Collapsing non-idempotent intersection types

Thomas Ehrhard

CNRS, PPS, UMR 7126, Univ Paris Diderot, Sorbonne Paris Cité
F–75205 Paris, France

Abstract

We proved recently that the extensional collapse of the relational model of linear logic coincides with its Scott model, whose objects are preorders and morphisms are downwards closed relations. This result is obtained by the construction of a new model whose objects can be understood as preorders equipped with a realizability predicate. We present this model, which features a new duality, and explain how to use it for reducing normalization results in idempotent intersection types (usually proved by reducibility) to purely combinatorial methods. We illustrate this approach in the case of the call-by-value lambda-calculus, for which we introduce a new resource calculus, but it can be applied in the same way to many different calculi.

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1 Introduction

The relational model of linear logic (LL) has been introduced implicitly by Girard in [12] as a model of the λ-calculus and recognized only later as a model of LL by several authors independently. Its objects are plain sets and a morphism from X to Y is a subset of $X \times Y$. Often despised because it identifies many logical constructions of LL (most dramatically $X^\perp = X$), this model is nevertheless extremely interesting as it preserves many relevant information about programs: it is *quantitative* in the sense that the interpretation of functions allows to recover how many times an argument is used to compute a given result. For that reason, computation time can be recovered from the interpretation of terms, as shown in [3, 4]. Arbitrary fixpoints of types are quite easy to compute and therefore many interesting relational models of the pure λ-calculus and of its variants and extensions are available: call-by-value (cbv) λ-calculus, $\lambda\mu$- calculus etc. Also, the relational model provides a natural interpretation of the differential and resource λ-calculi and LL [7, 8, 9, 21, 19].

Scott semantics is of course older. It has been recognized as a model of LL a few years after Girard's discovery of LL, by Michael Huth [13, 14] and independently by Glynn Winskel [22, 23]. In this model, types are interpreted as prime algebraic complete lattices, or equivalently as preorders, since any such lattice can be presented as the set of downwards closed subsets of a preorder. The Kleisli category associated with this model of LL is (equivalent to) the category of prime-algebraic complete lattices and Scott-continuous functions. This model forgets much more information about programs than the relational model: it is purely *qualitative* in the sense that the interpretation of a function tells which parts of the arguments are used to compute a given result, but not how many times they are used.

This difference between the relational model and the Scott model of LL materializes itself in the fact that the Kleisli category of the second model is well-pointed (intuitively: morphisms can be seen as functions), whereas the Kleisli category of the first model is not. We proved in [6] that the the second model is the *extensional collapse* of the first

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one. The extensional collapse logical relation is a partial equivalence relation (PER) which equates morphisms if they yield equal results when applied to equal arguments. We explain now briefly how we proved that the second model is the "quotient" of the first one by the extensional collapse PER.

An object in **Rel** (the relational model) is a plain set and an object in **Pol** (the preorder model) is a structure $S = (|S|, \leq_S)$ where $|S|$ is a set (the web) and $\leq_S$ is a preorder relation on $|S|$. We can define the Scott semantics of LL in such a way that the web of the **Pol** object interpreting a formula coincide with the **Rel** object (set) interpreting the same formula. Using this fact, we build a new model of LL whose objects – called preorder with projections (pop) – are pairs $E = (\langle E \rangle, \mathsf{D}(E))$ where $\langle E \rangle$ is a preorder and $\mathsf{D}(E)$ is a subset of $\mathcal{P}(|\langle E \rangle|)$ which satisfies a closure property defined by an orthogonality relation. This allows to define a PER on $\mathcal{P}(|\langle E \rangle|)$: u and v are E-equivalent if they both belong to $\mathsf{D}(E)$ and have the same $\langle E \rangle$-downwards closure. This PER coincides with the extensional collapse PER: given pops E and F, the pop $G = (E \Rightarrow F) = (!E \multimap F)$ is such that $w, w' \subseteq |\langle G \rangle|$ are G-equivalent iff, for any $u, u' \subseteq |\langle E \rangle|$, if u and u' are E-equivalent, then $w(u)$ and $w'(u')$ are F-equivalent (applications are computed in the relational model). The $\langle G \rangle$-downwards closure of w is a morphism in the Scott model, which represents the equivalence class of w. Since conversely any downwards closed subset of $|\langle G \rangle|$ belongs to $\mathsf{D}(G)$, the Scott model coincides with the extensional collapse of the relational model. The constructions of [6] allow also to extend this result to arbitrary fixpoints of types.

