Linear min-max relation between the treewidth of H-minor-free graphs and its largest grid minor

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Abstract

A key theorem in algorithmic graph-minor theory is a min-max relation between the treewidth of a graph and its largest grid minor. This min-max relation is a keystone of the Graph Minor Theory of Robertson and Seymour, which ultimately proves Wagner's Conjecture about the structure of minor-closed graph properties. In 2008, Demaine and Hajiaghayi proved a remarkable linear min-max relation for graphs excluding any fixed minor H: every H-minor-free graph of treewidth at least $c_H r$ has an $r \times r$-grid minor for some constant c_H. However, as they pointed out, there is still a major problem left in this theorem. The problem is that their proof heavily depends on Graph Minor Theory, most of which lacks explicit bounds and is believed to have very large bounds. Hence c_H is not explicitly given in the paper and therefore this result is usually not strong enough to derive efficient algorithms.

Motivated by this problem, we give another (relatively short and simple) proof of this result without using big machinery of Graph Minor Theory. Hence we can give an explicit bound for c_H (an exponential function of a polynomial of $|H|$). Furthermore, our result gives a constant $w = 2^{O(r^2 \log r)}$ such that every graph of treewidth at least w has an $r \times r$-grid minor, which improves the previously known best bound $2^{\Theta(r^5)}$ given by Robertson, Seymour, and Thomas in 1994.

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1 Introduction

One of the deepest and most far-reaching theories of the recent 20 years in the realm of discrete mathematics and theoretical computer science is Graph Minor Theory developed by Robertson and Seymour in a series of over 20 papers spanning the last 20 years. The original goal of this work, now achieved, was to prove Wagner's Conjecture [26], which can be stated as follows: every minor-closed graph property (preserved under taking of minors) is characterized by a finite set of forbidden minors. This theorem has a powerful algorithmic consequence:

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every minor-closed graph property can be decided by a polynomial-time algorithm. This follows from another important result in Graph Minor Theory which gives a polynomial time algorithm to test whether or not a given graph has a fixed graph as a minor. One of the most central concepts, introduced early on, is the notion of *treewidth* [24]. Treewidth has obtained immense attention ever since, especially because many NP-hard problems can be handled efficiently on graphs of bounded treewidth [1]. In fact, all problems that can be defined in monadic second-order logic are solvable for graphs of bounded treewidth [4]. But perhaps even more importantly, Graph Minor Theory gives a powerful and vast toolkit of concepts and ideas to handle graphs and understand their structure. Indeed, a huge body of work has evolved that applies and extends these ideas in various fields of discrete mathematics and computer science.

A keystone in the proof of these theorems, and many other theorems, is a grid-minor theorem [24]: any graph of treewidth at least some $f(r)$ is guaranteed to have the $r \times r$ grid graph as a minor. This gird-minor theorem played a key role for the graph minor algorithm (c.f., the disjoint paths problem [16, 17, 25, 27, 28]). It also played a key role for some other deep applications (e.g., [12, 14, 15, 20]).

Such grid-minor theorems have also played a key role for many algorithmic applications, in particular via the bidimensionality theory (e.g., [5, 6, 7, 9]), including many approximation algorithms, PTASs, and fixed-parameter algorithms. These include feedback vertex set, vertex cover, minimum maximal matching, face cover, a series of vertex-removal parameters, dominating set, edge dominating set, R-dominating set, connected dominating set, connected edge dominating set, connected R-dominating set, and unweighted TSP tour.

The grid-minor theorem of [24] has been extended, improved, and re-proved by Robertson, Seymour, and Thomas [29], Reed [22], and Diestel, Jensen, Gorbunov, and Thomassen [11]. The best bound known for general graphs is superexponential: every graph of treewidth more than 20^{2r^5} has an $r \times r$ grid minor [29]. We note that as a corollary of our main theorem in this paper, we improve this bound in Corollary 2. Robertson et al. [29] conjecture that the bound on $f(r)$ can be improved to a polynomial $r^{\Theta(1)}$; the best known lower bound is $\Omega(r^2 \log r)$.

A linear upper bound has been shown for planar graphs [29] and bounded genus graphs [6]. Recently this min-max relation is also established for graphs excluding any fixed minor H: every H-minor-free graph of treewidth at least $c_H\, r$ has an $r \times r$ grid minor for some constant c_H [8]. This bound leads to many powerful algorithmic results on H-minor-free graphs [3, 8, 9, 13] that are previously not known.

However, as Demaine and Hajiaghayi pointed out in [8] (also see [10]), there are still major problems left in this grid-minor theorem for H-minor-free graphs, in particular in algorithmic graph-minor theory. The biggest problem is how large the constant c_H in the grid-minor theorem for H-minor-free graphs is. In particular, how does it depend on H? This constant is particularly important because it is in the exponent of the running times of many algorithms, as mentioned in [8, 10]. The current results (e.g., [8]) heavily depend on Graph Minor Theory, most of which lacks explicit bounds and is believed to have very large bounds. Recently, there is a simplified proof of Graph Minor Theory [18], but the bound is still huge. For this reason, improving the constants, even for special classes of graphs, and presumably using different approaches from graph minors, is an important theoretical and practical challenge.

Perhaps, Demaine, Hajiaghayi and Kawarabayashi [10] are the first to try to attack this issue, and they gave explicit bounds for the case of $K_{3,k}$-minor-free graphs, an important class of apex-minor-free graphs extending bounded genus graphs. The bounds are not too small but are a vast improvement over previous bounds (in particular, much smaller than $2 \uparrow |V(H)|$, where $2 \uparrow n$ denotes a tower $2^{2^{2^{\cdot^{\cdot^{\cdot}}}}}$ involving n 2's).

In this paper, we resolve this issue. More precisely, our main theorem is the following.

▶ **Theorem 1.** *For any fixed graph H and for any positive integer r, there exists a constant $w = |V(H)|^{O(|E(H)|)} \cdot r$ satisfying the following. If G does not contain an H-minor but has treewidth is at least w, then G has an $r \times r$-grid minor. Moreover, there is an algorithm, whose running time is a polynomial in $|V(G)|$ and w, to output either a tree-decomposition of width at most w, an $r \times r$-grid minor, or an H-minor in a given graph.*

Let us emphasize that, unlike the algorithms using the graph minor theory [8], no huge function of $|H|$ is involved in the above algorithm.

