.

15 P. Lescanne, T. Heuillard, M. Dauchet, and S. Tison. Decidability of the confluence of ground term rewriting systems. Research Report RR-0675, INRIA, 1987.

16 T. Nagaya and Y. Toyama. Decidability for left-linear growing term rewriting systems. In *Proceedings of the 10th International Conference on Rewriting Techniques and Applications*, pages 256–270, London, UK, 1999. Springer-Verlag.

17 P. Réty and J. Vuotto. Tree automata for rewrite strategies. *J. Symb. Comput.*, 40(1):749–794, 2005.

18 H. Seki, T. Takai, Y. Fujinaka, and Y. Kaji. Layered transducing term rewriting system and its recognizability preserving property. In *Proceedings of the 13th International Conference on Rewriting Techniques and Applications*, volume 2378 of *LNCS*. Springer Verlag, 2002.

19 F. Seynhaeve, S. Tison, and M. Tommasi. Homomorphisms and concurrent term rewriting. In *FCT*, pages 475–487, 1999.

20 T. Takai, Y. Kaji, and H. Seki. Right-linear finite path overlapping term rewriting systems effectively preserve recognizability. In *Proceedings of the 11th International Conference on Rewriting Techniques and Applications*, pages 246–260, 2000.

21 T. Takai, Y. Kaji, and H. Seki. Termination property of inverse finite path overlapping term rewriting system is decidable. *IEICE transactions on information and systems*, 85(3):487–496, 2002-03-01.

22 T. Takai, Y. Kaji, and H. Seki. Right-linear finite-path overlapping term rewriting systems effectively preserve recognizability. *Scienticae Mathematicae Japonicae*, 2006. (to appear, preliminary version: IEICE Technical Report COMP98-45).

23 Terese. *Term Rewriting Systems by Terese*, volume 55. Cambridge University Press, 2003.

24 J. Waldmann. Matchbox: A Tool for Match-Bounded String Rewriting (system description). In *Proceedings of the 15th International Conference on Rewriting Techniques and Applications*, pages 85–94, 2004.

25 H. Zantema. TORPA: Termination of Rewriting Proved Automatically (system description). In *Proceedings of the 15th International Conference on Rewriting Techniques and Applications*, pages 95–104, 2004.