Report from Dagstuhl Seminar 12121

Applications of Combinatorial Topology to Computer Science

Edited by

Lisbeth Fajstrup[1], Dmitry Feichtner-Kozlov[2], and Maurice Herlihy[3]

1 **Aalborg University, DK,** fajstrup@math.aau.dk
2 **Universität Bremen, DE,** dfk@math.uni-bremen.de
3 **Brown University – Providence, US,** herlihy@cs.brown.edu

Abstract

This report documents the program of Dagstuhl Seminar 12121 "Applications of Combinatorial Topology to Computer Science". The seminar brought together researchers working on applications of combinatorial topology to various fields of computer science. The goal was to foster communication across these fields by providing researchers in each field the opportunity to explain their research programs to the others. The fields covered included distributed computing, persistent homology, semantics of concurrency, and sensor networks.

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1 Executive Summary

Lisbeth Fajstrup
Dmitry Feichtner-Kozlov
Maurice Herlihy

In recent years, concepts and techniques adapted from combinatorial and algebraic topology have led to a variety of promising new results in several areas of Computer Science, including distributed computing, sensor networks, semantics of concurrency, robotics, and vision.

The recent Dagstuhl seminar *Applications of Combinatorial Topology to Computer Science* (12121), brought together researchers in these fields, both to share ideas and experiences, and to establish the basis for a common research community. Because of differences in terminology and academic culture, it is often difficult for researchers in one area to become aware of work in other areas that may rely on similar mathematical techniques, sometimes resulting in duplication of effort. This Dagstuhl seminar provided a valuable opportunity to bring together researchers in both computer science and mathematics who share a common interest in emerging applications of combinatorial topology.

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3 Overview of Talks

3.1 Evasion paths in mobile sensor networks

Henry Adams (Stanford University, US)

Imagine that disk-shaped sensors wander in a planar domain. A sensor can't measure its location but does know when it overlaps a nearby sensor. We say that an evasion path exists in this sensor network if a moving evader can avoid detection. A theorem of Vin de Silva and Robert Ghrist gives a necessary condition, depending only on the time-varying connectivity graph of the sensor network, for an evasion path to exist. Can we sharpen this theorem? We'll consider examples that show the existence of an evasion path depends not only on the network's connectivity data but also on its embedding.

3.2 An equivariance theorem with applications to renaming

Armando Castañeda (IRISA / INSA – Rennes, FR)

Joint work of Castañeda, Armando; Herlihy, Maurice; Rajsbaum, Sergio

In the *M-renaming* task, each of $n+1$ processes is issued a unique name taken from a large namespace, and after coordinating with one another, each chooses a unique name taken from a (much smaller) namespace of size M. Processes are *asynchronous* (there is no bound on their relative speeds), and potentially *faulty* (any proper subset may halt without warning). Assuming processes communicate through a shared read- write memory, for which values of M can we devise a protocol that ensures that all non-faulty processes choose unique names?

To rule out trivial solutions, we require that any such protocol be *anonymous*: informally stated, in any execution, the name a process chooses can depend only on the name it was originally issued and how its protocol steps are interleaved with the others.

This problem was first proposed by Attiya et al. [1], who provided a protocol for $M = 2n+1$, and showed that there is no protocol for $M = n+2$. Later, Herlihy and Shavit [6] used chain complexes, a construct borrowed from Algebraic Topology, to show impossibility for $M = 2n$. Unfortunately, this proof, and its later refinements [2, 6, 7], had a flaw: because of a calculation error, the proof did not apply to certain dimensions satisfying a number-theoretic property described below. Castañeda and Rajsbaum [3] provided a new proof based on combinatorial properties of black-and-white simplicial colorings, and were able to show that in these dimensions, and only for them, protocols do exist for $M = 2n$. Nevertheless, this later proof was highly specialized for the weak symmetry breaking task, a task equivalent to renaming with $M = 2n$, so it was difficult to compare it directly to earlier proofs, either for renaming, or for other distributed problems. In the *weak symmetry breaking* task [4, 6], each of $n+1$ processes chooses a binary output value, 0 or 1, such that there is no execution in which the $n+1$ processes choose the same value.