Furthermore, by setting H as an $r \times r$-grid with r^2 vertices and $2r^2 - 2r$ edges, Theorem 1 implies the following as a corollary, which improves the previously known best bound 20^{2r^5} given in [29] for large r.

▶ **Corollary 2.** *There exists a constant $w = 2^{O(r^2 \log r)}$ such that every graph of treewidth at least w has an $r \times r$-grid minor.*

To the best of our knowledge, Theorem 1 is the only grid-minor theorem with an explicit bound other than for planar graphs [29], bounded-genus graphs [6], and $K_{3,k}$-minor-free graphs [10]. Our theorem also leads to several algorithms with explicit and improved bounds on their running time, as mentioned above, in particular via the bidimensionality theory (e.g., [5, 6, 7, 9]).

In addition, the proof techniques are interesting in their own right, for example, the path-intertwining technique used in many contexts (see, e.g., [2, 19]), together with some techniques from Diestel et al. [11].

This paper is organized as follows. In Section 2, we give notations and results that are needed in this paper. In Section 3, we adapt tools from Diestel et al. [11]. Our key lemmas are provided in Section 4. Finally in Section 5, we give our main proof of Theorem 1.

2 Preliminaries

In this paper, n and m always mean the number of vertices of a given graph and the number of edges of a given graph, respectively. For $X \subseteq V$ in a graph $G = (V, E)$, let $N_G(X)$ denote the set of vertices in $V \setminus X$ that are adjacent to X. For simplicity, for $v \in V$, $N_G(\{v\})$ is denoted by $N_G(v)$. A *separation* (A, B) is that $G = A \cup B$, there are no edges in $E(A) \cap E(B)$, and moreover both $A - B$ and $B - A$ are nonempty. The order of the separation (A, B) is $|V(A) \cap V(B)|$. An $r \times r$ *grid* is a graph which is isomorphic to the graph W_r obtained from Cartesian product of paths of length $r - 1$, with vertex set $V(W_r) = \{(i, j) \mid 1 \leq i \leq r,\ 1 \leq j \leq r\}$ in which two vertices (i, j) and (i', j') are adjacent if and only if $|i - i'| + |j - j'| = 1$.

A *tree decomposition* of a graph G is a pair $(T, \mathcal{W})$, where T is a tree and $\mathcal{W}$ is a family $\{W_t \mid t \in V(T)\}$ of vertex sets $W_t \subseteq V(G)$, such that the following two properties hold:

(1) $\bigcup_{t \in V(T)} W_t = V(G)$, and every edge of G has both ends in some W_t.
(2) If $t, t', t'' \in V(T)$ and t' lies on the path in T between t and t'', then $W_t \cap W_{t''} \subseteq W_{t'}$.

The *width* of a tree decomposition $(T, \mathcal{W})$ is $\max_{t \in V(T)} |W_t| - 1$. The *treewidth* of a graph G is the minimum width over all possible tree decompositions of G.

A *linkage* $\mathcal{P}$ is a set of mutually vertex-disjoint paths in a graph. For two vertex sets Z_1 and Z_2, $\mathcal{P}$ is a Z_1-Z_2 *linkage* if each member is a path from Z_1 to Z_2. The *order* of the linkage, denoted by $|\mathcal{P}|$ is the number of paths. In slightly sloppy notation, sometimes we will identify a linkage $\mathcal{P}$ with the subgraph consisting of the paths in $\mathcal{P}$. For a linkage $\mathcal{P} = \{P_1, \dots, P_p\}$ in G, a $\mathcal{P}$-*bridge* in G is either an edge $e \in E(G) \backslash E(\mathcal{P})$ whose endpoints are both in $\mathcal{P}$, or a subgraph of G consisting of a connected component C of $G - \mathcal{P}$ together with all edges joining C and $\mathcal{P}$. The vertices of a $\mathcal{P}$-bridge B in $\mathcal{P} \cap B$ are called *attachments* of B, and we say that B is *attached* to $\mathcal{P}$ at these vertices. Given any two subpaths P and Q contained in the linkage $\mathcal{P}$, we say that they are *adjacent* if there exists a $\mathcal{P}$-bridge which intersects with both P and Q.

Now, we present some known results on mesh and treewidth, which will be used in the next section. For an integer α, we call a set $X \subseteq V(G)$ α-*connected* in G if $|X| \geq \alpha$ and for all subsets $Y, Z \subseteq X$ with $|Y| = |Z| \leq \alpha$, there are $|Y|$ mutually vertex-disjoint paths in G from Y to Z. Note that the sets Y and Z are not required to be disjoint. If $X = V(G)$, then we say G is α-connected. An α-connected set X is *externally α-connected* if, in addition, the required paths do not contain any vertex in X except their endpoints. Following [11], let us call a separation (A, B) a *premesh* if all the edges with both end vertices in $V(A) \cap V(B)$ lie in A, and A contains a tree T with the following properties:

1. T has maximum degree at most three;
2. every vertex of $A \cap B$ lies in T and has degree at most two in T; and
3. T has a leaf in $A \cap B$.

A premesh (A, B) is called an α-*mesh* if $V(A \cap B)$ is externally α-connected in B, and the graph $G = A \cup B$ is said to *have* this premesh or α-mesh.

Among useful lemmas on the α-mesh, Diestel et al. [11] proved the following lemmas.

▶ **Lemma 3.** *Let G be a graph and let $\beta \geq \alpha \geq 1$ be integers. If G has no α-mesh of order β, then G has treewidth $< \alpha + \beta - 1$.*

▶ **Lemma 4.** *Let $\beta \geq 2$ be an integer. Let T be a tree of maximum degree ≤ 3 and $X \subseteq V(T)$ be a vertex set with $|X| \geq \beta$. Then T has an edge set $F \subseteq E(G)$ such that every component of $T - F$ has at least β vertices and at most $2\beta - 2$ vertices in X, except that one such component may have fewer vertices in X.*

3 Finding good linkages

In this section, we show that graphs with large treewidth have a pair of linkages with some good properties. Such linkages will be used to construct a grid-minor or an H-minor in Sections 4 and 5. The following lemma is obtained from the arguments in [11], but we describe the proof for completeness.