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Indeed, just as **Rel** and **Pol**, this new category **Pop** is a model of LL where all types have least fixpoints, for a suitable notion of inclusion between pops. In the present paper, we use this property to prove an adequacy result for a Scott model of the cbv λ-calculus, which is defined as the least fixpoint $\mathcal{U}_\mathsf{S}$ (for a suitable order relation on preorders) of the operation $S \mapsto !S \multimap !S$. We can solve the same domain equation in **Pop** and we get an object $\mathcal{U}_\mathsf{P}$ for which $\langle \mathcal{U}_\mathsf{P} \rangle = \mathcal{U}_\mathsf{S}$ and $\mathcal{U}_\mathsf{R} = |\langle \mathcal{U}_\mathsf{P} \rangle|$ satisfies $\mathcal{U}_\mathsf{R} = !\mathcal{U}_\mathsf{R} \multimap !\mathcal{U}_\mathsf{R}$ in **Rel**. Given a term M of the cbv λ-calculus, that we assume to be closed for simplicity, we can therefore compute its relational interpretation $[M]_\mathsf{R}$ which is a subset of $\mathcal{U}_\mathsf{R}$ *which belongs to* $\mathsf{D}(\mathcal{U}_\mathsf{P})$ and its Scott interpretation $[M]_\mathsf{S}$, which is a downwards closed subset of $\mathcal{U}_\mathsf{R} = |\langle \mathcal{U}_\mathsf{P} \rangle|$ (for the preorder relation of $\langle \mathcal{U}_\mathsf{P} \rangle = \mathcal{U}_\mathsf{S}$). By induction on M, and using crucially the properties of the model **Pop**, one proves that $[M]_\mathsf{S} = \downarrow [M]_\mathsf{R}$: this is an instance of the "extensional collapse property" of this model. Now, adequacy of $\mathcal{U}_\mathsf{R}$ for the cbv λ-calculus (that is: if $[M]_\mathsf{S} \neq \emptyset$ then M reduces to a value) can be proved purely combinatorially, introducing a cbv resource λ-calculus and the fact that **Rel** satisfies a version of the Taylor formula. If $[M]_\mathsf{S} \neq \emptyset$ then $[M]_\mathsf{R} \neq \emptyset$ since $[M]_\mathsf{S} = \downarrow [M]_\mathsf{R}$ and so M reduces to a value. Whereas the standard proofs of this kind of results for Scott semantics are based on reducibility (with the noticeable exception of [2]), the present approach provides a purely semantical reduction of this result to a combinatorial argument: all the reducibility argument has been encapsulated in the model **Pop**. This approach can be used in the same way for many different calculi (standard λ-calculus, PCF, $\lambda\mu$-calculus. . .).

This work can be understood as relating usual idempotent intersection typing systems – points in the **Pol** model can be seen as idempotent intersection types – with non-idempotent ones, which use points of the **Rel** model as types. We adopt this viewpoint in Sections 6.1 and 7.1 where we present the semantics of the cbv λ-calculus under consideration as typing systems. Actually, in both systems, types are pairs (p, q) where p and q are finite multisets of types but the systems differ by their typing rules. The type (p, q) could also nicely be written $p \multimap q$ and "intersection" corresponds to multiset concatenation.

Notations. We use $[a_1, \dots, a_n]$ for the multiset made of $a_1, \dots, a_n$, taking multiplicities into account. We use $[]$ for the empty multiset and standard algebraic notations such as $m + m'$ of $\sum_i m_i$ for sums of multisets. We use $|m|$ for the support of the multiset m, which is the set of elements which appear at least once in m.