In this talk we present an algebraic topology theorem that captures the impossibility of the renaming task. While this theorem requires more mathematical machinery than the specialized combinatorial arguments used by Castañeda and Rajsbaum, the chain complex formalism is significantly more general. While earlier work has focused on protocols for an asynchronous model where all processes but one may fail ("wait-free" protocols), the chain complex formalism applies to any model where one can compute the connectivity of the "protocol complexes" associated with that model. This approach has also proved broadly applicable to a range of other problems in Distributed Computing [5, 7]. In this way, we incorporate the renaming task in a broader framework of distributed problems. The second contribution is to point out where the flaw is in previous renaming lower bound proofs [6, 7].

As in earlier work [5, 7], the existence (or not) of a protocol is equivalent to the existence of a certain kind of chain map between certain chain complexes. Here, we replace the *ad-hoc* conditions used by prior work [6, 7] to capture the informal notion of anonymity with the well-established mathematical notion of *equivariance*. We prove a purely topological theorem characterizing when there exists an equivariant map between the chain complexes of an n-simplex and the chain complexes of an annulus. The desired map exists in dimension n if and only if $n+1$ is not a prime power. These are exactly the dimensions for which renaming is possible for $M = 2n$ [3].

In a more precisely way, the theorem is the following. Let σ^n be the simplex $\{P_0, \ldots, P_n\}$. For brevity, let σ^n denote the complex containing σ^n and all its faces. Let $\mathcal{S}_n$ be the symmetric group of order $n+1$. Clearly, $\mathcal{C}(\sigma^n)$ is an $\mathcal{S}_n$-chain complex: for each $\pi \in \mathcal{S}_n$, $\pi(\langle P_0 P_1 \ldots P_j \rangle) = \langle \pi(P_0)\pi(P_1) \ldots \pi(P_j) \rangle$. Now consider the following annulus $\mathcal{A}^n$ defined as follows. Each vertex has the form (P_i, b_i), where $P_i \in \sigma^n$ and v_i is 0 or 1. A set of vertexes $\{(P_0, v_0), \ldots, (P_j, v_j)\}$ defines a simplex of $\mathcal{A}^n$ if the P_i are distinct, and if $j = n$ then the b_i are not all 0 or all 1. Clearly, $\mathcal{C}(\mathcal{A}^n)$ is a $\mathcal{S}_n$-chain complex: for each $\pi \in \mathcal{S}_n$, $\pi(\langle (P_0, b_0) \ldots (P_j, b_j) \rangle) = \langle (\pi(P_0), b_0) \ldots (\pi(P_j), b_j) \rangle$.

▶ **Theorem 1.** *There exists a non-trivial $\mathcal{S}_n$-equivariant chain map*

$$a : \mathcal{C}(\sigma^n) \to \mathcal{C}(\mathcal{A}^n)$$

if and only if $n+1$ is not a prime power.

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3.3 Persistence based signatures for compact metric spaces

Frederic Chazal (INRIA Saclay – Orsay, FR)

Joint work of Chazal, Frederic; de Silva, Vin; Oudot, Steve
URL http://geometrica.saclay.inria.fr/team/Fred.Chazal/papers/RipsCompactTalk.pdf

We introduce a family of signatures for compact metric spaces, possibly endowed with real valued functions, based on the persistence diagrams of suitable filtrations built on top of these spaces. We prove the stability of these signatures with respect to the Gromov-Hausdorff metric. We illustrate their use through an application in shape classification.

3.4 Lower bounds on multiple sensor estimation

Frederick R. Cohen (University of Rochester, US)

Joint work of Moran, Bill; Cochran, Doug; Suvarova, Sofia; Howard, Stephen; Taylor, Tom
Main reference In preparation

This summary represents joint work with Bill Moran, Doug Cochran, Sofia Suvarova, Stephen Howard, and Tom Taylor.

Given sensor reports of counts of agents, a typical classical problem is to try to deduce the total number of agents reported by the sensors. One standard method is given by "inclusion-exclusion" as well as the Bonferroni inequalities. The main focus here is to refine techniques to provide estimates of minimum total numbers.

The new input here is the use of topology and geometry to give some estimates.