▶ **Lemma 5.** *For a graph H with h vertices and for integers k, p', there exists an integer $w = (kh)^{O(|E(H)|)} \cdot p'$ satisfying the following. If a graph G has treewidth at least w, then either G contains an H-minor or two linkages $\mathcal{P}$ and $\mathcal{Q}$ such that*

(C1) *$|\mathcal{P}| \geq p'$ and $|\mathcal{Q}| \geq 3k^2|\mathcal{P}|$,*
(C2) *each path in $\mathcal{Q}$ hits all but at most $|\mathcal{P}|/3k^2$ paths in $\mathcal{P}$, and*
(C3) *$\mathcal{P}$ is a Z_1-Z_2 linkage for some $Z_1, Z_2 \subseteq V(G)$ such that for each edge $e \in E(\mathcal{P})$, $(\mathcal{P} \cup \mathcal{Q}) - e$ has no Z_1-Z_2 linkage.*

Proof. Let $c = 3k^2h^2$ and let $\alpha = c^{2|E(H)|-1}p'$. We show that $w = (2h + 2)\alpha$ is a desired integer.

Suppose that G has treewidth at least w. By Lemma 3, there is an α-mesh of order at least $(2h+1)(\alpha-1)$. Let $T \subseteq A$ be a tree associated with the premesh (A, B). Let $X = V(A \cap B) \subseteq V(T)$. By Lemma 4, T has at least h disjoint subtrees each containing at least h vertices of X. Let $A_1, \ldots, A_h$ be the vertex sets of these subtrees. Then by the definition of k-mesh, B contains a set $\mathcal{P}_{ij}$ of k mutually vertex-disjoint paths between A_i and A_j that have no inner vertices in A.

Let us identify the index set $\{0, 1, \ldots, h-1\}$ and the vertex set of H, and let us impose a linear ordering on the index pairs ij by fixing a bijection $f : \{ij \mid 1 \leq i < j \leq h\}$ to $\{0, \ldots, \binom{h}{2} - 1\}$ such that $f(ij) < |E(H)|$ if and only if $ij \in E(H)$. Let $l^* \leq \binom{h}{2}$ be a maximum integer such that for all $0 \leq l < l^*$ and all i, j, there exist sets $\mathcal{P}_{ij}^l$ satisfying the following conditions.

1. $\mathcal{P}_{ij}^l$ is a set of mutually vertex-disjoint paths from A_i to A_j in B that hit A only in their end points.
2. If $f(ij) < l$, then $\mathcal{P}_{ij}^l$ has exactly one path P_{ij}, and P_{ij} does not meet any paths in $\mathcal{P}_{st}^l$ with $ij \neq st$.
3. If $f(ij) = l$, then $|\mathcal{P}_{ij}^l| = \alpha/c^{2l}$.
4. If $f(ij) > l$, then $|\mathcal{P}_{ij}^l| = \alpha/c^{2l+1}$.
5. If $l = f(st) < f(ij)$, then for every edge $e \in E(\mathcal{P}_{ij}^l) \setminus E(\mathcal{P}_{st}^l)$, there are no k/c^{2l+1} vertex-disjoint paths from A_i to A_j in the graph $(\mathcal{P}_{ij}^l \cup \mathcal{P}_{st}^l) - e$.

If $l^* \geq |E(H)|$, then we are done since there is an H-minor. Hence we may assume that $l^* < |E(H)|$.

We shall first prove that $l^* > 0$. Let $st = f^{-1}(0)$ and put $\mathcal{P}_{st}^0 := \mathcal{P}_{st}$. For any ij with $f(ij) > 0$, let $F_{ij} \subseteq E(\mathcal{P}_{ij}) \backslash E(\mathcal{P}_{st}^0)$ be a maximal edge set such that there are still α/c vertex-disjoint paths from A_i to A_j in $(\mathcal{P}_{ij} \cup \mathcal{P}_{st}^0) - F_{ij}$, and define $\mathcal{P}_{ij}^0$ as such a set of paths. Then it is easy to see that $\mathcal{P}_{ij}^0$ satisfies the above conditions. This proves that $l^* > 0$.

Since $l^* > 0$, by the maximality of l^*, the above five conditions are satisfied for $l < l^*$ but cannot be satisfied for $l = l^*$. Let $st = f^{-1}(l^* - 1)$. We claim that there is no path $P \in \mathcal{P}_{st}^{l^*-1}$ such that P avoids a set $\mathcal{L}_{ij}$ of some $|\mathcal{P}_{ij}^{l^*-1}|/c$ paths in $\mathcal{P}_{ij}^{l^*-1}$ for all ij with $f(ij) \geq l^*$. Suppose such a path P exists. Let $s't' := f^{-1}(l^*)$ and define $\mathcal{P}_{s't'}^{l^*} := \mathcal{L}_{s't'}$. Let $\mathcal{P}_{st}^{l^*} := \{P\}$ and $\mathcal{P}_{ij}^{l^*} := \mathcal{P}_{ij}^{l^*-1}$ for $f(ij) < l^* - 1$. For each ij with $f(ij) > l^*$, let $F_{ij} \subseteq E(\mathcal{L}_{ij}) \backslash E(\mathcal{L}_{s't'}^{l^*})$ be a maximal edge set such that there are still $|\mathcal{P}_{ij}^{l^*-1}|/c^2$ vertex-disjoint paths from A_i to A_j in $(\mathcal{L}_{ij} \cup \mathcal{L}_{s't'}^{l^*}) - F_{ij}$ and define $\mathcal{P}_{ij}^{l^*}$ as such a set of paths. Then these would give rise to a family of sets $\mathcal{P}_{ij}^{l^*}$, a contradiction to the maximality of l^*.