2 Categorical semantics of LL in a nutshell

Our main reference for categorical models of LL is the survey paper [16].

Let $\mathcal{C}$ be a Seely category. We recall briefly that such a structure consists of a category $\mathcal{C}$, whose morphisms should be thought of as linear maps, equipped with a symmetric monoidal structure for which it is closed and $*$-autonomous wrt. a dualizing object $\perp$. The monoidal product (tensor product) is denoted as $\otimes$, the linear function space object from X to Y is denoted as $X \multimap Y$. We use $\mathsf{ev} \in \mathcal{C}((X \multimap Y) \otimes X, Y)$ for the linear evaluation morphism and $\lambda(f) \in \mathcal{C}(Z, X \multimap Y)$ for the "linear curryfication" of a morphism $f \in \mathcal{C}(Z \otimes X, Y)$. The dual object $X \multimap \perp$ is denoted as $X^\perp$. Given an object X of $\mathcal{C}$ and a permutation $f \in \mathfrak{S}_n$, we use σ_f for the induced automorphism of $X^{\otimes n}$ in $\mathcal{C}$; the operation $f \to \sigma_f$ is a group homomorphism from $\mathfrak{S}_n$ to the group of automorphisms of $X^{\otimes n}$ in $\mathcal{C}$.

We also assume that $\mathcal{C}$ is cartesian, with a cartesian product denoted as $\&$ and a terminal object $\top$. By $*$-autonomy, this implies that $\mathcal{C}$ is also cocartesian; we use $\oplus$ for the coproduct and 0 for the initial object. If $\mathcal{C}$ has cartesian products of all countable families $(X_i)_{i \in I}$ of objects, we say that it is *countably cartesian*, and in that case, $\mathcal{C}$ is also countably cocartesian. If finite sums and finite products coincide, then each hom-set has a canonical commutative monoid structure and all operations defined so far (composition, tensor product, linear curryfication) are linear wrt. this structure. In that case we say that $\mathcal{C}$ is *additive*. We say that it is *countably additive* if this property extends to countable sums and products, and in that case hom-sets have countable sums. The corresponding operations are denoted using the standard mathematical notations for sums.

Last, we assume that $\mathcal{C}$ is equipped with an endofunctor $!\_$ which has a structure of comonad (unit $\mathsf{d}_X \in \mathcal{C}(!X, X)$ called *dereliction*, multiplication $\mathsf{p}_X \in \mathcal{C}(!X, !!X)$ called *digging*). Moreover, this functor must be equipped with a monoidal structure which turns it into a symmetric monoidal functor from the symmetric monoidal category $(\mathcal{C}, \&)$ to the symmetric monoidal category $(\mathcal{C}, \otimes)$: the corresponding isomorphisms $\mathsf{m} : 1 \to !\top$ and $\mathsf{m}_{X,Y} : !X \otimes !Y \to !(X \mathbin{\&} Y)$ are often called *Seely isomorphisms*. An additional diagram, relating digging and the Seely isomorphisms is required, see [16].

2.1 Structural natural transformations

Using these structures, we can define a *weakening* natural transformation $\mathsf{w}_X \in \mathcal{C}(!X, 1)$ and a *contraction* natural transformation $\mathsf{c}_X \in \mathcal{C}(!X, !X \otimes !X)$ as follows. Since $\top$ is terminal, there is a canonical morphism $\mathsf{t}_X \in \mathcal{C}(X, \top)$ and we set $\mathsf{w}_X = \mathsf{m}^{-1}\, !\mathsf{t}_X$. Similarly, we have a diagonal natural transformation $\Delta_X \in \mathcal{C}(X, X \mathbin{\&} X)$ and we set $\mathsf{c}_X = \mathsf{m}_{X,X}^{-1}\, !\Delta_X$.

One can also prove that the Kleisli category $\mathcal{C}_!$ of the comonad $!\_$ is cartesian closed, with $\&$ as cartesian product and $!X \multimap Y$ as function space object: this is a categorical version of Girard's translation of intuitionistic logic into linear logic.