1. With natural assumptions concerning the sensor regions, methods are given for minimum counts via topology.
2. Three features are an introduction of
   a. a universal solution,
   b. topological methods to give criteria for whether "atoms are represented", and
   c. an infinite polytope which has an action of an integral lattice with some describable vertices and which gives a potential list of vertices for testing of minima.

Specific examples arise from a hexagonal tesselation of the plane and the introduction of a universal polytope with data concerning the structure of some of the vertices.

3.5 Why so persistent?

Herbert Edelsbrunner (IST Austria – Klosterneuburg, AT)

[Abstract omitted.] Herbert Edelsbrunner gave a survey talk about his work on proteins with E.P. Mücke and C.J.A. Delfinado, persistence with D. Letscher and A. Zomorodian, and stability with D. Cohen-Steiner, J. Harer, and D. Morozov.

3.6 Ditopology: A short tutorial

Lisbeth Fajstrup (Aalborg University, DK)

3.6.1 Introduction

The objects of ditopology are d-spaces, topological spaces with a selected set of *directed* paths. Such spaces provide a geometric model for the most powerful model of concurrent computing, Higher Dimensional Automata [3]. Dipaths model executions and paths which are directed homotopic model equivalent executions.

3.6.2 Definitions

▶ **Definition 1.** A pair $(X, \vec{P})$, where X is a topological space and $\vec{P} \subset X^I$ is a set of paths, is a d-space if

- $\vec{P}$ contains all constant paths.
- $\vec{P}$ is closed under concatenation.
- For $\gamma \in \vec{P}$ and $\alpha : I \to I$ non-decreasing, $\gamma \circ \alpha \in \vec{P}$.

For $p, q \in X$, the set of directed paths $\vec{P}(X)(p, q)$ is a topological space with the compact-open topology.

The trace space is the quotient space $\vec{T}(X)(p, q) = \vec{P}(X)(p, q)/R$ where R is the relation generated by non-decreasing reparametrization. See [1].

▶ **Definition 2.** A trace $\sigma \in \vec{T}(X)(p', p)$ induces maps $\sigma^* : \vec{T}(X)(p, q) \to \vec{T}(X)(p', q)$ and $\sigma_* : \vec{T}(X)(r, p') \to \vec{T}(X)(r, p)$ by concatenation $\sigma^*([\gamma]) = [\gamma \circ \sigma]$ and $\sigma_*([\mu]) = [\sigma \circ \mu]$.

The directed topology of X is the (ordinary) topology of $\vec{T}(X)(p, q)$ for all pairs of points p, q, and of the induced maps.

▶ **Definition 3.** The fundamental category of a d-space $(X, \vec{P})$ has objects all points of X. The morphisms from p to q are $\vec{\pi}_1(X)(p, q)$, the directed homotopy classes of dipaths from p to q.

In other words: The morphisms are the connected components of $\vec{T}(X)(p, q)$. There are no inverses, so the dihomotopy classes and concatenation gives rise to a fundamental groupoid; not a group.

3.6.3 Examples

Prominent examples of d-spaces are built from cubes I^n with the coordinate wise order or as subsets of cubes:

▶ **Example 1.** The geometric model of a Higher Dimensional Automaton is a the geometric realization of a cubical complex. This gives rise to a d-space, where the directed paths in a cube are paths which increase in all coordinates. The space $\vec{P}$ is obtained by concatenation and non-decreasing reparametrization of d-paths in cubes.

▶ **Example 2.** In Dijkstra's PV-model, n processes share some resources $R_1, \dots, R_l$, which allow the access of a finite number $k_1, \dots, k_l$ of processes. Each process is modelled as a directed graph Γ_i. The geometric model of the concurrent execution is the product $Y = \Gamma_1 \times \cdots \times \Gamma_n$ representing the joint progress of each process. A subset of the product, the forbidden region, is removed – the points corresponding to states where more than k_i processes access resource R_i.

In the simpler case when processes neither loop nor branch, Γ_i is an interval and the concurrent model is a cube $[0,1]^n$. The forbidden region is a union of n-rectangles. When $k_j \geq n$ there is no conflict at R_j. When $k_i \leq n-1$, and at least k_i+1 processes want access to R_i, the forbidden rectangle is $\times_{j=1}^n J_j$ where $J_j =]a_j^i, b_j^i[$ if the process j wants access to R_i at time a_j^i and releases R_i at time b_j^i and $J_j = [0,1]$ else. It is a generalized cylinder.