Thus for every path $P \in \mathcal{P}_{st}^{l^*-1}$, P must intersect all but at most $|\mathcal{P}_{ij}^{l^*-1}|/c - 1$ paths in $\mathcal{P}_{ij}^{l^*-1}$ for some ij with $f(ij) \geq l^*$. By the pigeonhole principle, there are at least $|\mathcal{P}_{st}^{l^*-1}|/\binom{h}{2}$ paths (letting these paths $\mathcal{Q}$) in $\mathcal{P}_{st}^{l^*-1}$ each of which intersects all but $|\mathcal{P}_{ij}^{l^*-1}|/c - 1$ paths in $\mathcal{P}_{ij}^{l^*-1}$ for some ij with $f(ij) \geq l^*$ (letting such a set $\mathcal{P}_{ij}^{l^*-1}$ be $\mathcal{P}$).

Then, we have $|\mathcal{Q}| \geq |\mathcal{P}_{st}^{l^*-1}|/\binom{h}{2} \geq \alpha/(c^{2l^*}h^2)$ and $|\mathcal{P}| = \alpha/c^{2l^*+1}$, which implies that $|\mathcal{P}| \geq p'$ and $|\mathcal{Q}| \geq 3k^2|\mathcal{P}|$. Furthermore, by the definitions of $\mathcal{P}$ and $\mathcal{Q}$ and by condition 5, we obtain the following:

1. each path in $\mathcal{Q}$ meets all but at most $|\mathcal{P}|/c \leq |\mathcal{P}|/3k^2$ paths in $\mathcal{P}$, and
2. $\mathcal{P}$ is a Z_1-Z_2 linkage for some $Z_1 \subseteq A_i$ and $Z_2 \subseteq A_j$ such that for each edge $e \in E(\mathcal{P})$, $(\mathcal{P} \cup \mathcal{Q}) - e$ has no Z_1-Z_2 linkage.

This completes the proof of Lemma 5. ◀

Later, we will use this lemma in which $h = k$. The next lemma is a key lemma in this section. Its proof is inspired by [11].

▶ **Lemma 6.** *Suppose that $k, p', \mathcal{P}$, and $\mathcal{Q}$ satisfy the conditions (C1)-(C3) in Lemma 5, and $G = \mathcal{P} \cup \mathcal{Q}$. Each path $P_j \in \mathcal{P}$ has vertices $p_{j,1}, p_{j,2}, \ldots, p_{j,2k}$ which appear in this order from Z_1 to Z_2 such that the following holds:*

(C4) *For all j, let i*th segment *of P_j be the subpath of P_j between $p_{j,i}$ and $p_{j,i+1}$, and let i*th interval *of $\mathcal{P}$ be the union of the ith segment of P_j. Then, for each i, there is a subset $\mathcal{Q}_i \subseteq \mathcal{Q}$ with $|\mathcal{Q}_i| \geq (k-2)|\mathcal{P}|$ such that each path in $\mathcal{Q}_i$ intersects all but at most $|\mathcal{P}|/3k^2$ paths of $\mathcal{P}$ only in their ith segments.*

Proof. Let $p = |\mathcal{P}|$. Since each path in $\mathcal{Q}$ hits all but at most $p/3k^2$ paths, and $|\mathcal{Q}| \geq 3k^2p$, we may assume that P_1 intersects at least $(1 - 1/3k^2)3k^2p \geq 2k^2p$ paths in $\mathcal{Q}$.

Walk along P_1 from one end vertex until encountered kp paths in $\mathcal{Q}$, then pick up $e_1 \in E(P_1) - \bigcup_{Q \in \mathcal{Q}} E(Q)$. Then walk along P_1 until encountered another kp paths in $\mathcal{Q}$, then pick up $e_2 \in E(P_1) - \bigcup_{Q \in \mathcal{Q}} E(Q)$, and so on. Hence we pick up such edges $e_1, e_2, \ldots, e_{2k}$.

By our assumption and Menger's theorem, there exists a vertex set of size at most $p-1$ separating Z_1 and Z_2 in $G - e_i$ for each i. Clearly each path P_j contains exactly one vertex in this cut for $2 \leq j \leq p$. Let $\{p_{2,i}, p_{3,i}, \ldots, p_{2k,i}\}$ be the set of vertices consisting of the cut in $G - e_i$ such that P_j contains $p_{j,i}$ for $2 \leq j \leq p$ and $1 \leq i \leq 2k$. We may define $p_{1,i}$ as one of the end vertices of e_i. Let us define the segment $P_j[i, i+1]$ which is the subpath of P_j between $p_{j,i}$ and $p_{j,i+1}$, for $j = 1, \ldots, p$ and for $i = 1, \ldots, 2k-1$. Note that some of $P_j[i, i+1]$ could be a single vertex. The vertex set $\{p_{1,i}, \ldots, p_{p,i}\}$ divides $\mathcal{P}$ into two parts $\mathcal{P}^{\mathrm{R}_i}$ and $\mathcal{P}^{\mathrm{L}_i}$ such that P^{R_i} is a linkage from Z_1 to $\{p_{1,i}, \ldots, p_{p,i}\}$, and P^{L_i} is a linkage from Z_2 to $\{p_{1,i}, \ldots, p_{p,i}\}$, respectively. Let us remind that at least kp paths in $\mathcal{Q}$ hit $P_1[i, i+1]$ for each i.

Recall that the *ith interval* is defined by $\bigcup_{j=1}^{p} P_j[i, i+1]$. We claim that at least $(k-2)p$ of the kp paths in $\mathcal{Q}$ encountered on $P_1[i, i+1]$ do not leave the ith interval. Since there is no path from Z_1 to Z_2 in $G - \{p_{1,i}, \ldots, p_{p,i}\}$, at most p paths of the kp paths in $\mathcal{Q}$ leave for $\mathcal{P}^{\mathrm{R}_i} - \{p_{1,i}, \ldots, p_{p,i}\}$ through $\{p_{1,i}, \ldots, p_{p,i}\}$. Similarly, at most p paths of the kp paths in $\mathcal{Q}$ leave for $\mathcal{P}^{\mathrm{L}_{i+1}} - \{p_{1,i+1}, \ldots, p_{p,i+1}\}$ through $\{p_{1,i+1}, \ldots, p_{p,i+1}\}$. Therefore, at least $(k-2)p$ of the kp paths in $\mathcal{Q}$ encountered on $P_1[i, i+1]$ do not leave the ith interval. Hence, at least $(k-2)p$ paths in $\mathcal{Q}$ stay strictly inside the ith interval.