We use $\mathsf{c}_X^n : !X^{\otimes n} \to !X^{\otimes n} \otimes !X^{\otimes n}$ for the generalized contraction morphism which is defined as the composition $(!X)^{\otimes n} \xrightarrow{(\mathsf{c}_X)^{\otimes n}} (!X \otimes !X)^{\otimes n} \xrightarrow{\sigma_f} (!X)^{\otimes n} \otimes (!X)^{\otimes n}$ where $f \in \mathfrak{S}_{2n}$ is the bijection which maps $2k-1$ to k and $2k$ to $n+k$ (for $k = 1, \dots, n$).

Similarly, we define a generalized weakening morphism w_X^n as the composition of morphisms $(!X)^{\otimes n} \xrightarrow{(\mathsf{w}_X)^{\otimes n}} (1)^{\otimes n} \xrightarrow{\nu} 1$ where ν is the unique canonical isomorphism induced by the monoidal structure. Given $f \in \mathcal{C}((!X)^{\otimes n}, X)$, it is standard to define $f^! \in \mathcal{C}((!X)^{\otimes n}, !X)$, using the comonad and the monoidal structure of the functor $!\_$. This operation is usually called *promotion* in LL. Given two LL models $\mathcal{C}$ and $\mathcal{D}$, an *LL functor* from $\mathcal{C}$ to $\mathcal{D}$ is a functor $F : \mathcal{C} \to \mathcal{D}$ which preserves all the structure defined above. For instance, we must have $F(f \otimes g) = F(f) \otimes F(g)$, $F(\mathsf{p}_X) = \mathsf{p}_{F(X)}$ etc.

2.2 Weak differential LL models

The notion of categorical model recalled above allows to interpret standard classical linear logic. If one wishes to interpret differential constructs as well (in the spirit of the differential λ-calculus or of differential linear logic), more structure and hypotheses are required. Basically, we need:

- that the cartesian and cocartesian category $\mathcal{C}$ be additive
- and that the model be equipped with a *codereliction* natural transformation $\overline{\mathsf{d}}_X \in \mathcal{C}(X, !X)$ such that $\mathsf{d}_X \overline{\mathsf{d}}_X = \mathrm{Id}_X$.

More conditions are required if one wants to interpret the full differential λ-calculus of [7] or full differential LL as presented in e.g. [18]: these conditions are a categorical axiomatization of the usual chain rule of calculus, but this rule is not required in the present paper, see [10] for a complete axiomatization. When these additional conditions hold, we say that the chain rule holds in $\mathcal{C}$.

If $\mathcal{C}$ is a weak differential LL model, we can define a coweakening morphism $\overline{\mathsf{w}}_X \in \mathcal{C}(1, !X)$ and a cocontraction morphism $\overline{\mathsf{c}}_X \in \mathcal{C}(!X \otimes !X, !X)$ as we did for w_X and c_X. Similarly we also define $\overline{\mathsf{c}}_X^n \in \mathcal{C}((!X)^{\otimes n}, !X)$. Due to the naturality of $\overline{\mathsf{d}}_X$ we have $\mathsf{w}_X \overline{\mathsf{d}}_X = 0$ and $\mathsf{c}_X \overline{\mathsf{d}}_X = \overline{\mathsf{d}}_X \otimes \overline{\mathsf{w}}_X + \overline{\mathsf{w}}_X \otimes \overline{\mathsf{d}}_X$. We also define $\mathsf{d}_X^n = \mathsf{d}_X^{\otimes n} \mathsf{c}_X^n \in \mathcal{C}(!X, X^{\otimes n})$ and $\overline{\mathsf{d}}_X^n = \overline{\mathsf{c}}_X^n \overline{\mathsf{d}}_X^{\otimes n} \in \mathcal{C}(X^{\otimes n}, !X)$.