▶ **Example 3.** Dipaths may be homotopy equivalent but not dihomotopy equivalent:

Let X be $I^3 \setminus F$ where $F = R_1 \cup R_2 \cup R_3$ $R_1 =]1/7, 2/7[\times]1/7, 2/7[\times[0,1]$, $R_2 =]5/7, 6/7[\times]5/7, 6/7[\times[0,1]$, $R_3 =]3/7, 4/7[\times]1/7, 6/7[\times]1/4, 3/4[$. The piecewise linear directed paths γ_1 through $(0,0,0)$, $(1/7, 2/7, 0)$, $(1, 2/7, 0)$, $(1,1,1)$ and γ_2 through $(0,0,0)$, $(1/7, 2/7, 1)$, $(1, 2/7, 1)$, $(1,1,1)$ are homotopy equivalent, but not dihomotopy equivalent, since a homotopy between them will include a path whose second coordinate either passes $2/7$ before $1/7$ (to get over R_1, then under R_3) or it passes $6/7$ before $5/7$ (to get over R_3 then under R_2).

3.6.4 Calculations

When $X = I^n \setminus F$ and F is the union of a finite set of rectangles, Raussen's algorithm [5] provides a prod-simplicial model of the trace space. This has been implemented and there is a preliminary version of an extension to the case with loops [2]. The connected components of the trace space are calculated and used for static analysis. Moreover, calculation of higher homology is being implemented by M.Juda with the coreduction technique of M.Mrozek and B. Batko [4].

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3.7 Random manifolds and random simplicial complexes

Michael Farber (University of Warwick, GB)

In the talk I described the construction of random closed smooth manifolds arising as configuration spaces of linkages with random bar lengths. I also stated and explained theorems of M. Farber, T. Kappeler, C. Mazza, and C. Dombry on the asymptotic values of

Betti number of these random manifolds. In the second part of the talk I considered the Linial-Meshulam model of random simplicial complexes. I stated a recent joint result with A. Costa stating that in certain range of the probability parameter p a random complex can be made aspherical by puncturing all contained in it tetrahedral; the obtained punctured complex satisfied the Whitehead conjecture, a.a.s.

3.8 Combinatorial algebraic topology

Dmitry Feichtner-Kozlov (Universität Bremen, DE)

Combinatorial Algebraic Topology is concerned with computing algebraic invariants for combinatorial complexes with combinatorial means, and more generally to study properties of such complexes.

A number of applications in theoretical computer science (in particular, recently in theoretical distributed computing) use such combinatorial complexes, and the methods of combinatorial algebraic topology turn out to be quite useful in this context.

This talk is a survey, in part following my textbook, and is aimed at computer scientists as well as interested mathematicians working in related areas.

3.9 Some research notes on G-invariant persistent homology

Patrizio Frosini (University of Bologna, IT)

In this talk we would like to illustrate a current research about the problem of adapting Persistent Homology, in order to obtain a theory that is invariant with respect to a given subgroup G of the group of all the homeomorphisms from a compact topological space to itself. This research is motivated both by applications in shape comparison and by the need of mathematical tools to compute lower bounds for the natural pseudo-distance associated with the group G.

3.10 Some elements on Static Analysis and Geometry

Eric Goubault (CEA LIST and Ecole Polytechnique, France)

[Abstract omitted.] Eric Goubault's talk began with a tour of semantics/static analysis of sequential programs. He then described techniques for geometric analysis of concurrent programs and the inherent difficulties in analysis due to the interleaving semantics.

3.11 Introduction to combinatorial topology and distributed computing

Maurice Herlihy (Brown University – Providence, US)

This talk describes how simplicial complexes can be used to describe many kinds of distributed computing.

3.12 Torsion in computations

Anil N. Hirani (Univ. of Illinois – Urbana, US)

The absence of relative torsion in a simplicial complex leads to a polynomial time algorithm for finding smallest chains homologous to a given chain. This seems to be the first appearance of torsion in computations. I will give a brief exposition of what torsion is and how it is related to the constraint polyhedron of linear programming. Then I will describe a few variants of the problem and show an application to finding least spanning area surface of a knot. This is joint work with T. Dey, N. Dunfield, and B. Krishnamoorthy.