Thus, the cuts $\{p_{1,i}, \ldots, p_{p,i}\}$ for $1 \leq i \leq 2k-1$ will break the elements of $\mathcal{P}$ into $2k$ intervals. Moreover, each interval contains at least $(k-2)p$ paths in $\mathcal{Q}$ that stay strictly in the interval. These paths form the set $\mathcal{Q}_i$. This completes the proof. ◀

4 Main Lemmas

Suppose that $\mathcal{P}$ and $\mathcal{Q}$ are linkages satisfying the conditions (C1)-(C4) in Lemmas 5 and 6, and let $G = \mathcal{P} \cup \mathcal{Q}$ and $p = |\mathcal{P}|$. For each $i = 1, \ldots, 2k$, define G'_i to be the induced subgraph of G in the ith interval. We say that an index set $X \subseteq \{1, 2, \ldots, p\}$ is *good in G'_i* if it satisfies the following: for any subsets $Y_1, Y_2 \subseteq X$ with $|Y_1| = |Y_2| = 2r$, there are $2r$ mutually vertex-disjoint paths from $\{p_{j,i} \mid j \in Y_1\}$ to $\{p_{j,i+1} \mid j \in Y_2\}$ in G'_i.

Our first lemma in this section is the following.

▶ **Lemma 7.** *Let r and k be integers, and set $p' = 400k^2r$. Suppose that $\mathcal{P}$ and $\mathcal{Q}$ are linkages that satisfy conditions (C1)-(C4) in Lemmas 5 and 6, and let $p = |\mathcal{P}|$. For each i, there is a good X_i in G'_i with $|X_i| \geq 3p/4$. Moreover, $|X_{i-1} \cap X_i \cap X_{i+1}| \geq 100k^2r$ for $i = 2, \ldots, 2k-1$.*

Proof. Define $X = \{j \mid P_j \in \mathcal{P}$ hits at least $2r$ paths of $\mathcal{Q}_i\}$. Then, by simple counting argument, we have $|X| \geq 3p/4$, because $(p - p/3k^2)(k-2)p > (k-2)p(3p/4) + 2r(p/4)$.

Assume that X is not a good set in G'_i. Then, for some subsets $Y_1, Y_2 \subseteq X$ with $|Y_1| = |Y_2| = 2r$, there is a separation (A, B) of order at most $2r-1$ in G'_i with $\{p_{j,i} \mid j \in Y_1\} \subseteq V(A)$ and $\{p_{j,i+1} \mid j \in Y_2\} \subseteq V(B)$. We now consider $Z_A := \{j \mid V(P_j) \cap V(A - B) \neq \emptyset\}$ and $Z_B := \{j \mid V(P_j) \cap V(B - A) \neq \emptyset\}$. Since $P_j \in \mathcal{P}$ hits at least $2r > |V(A) \cap V(B)|$ paths of $\mathcal{Q}_i$ for each $j \in X$ and moreover each path in $\mathcal{Q}$ intersects at least $(1 - 1/3k^2)p \geq 3p/4$ paths of $\mathcal{P}$, both $|Z_A|$ and $|Z_B|$ are at least $3p/4$. Since $|Z_A \cap Z_B| \leq |V(A) \cap V(B)| \leq 2r - 1$, we have $|Z_A \cup Z_B| = |Z_A| + |Z_B| - |Z_A \cap Z_B| > p$, which is a contradiction.

Since $|X_i| \geq 3p/4$ for each i, we have $|X_{i-1} \cap X_i \cap X_{i+1}| \geq p - 3 \cdot (p/4) \geq 100k^2 r$. ◀

We say that a *leaf* of a connected graph is a vertex of degree one, and a $K_{1,k}$-minor (or a *k-star-minor*) is a connected subgraph with at least k leaves. For a linkage $\mathcal{P}' = \{P'_1, \ldots, P'_{|\mathcal{P}'|}\}$ in a graph G, a $K_{1,k}$-minor S in G is said to be *attached to* $\mathcal{P}'$ if every leaf of S is contained in $\mathcal{P}'$, and $|V(S) \cap V(P'_j)| = 1$ holds whenever $V(P'_j)$ contains a leaf of S.

The next lemma is the key lemma in our proof. It roughly says that one can either find an $r \times r$-grid minor in G'_i or else, given a good set X in G'_i, construct a minor of a "star-like graph" with at least k leaves in X. This gives us a $K_{1,k}$-minor with some condition in G'_i. This lemma allows us to "weave" the paths in $\mathcal{P}$ and construct a K_k-minor. Some idea in our proof can be found in [2].

▶ **Lemma 8.** *For each i, we have the following. Let $r, k, p, \mathcal{P}$, $\mathcal{Q}$, and G'_i be as above, and let X be a good set in G'_i with $|X| \geq 100k^2 r$. Then, either*

1. *G'_i has an $r \times r$-grid minor, or*
2. *there exist $Y_1, Y_2 \subseteq X$ with $|Y_1| = |Y_2| = k$ such that G'_i has a linkage $\mathcal{P}'$ from $\{p_{j,i} \mid j \in Y_1\}$ to $\{p_{j,i+1} \mid j \in Y_2\}$ and a $K_{1,k}$-minor S' attached to $\mathcal{P}'$.*

Proof. Since we only consider the ith interval of $\mathcal{P}$, we omit the index i in this proof for simplicity if no confusion may arise. That is, we denote $P_j[i, i+1]$ and G'_i by P_j and G', respectively.

Let $\mathcal{P}_X = \bigcup_{j \in X} P_j$ be the linkage that consists of the paths corresponding to X. Since X is a good set in G', we shall only focus on the unique connected component of G' containing $\mathcal{P}_X$. For our convenience, let us assume that G' itself is such a unique component.