2.3 The Taylor formula

Let $\mathcal{C}$ be a weak differential LL model which is countably additive. Remember that each hom-set $\mathcal{C}(X, Y)$ is endowed with a canonical structure of commutative monoid in which countable families are summable. We assume moreover that these monoids are idempotent. This means that, if $f \in \mathcal{C}(X, Y)$, then $f + f = f$. We say that *the Taylor formula holds in* $\mathcal{C}$ if, for any morphism $f \in \mathcal{C}(X, Y)$, we have $!f = \sum_{n=0}^{\infty} \overline{\mathsf{d}}_Y^n f^{\otimes n} \mathsf{d}_X^n$

▶ Remark. If the idempotency condition does not hold in $\mathcal{C}$, one has to require the homsets to have a module structure over the rig of non-negative real numbers, and the Taylor condition must be written in the more familiar way $!f = \sum_{n=0}^{\infty} \frac{1}{n!} \overline{\mathsf{d}}_Y^n f^{\otimes n} \mathsf{d}_X^n$.

▶ Remark. If the chain rule holds in $\mathcal{C}$, the Taylor condition reduces to the particular case of identity morphisms: one has just to require that $\mathrm{Id}_{!X} = \sum_{n=0}^{\infty} \frac{1}{n!} \overline{\mathsf{d}}_X^n \mathsf{d}_X^n$ (in the non-idempotent case) or $\mathrm{Id}_{!X} = \sum_{n=0}^{\infty} \overline{\mathsf{d}}_X^n \mathsf{d}_X^n$ (in the idempotent case).

3 The extensional collapse

We present the extensional collapse construction developed in [6].

3.1 The relational model of LL

The model. The base category is **Rel**, the category of sets and relations. Identities are diagonal relations and composition is the standard composition of relations. In this category, the isomorphisms are the bijections. The symmetric monoidal structure is given by $1 = \{*\}$ (arbitrary singleton set) and $X \otimes Y = X \times Y$, we do not give the monoidal isomorphisms which are obvious. This symmetric monoidal category (SMC) is closed, with $X \multimap Y = X \times Y$ and $\mathsf{ev} = \{(((a,b),a),b) \mid a \in X \text{ and } b \in Y\}$. It is $*$-autonomous with dualizing object $\perp = 1$ so that $X^\perp = X$ up to an obvious isomorphism.

Rel is countably cartesian with $\&_{i\in I} X_i = \bigcup_{i\in I}\{i\} \times X_i$ (disjoint union) and projections $\pi_i = \{((i,a),a) \mid a \in X_i\}$. It is also countably additive with $\bigoplus_{i\in I} X_i = \&_{i\in I} X_i$. The sum of a countable family of elements of $\mathbf{Rel}(X,Y)$ is its union, so that hom-sets are idempotent monoids.

The exponential functor is given by $!X = \mathcal{M}_{\mathrm{fin}}(X)$ (finite multisets of elements of X) and, if $R \in \mathbf{Rel}(X,Y)$, one sets $!R = \{([a_1,\dots,a_n],[b_1,\dots,b_n]) \mid n \in \mathbb{N} \text{ and } (a_1,b_1),\dots,(a_n,b_n) \in R\}$. The Seely isomorphism $\mathsf{m} \in \mathbf{Rel}(1,!\top)$ is $\{(*,[])\}$ and the Seely natural isomorphism $\mathsf{m}_{X,Y} \in \mathbf{Rel}(!X \otimes !Y, !(X \& Y))$ is the bijection which maps $([a_1,\dots,a_n],[b_1,\dots,b_p])$ to $[(1,a_1),\dots,(1,a_n),(2,b_1),\dots,(2,b_p)])\}$. Dereliction is $\mathsf{d}_X \in \mathbf{Rel}(!X,X)$ defined by $\mathsf{d}_X = \{([a],a) \mid a \in X\}$ and digging is $\mathsf{p}_X \in \mathbf{Rel}(!X,!!X)$ defined by $\mathsf{p}_X = \{(m_1 + \cdots + m_k,[m_1,\dots,m_k]) \mid m_1,\dots,m_k \in !X\}$. As easily checked, weakening is given by $\mathsf{w}_X = \{([],*)\} \in \mathbf{Rel}(!X,1)$ and binary contraction is $\mathsf{c}_X = \{(m_1+m_2,(m_1,m_2))) \mid m_1,m_2 \in !X\}$.