3.13 CAPD::RedHom – Homology software based on reduction algorithms

Mateusz Juda (Jagiellonian University – Krakow, PL)

URL http://redhom.ii.uj.edu.pl/

In the talk I presented CAPD::RedHom software (http://redhom.ii.uj.edu.pl/) – a software for efficient computation of the homology of sets.

As an input we use cubical, simplicial, or in some cases CW complexes. The software uses geometric and algebraic reduction to speed up classical Smith diagonalization or even the diagonalization is not required. During the talk we discussed following methods:

- acyclic subspace construction,
- elementary reductions and coreductions,
- discrete Morse theory.

The presentation contained also numerical experiments, comparison with other packages, and latest results for huge data sets.

3.14 Spectral methods in probabilistic topology

Matthew Kahle (Ohio State University, US)

There has been quite a bit of interest in recent years in the study of the expected topological properties of various kinds of random spaces. Dunfield and Thurston constructed random 3-manifolds from random walks on mapping class groups. Linial and Meshulam introduced the study of random simplicial complexes with independent faces, providing higher-dimensional analogues of Erdos-Renyi random graphs.

Some of my recent work has focused on using spectral methods to prove theorems about random simplicial complexes. These methods depend on theorems of Ballman and Swiatkowski, and of Zuk, and the main idea goes back to foundational work of Garland, where he introduced the notion of p-adic curvature.

In joint work with Hoffman and Paquette, we found a sharp threshold for Property (T) of the fundamental group of random 2-complexes. This work requires new results for the spectral gap of random graphs near the connectivity threshold. Using similar techniques, I was recently able to show that with high probability, a random d-dimensional flag complex has nontrivial homology only in middle degree.

This most recent result helps make measure-theoretic sense of the fact that so many complexes arising in combinatorics have homology concentrated in a small number of degrees.

3.15 Distributed computing mishmash: the operational perspective

Petr Kuznetsov (TU Berlin, DE)

Joint work of Gafni, Eli; Kuznetsov, Petr

One difficulty in addressing computability questions in distributed computing is the huge diversity of existing models of distributed systems, abstractions for distributed programming, and complexity metrics, with no apparent connection. In particular, the computational power of a model depends on synchrony assumptions, communications primitives, and (possibly non-uniform) patterns in which processes may fail.

In this talk, we focus on a large class of shared-memory adversarial models. In these models, processes communicate via reading and writing in the shared memory and their failure patterns are described as a set system on the set of process subsets. In every run of the model, the set of correct processes must belong to the set system.

We overview a set of recent (operational) simulations that allow reducing the question of colorless task solvability given an arbitrary adversary to a similar question in the more studied and better understood wait-free model. We speculate how topological methods can be used to extend these results to more general classes of distributed computing problems.

3.16 Persistence for shape comparison

Claudia Landi (University of Modena e Reggio Emilio, IT)

Main reference S. Biasotti, L. De Floriani, B. Falcidieno, P. Frosini, D. Giorgi, C. Landi, L. Papaleo, M. Spagnolo, "Describing shapes by geometrical- topological properties of real functions," ACM Computing Surveys, Vol 40, No. 4, Article No. 12, pp. 1–87, 2008.
URL http://dx.doi.org/10.1145/1391729.1391731

Persistence is a theory for Topological Data Analysis based on analyzing the scale at which topological features of a topological space appear and disappear along a filtration of the space itself. As such, it is particularly suited for handling qualitative rather than quantitative information about the studied space. Moreover, persistence deals with noise consistently, in that noisy data do not need to be smoothed out in advance. Last but not least, it is modular, meaning that different filtrations give insights from different perspectives on the space under study.

For all these reasons persistence turns out to be a well-suited tool for shape comparison, i.e. the task of assessing similarity between digital shapes.

In particular, persistence provides a shape descriptor, the persistence diagram, and a distance between these diagrams, the bottleneck distance. Thus the similarity between two shapes, represented by spaces endowed by functions, is measured by the bottleneck distance between the corresponding persistence diagrams.