Let Y be the set of connected components of $G' - \mathcal{P}_X$. We consider the auxiliary graph G^* with the vertex set $X \cup Y$ such that there exists an edge connecting $j \in X$ and $y \in Y$ if a $\mathcal{P}_X$-bridge y has attachments in P_j, and there exists an edge connecting $j_1, j_2 \in X$ if G' has an edge connecting P_{j_1} and P_{j_2}. We note that G^* is connected, since we assume the connectivity of G'.

We say that a $K_{1,t}$-minor S' in G^* with t leaves is a *good $K_{1,t}$-minor* if all leaves are in X and $|V(S') \cap X| \leq 3t$. We take disjoint subgraphs $S_1, \ldots, S_l$ in G^* such that S_i is a good K_{1,t_i}-minor with $t_i \geq 3$ for $i = 1, \ldots, l$, and

the total number of leaves $\sum_{i=1}^{l} t_i$ is as large as possible.

We show the following claim.

▶ **Claim 9.** *If $\sum_{i=1}^{l} t_i \geq 3k$, then there is a $K_{1,k}$-minor S^* in G^* such that all the leaves of S^* are in X.*

Proof. For any two subgraphs S_i, S_j with t_i, t_j leaves, respectively, if there is a path between S_i and S_j, then we can obtain a K_{1,t_i+t_j-2}-minor whose all leaves are in X. Note that $t_i + t_j - 2 > t_i$ and $t_i + t_j - 2 > t_j$.

Having proved this, we just greedily construct a star-minor such that all the leaves of the star-minor are in X. At the first step, we pick up one graph $S_i \in \{S_1, \ldots, S_l\}$. Then

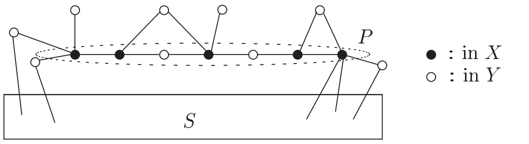

Figure 1 A connected component of $G^* - S$

we find a path between S_i and $\{S_1, \dots, S_l\} \setminus \{S_i\}$. Such a path must exist because G^* is connected. Suppose that the path connects S_i and S_j with $i \neq j$. Then we merge S_i and S_j as above to obtain a K_{1,t_i+t_j-2}-minor with all the leaves in X. Next, we find a path between the K_{1,t_i+t_j-2}-minor and $\{S_1, \dots, S_l\} \setminus \{S_i, S_j\}$, and we repeat this process until the end.

By the above remark, in each iteration, we get a star-minor with more leaves (in X) than the star-minor in the previous iteration. In fact, since the total number of leaves $\sum_{i=1}^{l} t_i$ is at least $3k$ at the beginning, in the final iteration, we get a star-minor with at least $\sum_{i=1}^{l}(t_i - 2) \geq k$ leaves in X. Note that we use the assumption $t_i \geq 3$ in this inequality. This completes the proof of Claim 9. ◀

We note that if there is a $K_{1,k}$-minor S^* in G^* such that all the leaves are in X, then we have the second conclusion of Lemma 8, in which $\mathcal{P}' = \mathcal{P}_X$ and S' is a minimal subgraph corresponding to S^*. Hence, in what follows, we assume that $\sum_{i=1}^{l} t_i < 3k$. By the definition of a good $K_{1,t}$-minor, this implies that $|V(S) \cap X| < 9k$ for $S := \bigcup_{i=1}^{l} S_i$. Now we show the following.

▶ **Claim 10.** *Let $S = \bigcup_{i=1}^{l} S_i$. As shown in Figure 1, each connected component of $G^* - S$ consists of a path P, a vertex set $Y' \subseteq Y - V(S)$, and edges between $V(P)$ and Y' such that for every $y \in Y'$, either*

- *$N_{G^*}(y) \cap V(P)$ consists of one vertex, or*
- *$N_{G^*}(y) \cap V(P)$ consists of two vertices v_1, v_2 with either $v_1v_2 \in E(P)$ or $v_1v_3, v_3v_2 \in E(P)$ for some $v_3 \in V(P) \cap Y$.*

Furthermore, each internal vertex of P is not adjacent to S, and each vertex in Y' adjacent to an internal vertex of P is not adjacent to S.

Proof. Let C be a connected component of $G^* - S$. If $|V(C) \cap X| \leq 2$, then the claim is obvious, because each vertex $y \in Y$ is not adjacent to a vertex in Y by the definition of G^*.

Suppose that $|V(C) \cap X| \geq 3$. By our choice of $\{S_1, \dots, S_l\}$, we observe that

- each vertex $y \in V(C) \cap Y$ is adjacent to at most two vertices in $V(C)$, and
- if a vertex $y \in V(C) \cap Y$ is adjacent to two vertices in $V(C)$, then y is not adjacent to S.

Again, we note that each vertex $y \in Y$ is not adjacent to a vertex in Y. While C contains a vertex $y \in V(C) \cap Y$ that is adjacent to two vertices v_1, v_2 in $V(C)$, we remove y (together with edges yv_1 and yv_2) and add an edge v_1v_2. Then, the obtained graph C' contains vertices in Y of degree one and vertices in X.

If there exists a vertex $x \in V(C') \cap X$ adjacent to three vertices in $V(C') \cap X$, then by adding this $K_{1,3}$-minor to S, we obtain a new set of star-minors with more total number of leaves, which contradicts the choice of S. Hence, the subgraph of C' induced by $V(C') \cap X$ forms a path or a cycle with multiple edges. Let $x_1, x_2, \dots, x_q$ be vertices of $V(C') \cap X$ that appear along this path (or cycle) in this order.

If x_j is adjacent to a vertex v in S for some $j = 2, 3, \dots, q-1$, then we can increase the total number of leaves of S by adding a $K_{1,3}$-minor whose leaves are x_{j-1}, x_{j+1}, and v. Therefore, x_j is not adjacent to S for $j = 2, 3, \dots, q-1$. Similarly, if there exists a vertex

$y \in V(C') \cap Y$ that is adjacent to x_j for some $j = 2, 3, \ldots, q-1$, then y is not adjacent to S. Note that, by this argument, we can also see that the subgraph of C' induced by $V(C') \cap X$ is not a cycle but a path, because G^* is connected.