This structure can also be extended to a weak differential LL model, codereliction being defined as $\overline{\mathsf{d}}_X = \{(a,[a]) \mid a \in X\} \in \mathbf{Rel}(X,!X)$. In this model, the Taylor formula holds as easily checked.

Fixpoints of types. Let $\mathbf{Rel}^\subseteq$ be the class of sets, ordered by inclusion. It is closed under arbitrary unions. A function $(\mathbf{Rel}^\subseteq)^n \to \mathbf{Rel}^\subseteq$ is continuous if it is monotone wrt. inclusion and preserves all directed lubs. Any continuous function $\Phi : \mathbf{Rel}^\subseteq \to \mathbf{Rel}^\subseteq$ admits a least fixpoint defined as usual as $\bigcup_{n\in\mathbb{N}} \Phi^n(\emptyset)$. All the LL constructions defined above are continuous functions.

3.2 The Scott model of LL

The model. A preordered set is a pair $S = (|S|, \leq_S)$ where $|S|$ is a countable set and $\leq_S$ is a transitive and reflexive binary relation on $|S|$. We denote as $\mathcal{I}(S)$ the set of all subsets of $|S|$ which are downwards closed wrt. the $\leq_S$ relation. We set $S^{\mathsf{op}} = (|S|, \geq_S)$. We use $S \times T$ for the product preorder. Scott semantics can also be presented as a model of LL. The base category is **Pol**, the category whose objects are preordered sets and where $\mathbf{Pol}(S,T) = \mathcal{I}(S^{\mathsf{op}} \times T)$. The identity morphism at S is $\mathrm{Id}_S = \{(a,a') \in |S| \times |S| \mid a' \leq_S a\}$. Composition is just the usual composition of relations.

▶ **Lemma 1.** *There is an order isomorphism from* $\mathbf{Pol}(S,T)$ *to the set of functions* $\mathcal{I}(S) \to \mathcal{I}(T)$ *which preserve arbitrary unions, ordered under the pointwise order. This isomorphism maps the relation* R *to the function* $\xi \mapsto R\,\xi = \{b \in |T| \mid \exists a \in \xi\ (a,b) \in R\}$.

This is quite easy to prove, and this mapping from relation to functions is functorial. We equip **Pol** with a symmetric monoidal structure, taking $1 = (\{*\},=)$ and $S \otimes T = S \times T$ (product preorder).

If two preorders S and S' are isomorphic as preorders through a bijection $\varphi : |S| \to |S'|$, then they are isomorphic in **Pol** by the relation $\{(a,a') \mid a' \leq_{S'} \varphi(a)\}$ but the converse is far from being true. In the first case we say that φ is a *strong isomorphism* from S to S'. The isomorphisms of the symmetric monoidal structure of **Pol** are the obvious strong ones. This SMC is closed, with $S \multimap T = S^{\mathsf{op}} \times T$ and linear evaluation $\mathsf{ev} \in \mathbf{Pol}((S \multimap T) \otimes S, T)$ given by $\mathsf{ev} = \{(((a', b), a), b') \mid a' \leq_S a \text{ and } b' \leq_T b\}$. **Pol** is $*$-autonomous with dualizing object $\bot = 1$, so that, up to an obvious strong isomorphism, $S^\bot = S^{\mathsf{op}}$. Observe that as in **Rel**, the cotensor product $⅋$ coincides with the tensor product $\otimes$; both categories **Rel** and **Pol** are compact closed.