Persistence diagrams are very concise descriptors, consisting of finitely many points of the plane. Moreover, the bottleneck distance between persistence diagrams is stable in the sense that small changes in the filtration imply small changes in the bottleneck distance. Finally, the bottleneck distance between persistence diagrams bounds from below the natural pseudo-distance between the original shapes.

3.17 Random methods in discrete topology: Discrete Morse functions and the complicatedness of triangulations

Frank H. Lutz (TU Berlin, DE)

We introduce a measure for the *complicatedness* of triangulations. For this, we define the *discrete Morse spectrum* of a simplicial complex as the distribution of discrete Morse vectors that are obtained by choosing free faces for collapses and critical faces uniformly at random. The complicatedness then is the expected number of critical cells.

It is hopeless to compute the discrete Morse spectrum for larger complexes, but it can easily be approximated by random experiments. In particular, the concept works well for manifolds and allows to compute optimal discrete Morse vectors in many cases. For example, we showed collapsibility of a nontrivial 5-manifold with f-vector $(5013, 72300, 290944, 495912, 383136, 110880)$.

3.18 Topology of random complexes

Roy Meshulam (Technion – Haifa, IL)

Joint work of Aronshtam, Lior; Linial, Nathan; Luczak, Tomasz; Meshulam, Roy; Wallach, Nathan

Let Y be a random d-dimensional subcomplex of the $(n-1)$-simplex S obtained by starting with the full $(d-1)$-dimensional skeleton of S and then adding each d-simplex independently with probability p.

For $d = 1$ this coincides with the Erdos-Renyi model $G(n, p)$ of random graphs, and the topology of Y in $G(n, p)$ is thoroughly understood. We'll survey some recent work on the topology of Y for $d > 1$, where much less is known. In particular, we'll discuss results concerning:

1. The threshold probability for vanishing of the $(d-1)$-dimensional homology of Y (Joint work with N. Linial and with N. Wallach).
2. The threshold probabilities for the vanishing of the d-dimensional homology of Y and for the d-collapsibility of Y (Joint work with L. Aronshtam, N. Linial and T. Luczak).

3.19 Homology and robustness of levelsets

Dmitriy Morozov (Stanford University, US)

Given a function $f : X \to R$ on a topological space, we consider its levelsets and their homology groups. We quantify the robustness of the homology classes under perturbations of f using well groups, and we show how to read the ranks of these groups from the extended persistence diagram. The special case $X = R^3$ has ramifications in the fields of medical imaging and scientific visualization.

3.20 Impossibility of set agreement and renaming

Ami Paz (Technion – Haifa, IL)

We present new proofs for two impossibility results for wait-free computation in asynchronous shared-memory systems, with only read / write operations. The results apply to two fundamental problems for n processes:

- $(n-1)$-set agreement, and
- renaming with a rank-based algorithm, when n is a prime power.

Both proofs are purely combinatorial and rely on simple counting arguments, and on results about the structure of restricted executions.

3.21 Locality and checkability in wait-free computing

Sergio Rajsbaum (Universidad Nacional Autonoma – Mexico, MX)

Main reference P. Fraigniaud, S. Rajsbaum, C. Travers, "Locality and Checkability in Wait-Free Computing," in Proc. of 25th Int'l Symp. on Distributed Computing (DISC'11), LNCS, Vol. 6950, pp. 333-347, Springer, 2011.
URL http://dx.doi.org/10.1007/978-3-642-24100-0_34

Given a task $T = (I, O, \Delta)$ and a black box protocol that claims to solve it, a distributed *checker* tries to find out whether the result of an execution is correct. Each process p_i gets as input (s_i, t_i), the i-th entries of an input-output pair $(s, t) \in I \times O$ produced by the black box, that is supposedly correct, i.e., $t \in \Delta(s)$. In a DISC 2011 paper we introduced AND-checkers, namely after communicating wait-free with the other processes, each process must output either "yes" or "no", with the following interpretation: every process says "yes" if and only if $t \in \Delta(s)$. We showed that there are many tasks that are AND-checkable. Yet, important tasks such as consensus and set agreement, are not.