Since each vertex $y \in Y$ is not adjacent to a vertex in Y, the original component C is obtained from C' by subdividing some edges into two edges. Thus, the claim holds by the above properties of C'. ◀

▶ **Claim 11.** *Suppose that $|V(S) \cap X| < 9k$. Then, some connected component of $G^* - S$ contains at least $4r + 2k(r+4)$ vertices in X.*

Proof. Let $\mathcal{C}$ be the set of connected components of $G^* - S$ each containing a vertex in X. By the choice of S, we can see that following:

- for each $x \in V(S) \cap X$, x is adjacent to at most one component of $\mathcal{C}$, and
- for each $y \in V(S) \cap Y$, y is adjacent to no component of $\mathcal{C}$.

This means that $|\mathcal{C}| \leq |V(S) \cap X| < 9k$, because G^* is connected. Since $|X| \geq 100k^2 r > 9k \cdot (4r + 2k(r+4))$, at least one connected component of $G^* - S$ contains at least $4r + 2k(r+4)$ vertices in X. ◀

By Claims 10 and 11, we can see that $G^* - S$ contains a long path. The following claim shows that each subgraph of G' corresponding to a long path with some condition contains either an $r \times r$-grid minor or "crossing paths".

▶ **Claim 12.** *Suppose that $0, 1, 2, \ldots, r+3 \in X$ appear in a path of $G^* - S$ in this order, and suppose also that there exist mutually vertex-disjoint paths $R_1, \ldots, R_r$ from $V(P_1)$ to $V(P_{r+2})$ in $G' - (P_0 \cup P_{r+3})$. Then, either G' contains an $r \times r$-grid minor or there exist two vertex-disjoint paths P' and R' in $G' - (P_0 \cup P_{r+3})$ such that P' connects $p_{j_1,i}$ and $p_{j_2,i+1}$ for some $j_1, j_2 \in \{2, 3, \ldots, r+1\}$, P' does not intersect with $V(P_1) \cup V(P_{r+2})$, and R' connects $V(P_1)$ and $V(P_{r+2})$. Furthermore, if such paths P' and R' exist, then $G' - (P_0 \cup P_{r+3})$ contains a linkage $\mathcal{P}' = \{P_1, P_{r+2}, P'\}$ and a $K_{1,3}$-minor attached to $\mathcal{P}'$.*

Proof. By the latter half of Claim 10, each of $R_1, \ldots, R_r$ intersects with $P_1, P_2, \ldots, P_{r+2}$ but does not intersect with the subgraph corresponding to S. Let D be the graph obtained from $(\bigcup_{1 \leq j \leq r+2} P_j) \cup (\bigcup_{1 \leq i \leq r} R_i)$ by executing the following procedure: contract P_1 to a single vertex s_1, contract P_{r+2} to a single vertex t_1, add a vertex s_2 and edges $s_2 p_{j,i}$ for $j = 2, 3, \ldots, r+1$, and add a vertex t_2 and edges $t_2 p_{j,i+1}$ for $j = 2, 3, \ldots, r+1$ (see Figure 2). Then, by a characterization of the existence of 2 vertex-disjoint paths (see [30]), either there exist a s_1-t_1 path and a s_2-t_2 path that are mutually vertex-disjoint, or D contains pairwise disjoint vertex sets $U_1, \ldots, U_q$ ($q \geq 0$) containing none of $\{s_1, t_1, s_2, t_2\}$ such that

(1) for $1 \leq i, j \leq q$ with $i \neq j$, $N_D(U_i) \cap U_j = \emptyset$,
(2) for $1 \leq i \leq q$, $|N_D(U_i)| \leq 3$, and
(3) if $\bar{D}$ is the graph obtained from D by contracting each component U_i to a single vertex for each i, then $\bar{D}$ can be embedded in a plane so that s_1, s_2, t_1 and t_2 are on the outer face boundary in this order.

If there exist a s_1-t_1 path and a s_2-t_2 path that are mutually vertex-disjoint, then the corresponding paths are two vertex-disjoint paths P' and R' in $G' - (P_0 \cup P_{r+3})$ such that P' connects $p_{j_1,i}$ and $p_{j_2,i+1}$ for some $j_1, j_2 \in \{2, 3, \ldots, r+1\}$, P' does not intersect with $V(P_1) \cup V(P_{r+2})$, and R' connects $V(P_1)$ and $V(P_{r+2})$. The existence of a $K_{1,3}$-minor attached to $\mathcal{P}' = \{P_1, P_{r+2}, P'\}$ is guaranteed by the existence of R'.

Suppose that there exist disjoint vertex sets $U_1, \ldots, U_q$ ($q \geq 0$) as above. By the construction of $\bar{D}$ in the condition (3), the paths in $\bar{D}$ corresponding to $P_2, \ldots, P_{r+1}$ are mutually vertex-disjoint except their end points, and the same thing holds for the paths in $\bar{D}$ corresponding to $R_1, \ldots, R_r$. By the planarity of $\bar{D}$, these paths form an $r \times r$-grid minor (see [23]). Since G' contains $\bar{D}$ as a minor, we have an $r \times r$-grid minor of G'. ◀

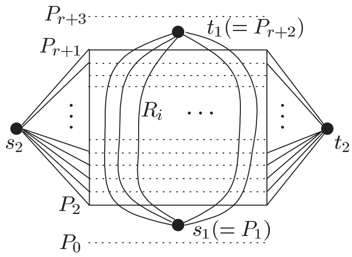

Figure 2 Construction of D

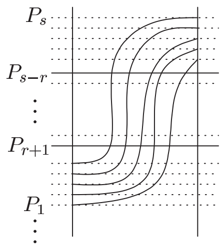

Figure 3 Paths from P_{r+1} to P_{s-r}

Now we are ready to prove Lemma 8. By Claims 10 and 11, $G^* - S$ contains a path containing at least $4r + 2k(r+4)$ vertices in X. We may assume that $-s, \ldots, -2, -1, 1, 2, \ldots, s \in X$ appear in the path in this order, where $s = 2r + k(r+4)$. Since X is a good set, there are $2r$ mutually vertex-disjoint paths from $\{p_{j,i} \mid j \in \pm\{1, 2, \ldots, r\}\}$ to $\{p_{j,i+1} \mid j \in \pm\{s, s-1, \ldots, s-r+1\}\}$. By the latter half of Claim 10, this means that G' contains either r vertex-disjoint paths from P_{r+1} to P_{s-r} or r vertex-disjoint paths from P_{-r-1} to P_{-s+r} that do not intersect with the subgraph corresponding to S. By symmetry, we may assume that G' contains r vertex-disjoint paths from P_{r+1} to P_{s-r} that do not intersect with the subgraph corresponding to S (see Figure 3).