Pol is countably cartesian: the cartesian product of a countable family $(S_i)_{i \in I}$ of preorders is $S = \&_{i\in I} S_i$ defined by $|S| = \bigcup_{i \in I}\{i\} \times |S_i|$ preordered as follows: $(i, a) \leq_S (j, b)$ if $i = j$ and $a \leq_{S_i} b$. The projections are $\pi_i = \{((i, a), a') \mid a' \leq_{S_i} a\}$. In particular, the terminal object is $(\emptyset, \emptyset)$. The category **Pol** is therefore also cocartesian, and it is countably additive with sums of morphisms defined as unions. We define the exponential functor by $!S = (\mathcal{M}_{\mathrm{fin}}(|S|), \leq_{!S})$ where the preorder is defined by $p \leq_{!S} q$ if $\forall a \in |p|\, \exists b \in |q|\; a \leq_S b$. Given $R \in \mathbf{Pol}(S, T)$, we set $!R = \{(p, q) \in |!S| \times |!T| \mid \forall b \in |q|\, \exists a \in |p|\; (a, b) \in R\}$ and it is quite easy to check that $!R \in \mathbf{Pol}(!S, !T)$, and that this operation is functorial.

▶ Remark. The crucial point in the definition of $!S$ is that $\leq_{!S}$ does not take multiplicities into account. Indeed, there is another possible definition, for which we use another notation: we can set $!_{\mathsf{s}}S = (\mathcal{P}_{\mathrm{fin}}(|S|), \leq_{!_{\mathsf{s}}S})$, with preorder defined just as above: $\mu \leq_{!_{\mathsf{s}}S} \nu$ if $\forall a \in \mu\, \exists b \in \nu\; a \leq_S b$. The preorders $!S$ and $!_{\mathsf{s}}S$ are isomorphic (but not strongly isomorphic) through the relation $\mathsf{e}_S \in \mathbf{Pol}(!S, !_{\mathsf{s}}S)$ defined by $\mathsf{e}_S = \{(p, \mu) \mid \forall a \in \mu\, \exists a' \in |p|\; a \leq_S a'\}$. The important point is that this natural isomorphism is compatible with all the structures of both exponentials, so that the models defined by these exponentials are equivalent. We prefer to use the multiset-based construction to present the model because it is closer to the exponential of the relational model – this simplifies greatly the presentation of the extensional collapse as we shall see – but keep in mind that we could give the same definitions with the other version.

The Seely isomorphism $\mathsf{m} \in \mathbf{Pol}(1, !\top)$ is $\{(*, [])\}$ and the Seely natural isomorphism $\mathsf{m}_{S_1,S_2} \in \mathbf{Pol}(!S_1 \otimes !S_2, !(S \,\&\, T))$ is $\{((p_1, p_2), q) \mid (i, a) \in |q| \Rightarrow \exists a' \in |p_i|\; a \leq_{S_i} a'\}$. Dereliction $\mathsf{d}_S \in \mathbf{Pol}(!S, S)$ is $\mathsf{d}_S = \{(p, a) \mid \exists a' \in |p|\; a \leq_S a'\}$ and digging $\mathsf{p}_S \in \mathbf{Pol}(!S, !!S)$ is $\mathsf{p}_S = \{(p, [p_1, \dots, p_n]) \mid i \in \mathbb{N} \text{ and } \forall i\; p_i \leq_{!S} p\}$. As easily checked, weakening is given by $\mathsf{w}_S = \{(p, *) \mid p \in |!S|\} \in \mathbf{Rel}(!S, 1)$ and binary contraction is $\mathsf{c}_S = \{(p, (p_1, p_2))) \mid p, p_1, p_2 \in |!S|\; p_1 \leq_{!S} p \text{ and } p_2 \leq_{!S} p\}$. Unlike the relational model, this structure cannot be extended into a weak differential LL model.

▶ Proposition 2. There is no natural transformation $\overline{\mathsf{d}}_S \in \mathbf{Pol}(S, !S)$ such that $\mathsf{d}_S\, \overline{\mathsf{d}}_S = \mathrm{Id}_S$.