In a new paper we generalize the AND-checker notion as a pair (E, D), respectively called the *encoder* and the *decoder*. The encoder E is a *wait-free distributed* protocol that takes as input a pair $(s, t) \in I \times O$, where each process p_i receives as input a pair (s_i, t_i), communicates with the others, and eventually returns an output value $u_i \in U$, where U is the range of E. The decoder D is a *centralized* algorithm that takes as input any multiset S of values from U output by the processes, and returns either "yes" or "no." For every pair $(s, t) \in I \times O$, it is required that $t \in \Delta(s)$ if and only if $D(E(s, t)) =$ "yes".

We show that every task has a *parsimonious* checker, based on a set U, independent of the task, and of small size. Tasks that are more difficult to check require a set U of larger size. We show that, for every task T on n processes, there exists a checker with range of size at most $n + 1$. The main result is a tight bound on the size $|U|$ of the encoder's range enabling every task on n processes to be checked. As a consequence, a classification of tasks in terms of their checkability difficulty is provided. We thus explain why consensus and set agreement are not AND-checkable: a range of three values is necessary to check consensus, while for k-set agreement the range of values needed depends on k.

3.22 Directed algebraic topology – with an eye to concurrency theory

Martin Raussen (Aalborg University, DK)

Higher-Dimensional Automata (HDA) are a framework for concurrency theory generalizing mutual exclusion using semaphores. These models consist of a geometric space (given combinatorially as a pre-cubical complex) with preferred directions, a so-called d-space. The space models the allowable (non-forbidden) states of all program counters. Not all continuous paths in that space are allowed; only so-called d-paths through the interleaving states, progressing with time.

A 1-parameter family of such d-paths (preserving the time constraint) is called a *dihomotopy*. Dihomotopic d-paths represent schedules that will always give the same result for a concurrent calculation. Therefore it is relevant to study d-paths up to dihomotopy; likewise to study d-spaces and d-maps between them (preserving d-paths) up to the dihomotopy relation.

Algebraic Topology offers a rich kit of insights, methods and tools to handle continuous geometric spaces (and their combinatorial counterparts) up to homotopy; in particular translating questions of a geometric flavor into algebraic problems that can solve the question or prove non-existence/unsolvability. We try to add a toolbox to the discipline taking explicit care of directedness. The algebra gets more complicated, since d-paths most often are not invertible.

Therefore, group theoretic constructions (like the fundamental group) have to be replaced by categorical constructions (like the fundamental category).

In general, one would like to get hold on properties of the space of all d-paths (or traces, i.e., d-paths up to directed reparametrization) in a d-space. One would like to calculate the number of components, to describe the homotopy types or at least some topological invariants of these components. For that purpose, we have constructed at least for simple HDA an algorithmic method yielding a description of the space of all d-paths (schedules) in such an automaton between given start and end points – as a *simplicial complex*. In principle, it is therefore possible to calculate invariants by known (computer) algorithms. In praxis, these complexes tend to be huge, and this is why we work on

- smaller representations yielding the same homotopy type,
- adaptations that work well when directed loops are part of the model, and
- general results concerning, e.g., the (higher) connectivity of the resulting spaces of d-paths.

At least formally, there are relations to *multidimensional persistence* to understand and to develop. These arise when the start and end point of a computation (schedule) are allowed to vary. Hence, one needs to understand, at what thresholds and how the trace spaces change under variations at end points. The goal is to subdivide the state space (or rather, its square) into a number of *components*: Trace spaces with end points in the same component should be homotopy equivalent to each other.

Moreover, we would like to explore relations to the methods from combinatorial algebraic topology used in *distributed computing*. This involves modeling further communication primitives and associated HDA. Moreover, one would need to compare d-spaces and their schedules for a variety of (live/dead) processors participating in the solution of a task.

3.23 A spectral sequence for parallelized persistence

Mikael Vejdemo-Johansson (University of St Andrews, GB)

Joint work of Vejdemo-Johansson, Mikael; Skraba, Primoz; Lipsky, David
Main reference D. Lipsky, P. Skraba, M. Vejdemo-Johansson, "A spectral sequence for parallelized persistence," arXiv:1112.1245v1 [cs.CG]
URL http://arXiv.org/abs/1112.1245v1

We describe a spectral sequence approach to a parallel algorithm to compute persistent homology. The spectral sequence of the double complex C_{**} with $C_{ij} = \bigoplus_{\sigma\in\mathcal{N}(\mathcal{U})_j} C_i \bigcap_{k\in\sigma} U_k$, where $\mathcal{U} = \{U_j\}$ is a covering of $\mathbb{X}$, will converge to the homology $H_*\mathbb{X}$ of the total space.