We partition $\{r+1, r+2, \ldots, s-r\}$ into k disjoint sets $U_1, U_2, \ldots, U_k$ by setting $U_l := \{l(r+4)-3, l(r+4)-2, \ldots, l(r+4)+r\}$. Note that $|U_l| = r+4$ for each l. Then, by the assumption that $1, 2, \ldots, s \in X$ appear in this order, there exist r vertex-disjoint paths from $P_{l(r+4)-2}$ to $P_{l(r+4)+r-1}$ in $G' - (P_{l(r+4)-3} \cup P_{l(r+4)+r})$ for each l. We now apply Claim 12 for each U_l. If we can find an $r \times r$-grid minor for some l, then we are done. Otherwise, by Claim 12, for each l, we can take a linkage $\mathcal{P}'_l = \{P_{l(r+4)-2}, P_{l(r+4)+r-1}, P'_l\}$ and a $K_{1,3}$-minor attached to $\mathcal{P}'_l$.

Let $\mathcal{P}' = \bigcup_l \mathcal{P}'_l$. Then, we have k disjoint $K_{1,3}$-minors attached to $\mathcal{P}'$. Since the total number of leaves is $3k$, by the same argument as Claim 9, we can construct a $K_{1,k}$-minor attached to $\mathcal{P}'$ that is the second conclusion of Lemma 8. ◀

5 Main Proof

In this section, we give a proof of Theorem 1. That is, we show that there exists a constant $w = |V(H)|^{O(|E(H)|)} \cdot r$ such that every graph with treewidth at least w has either an H-minor or an $r \times r$-grid minor.

By applying Lemma 5 with $k = h = |V(H)|$ and $p' = 400k^2 r$, we obtain an integer $w = k^{O(|E(H)|)} \cdot r$. If the graph G has treewidth at least w, then either G contains an H-minor or two linkages $\mathcal{P}$ and $\mathcal{Q}$ satisfying (C1)-(C3). By Lemma 6, the linkage $\mathcal{P}$ can be partitioned into $2k$ intervals with the condition (C4). By Lemma 7, for each $i = 1, 2, \ldots, 2k$, there is a good set X_i in G'_i such that $|X_i| \geq 3p/4$ and $|X_{i-1} \cap X_i \cap X_{i+1}| \geq 100k^2 r$.

For each $i = 1, 3, 5, \ldots, 2k-1$, we apply Lemma 8 with $X = X_{i-1} \cap X_i \cap X_{i+1}$ (where we define $X_0 = \{1, 2 \ldots, p\}$). If an $r \times r$-grid minor is obtained, then we are done. Thus, we may assume that there exist $Y_{1,i}, Y_{2,i} \subseteq X_{i-1} \cap X_i \cap X_{i+1}$ with $|Y_{1,i}| = |Y_{2,i}| = k$ such that G'_i has a linkage $\mathcal{P}'_i$ from $\{p_{j,i} \mid j \in Y_{1,i}\}$ to $\{p_{j,i+1} \mid j \in Y_{2,i}\}$ and a $K_{1,k}$-minor S'_i attached to $\mathcal{P}'_i$.

For $i = 2, 4, 6, \ldots, 2k-2$, since $Y_{2,i-1}, Y_{1,i+1} \subseteq X_i$, there exist k vertex-disjoint paths from $\{p_{j,i} \mid j \in Y_{2,i-1}\}$ to $\{p_{j,i+1} \mid j \in Y_{1,i+1}\}$ by the definition of good sets. That is, we can connect $\mathcal{P}'_{i-1}$ and $\mathcal{P}'_{i+1}$ in the ith interval. By adding these $(k-1) \times k$ paths to $\bigcup_i \mathcal{P}'_i$, we obtain a linkage $\mathcal{P}'$ from $Y_{1,1}$ to $Y_{2,2k-1}$ and $K_{1,k}$-minors $S'_1, S'_3, \ldots, S'_{2k-1}$ attached to $\mathcal{P}'$. This graph contains a complete bipartite graph $K_{k,k}$ as a minor, which implies that it contains a K_k-minor. Since H is a subgraph of K_k, this completes the proof of the first half of Theorem 1.

By following the above arguments, Lemmas 6, 7, and 8 can be translated in polynomial time algorithms by using known algorithms for finding constant number of disjoint paths between two disjoint sets, for finding a minimum vertex cut, and for solving the 2 paths problem (e.g. [30, 31, 32]). We note that, in the proof of Lemma 8, we do not have to maximize the total number of leaves $\sum_{i=1}^{l} t_i$ at the beginning. This is because, if we cannot obtain the desired objects in Claims 10, 11, and 12, then we can find a set of star-minors with more total number of leaves. Hence, we only have to apply these claims, repeatedly.

To translate Lemma 5 to a polynomial time algorithm, it suffices to translate Lemmas 3 and 4 to polynomial time algorithms. Given a tree T and a vertex set $X \subseteq V(T)$, we can easily find an edge set F as in Lemma 4 in linear time by a simple greedy algorithm. On the other hand, we have no polynomial time algorithm to compute either a tree decomposition of G of width $< \alpha + \beta - 1$ or an α-mesh of order β in G as in Lemma 3. However, by the arguments in [11] (see also [21, Lemma 3.10]), we can find in polynomial time either a tree decomposition of G of width $< w$ or h vertex sets $A_1, \ldots, A_h$ as in the proof of Lemma 5. Therefore, all the procedures in the proof can be done in polynomial time in n and w. ◀

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