We are able to describe all higher differentials in the spectral sequence, and to adapt the computation to persistence modules, which we hope will yield parallelizable algorithms for computing persistent homology.

3.24 Directed paths in d-simplicial complexes

Krzysztof Ziemianski (University of Warsaw, PL)

Main reference K.Ziemiański, "A cubical model for path spaces in d-simplicial complexes," Topology and its Applications, vol. 159, issue 8, pp. 2127–2145. 2012.
URL http://dx.doi.org/10.1016/j.topol.2012.02.005

A d-simplicial complex is a simplicial complex equipped with a suitable relation on the set of its vertices which allows one to define a d-structure on its geometric realization. Given a d-simplicial complex $\vec{K}$ and two of its vertices v and w I will construct a cubical complex $CT(\vec{K})$ which is homotopy equivalent (under some mild conditions) to the space of directed paths on $|\vec{K}|$ from v to w. This construction gives the minimal functorial model for spaces of directed paths. Then, I will present a similar construction for cubical complexes; in this case the model for directed paths is a CW-complex which has a structure of permutohedral complex.

4 Panel Discussions

4.1 Persistent homology

Herbert Edelsbrunner and Dmitriy Morozov served on a panel for a discussion about persistent homology and its history.

Participants

- Henry Adams
Stanford University, US
- Sergio Cabello
University of Ljubljana, SI
- Armando Castaneda
IRISA / INSA - Rennes, FR
- Bernadette Charron-Bost
Ecole Polytechnique – Palaiseau, FR
- Frederic Chazal
INRIA Saclay – Orsay, FR
- Frederick R. Cohen
University of Rochester, US
- Armindo Emanuel Costa
University of Warwick, GB
- Carole Delporte
University Paris-Diderot, FR
- Jean-Marie Droz
Universität Bremen, DE
- Herbert Edelsbrunner
IST Austria – Klosterneuburg, AT
- Ulrich Fahrenberg
IRISA / INSA – Rennes, FR
- Lisbeth Fajstrup
Aalborg University, DK
- Michael Farber
University of Warwick, GB
- Hugues Fauconnier
University Paris-Diderot, FR
- Eva-Maria Feichtner
Universität Bremen, DE
- Dmitry Feichtner-Kozlov
Universität Bremen, DE
- Pierre Fraigniaud
University Paris-Diderot, FR
- Patrizio Frosini
University of Bologna, IT
- Eric Goubault
Centre d'Etudes Nucleaires de Saclay, FR
- Tobias Heindel
CEA – Gif sur Yvette, FR
- Maurice Herlihy
Brown Univ. – Providence, US
- Anil N. Hirani
Univ. of Illinois – Urbana, US
- Marc Jeanmougin
ENS - Paris, FR
- Mateusz Juda
Jagiellonian Univ. – Krakow, PL
- Matthew Kahle
Ohio State University, US
- Petr Kuznetsov
TU Berlin, DE
- Claudia Landi
University of Modena e Reggio Emilio, IT
- Frank H. Lutz
TU Berlin, DE
- Facundo Memoli
Stanford University, US
- Roy Meshulam
Technion – Haifa, IL
- Alessia Milani
Université Bordeaux, FR
- Samuel Mimram
Commissariat a l'Energie Atomique – Gif-sur-Yvette, FR
- Dmitriy Morozov
Stanford University, US
- Thomas Nowak
TU Wien, AT
- Ami Paz
Technion – Haifa, IL
- Sergio Rajsbaum
Universidad Nacional Autonoma – Mexico, MX
- Martin Raussen
Aalborg University, DK
- Srivatsan Ravi
TU Berlin, DE
- Matthieu Roy
LAAS – Toulouse, FR
- Primoz Skraba
Jozef Stefan Institute – Ljubljana, SI
- Christine Tasson
University Paris-Diderot, FR
- Corentin Travers
Université Bordeaux, FR
- Mikael Vejdemo-Johansson
University of St Andrews, GB
- Peter Widmayer
ETH Zürich, CH
- Krzysztof Ziemianski
University of Warsaw, PL