Proof. We prove first that necessarily $\overline{\mathsf{d}}_S = \{(a, p) \in |S| \times |!S| \mid p \leq_{!S} [a]\}$. First, let $(a, p) \in \overline{\mathsf{d}}_S$. Let $a' \in |p|$. By definition of d_S, we have $(p, a') \in \mathsf{d}_S$, and hence $(a, a') \in \mathsf{d}_S\, \overline{\mathsf{d}}_S = \mathrm{Id}_S$. Therefore $a' \leq_S a$ and hence $p \leq_{!S} [a]$. Conversely, let $a \in |S|$. We have $(a, a) \in \mathrm{Id}_S$ and therefore there exists p such that $(a, p) \in \overline{\mathsf{d}}_S$ and $(p, a) \in \mathsf{d}_S$. By the second property, we can find $a' \in |p|$ such that $a \leq_S a'$. We have $[a] \leq_{!S} p$ and $(a, p) \in \overline{\mathsf{d}}_S \in \mathbf{Pol}(S, !S)$. Therefore $(a, [a]) \in \overline{\mathsf{d}}_S$. It follows that, for any p such that $p \leq_{!S} [a]$, one has $(a, p) \in \overline{\mathsf{d}}_S$.

Let $S = (\{0\}, =)$ and $T = (\{1, 2\}, =)$. Let $R = \{(0, 1), (0, 2)\}$, we have $R \in \mathbf{Pol}(S, T)$. Observe that $([0], [1, 2]) \in !R$ (warning: this is of course not true in **Rel**) so that $(0, [1, 2]) \in !R\, \overline{\mathsf{d}}_S$. But there is no $b \in |T|$ such that $(b, [1, 2]) \in \overline{\mathsf{d}}_T$ and hence we do not have $(0, [1, 2]) \in \overline{\mathsf{d}}_T\, R$, and this shows that $\overline{\mathsf{d}}_S$ is not a natural transformation. ◻

Fixpoints of types. Let S and T be preorders, we write $S \subseteq T$ if $|S| \subseteq |T|$ and, for any $a, a' \in |S|$, one has $a \leq_S a'$ iff $a \leq_T a'$. This is an order relation on the class of preorders and we use $\mathbf{Pol}^{\subseteq}$ for this partially ordered class. It is clear that any countable directed family in $\mathbf{Pol}^{\subseteq}$ has a lub and that all the LL constructions presented above are continuous. It is also clear that any continuous function $\Phi : \mathbf{Pol}^{\subseteq} \to \mathbf{Pol}^{\subseteq}$ has a least fixpoint.

3.3 The collapsing model of LL

Our last model combines the two models above. It is based on a new duality.

The model. Let S be a preorder and let $u, u' \subseteq |S|$. We write $u \perp u'$ if $u \cap u' = \emptyset \Rightarrow (\downarrow_S u) \cap u' = \emptyset$.

Observe that $(\downarrow_S u) \cap u' = \emptyset$ holds iff $(\downarrow_S u) \cap (\downarrow_{S^{\mathsf{op}}} u') = \emptyset$ so that $u \perp u'$ holds relatively to S iff $u' \perp u$ holds relatively to S^{op}. Given $D \subseteq \mathcal{P}(|S|)$, we define $D^{\perp(S)} \subseteq \mathcal{P}(|S|)$ by $D^{\perp(S)} = \{u' \subseteq |S| \mid \forall u \in D \; u \perp u'\}$. It is clear that $D \subseteq D^{\perp(S)\perp(S^{\mathsf{op}})}$ and that $D_1 \subseteq D_2 \Rightarrow {D_2}^{\perp(S)} \subseteq {D_1}^{\perp(S)}$, so that $D^{\perp(S)} = D^{\perp(S)\perp(S^{\mathsf{op}})\perp(S)}$. Observe that $\mathcal{I}(S^{\mathsf{op}}) \subseteq D^{\perp(S)} \subseteq \mathcal{P}(|S|)$ so that, when D is "closed" in the sense that $D = D^{\perp(S)\perp(S^{\mathsf{op}})}$, one has $\mathcal{I}(S) \subseteq D \subseteq \mathcal{P}(|S|)$.

The objects of the model are called *preorders with projections* (pop) and are pairs $E = (\langle E\rangle, \mathsf{D}(E))$ where $\langle E\rangle$ is a preorder and $\mathsf{D}(E) \subseteq \mathcal{P}(|\langle E\rangle|)$ satisfies $(\mathsf{D}(E))^{\perp(\langle E\rangle)\perp(\langle E\rangle^{\mathsf{op}})} \subseteq \mathsf{D}(E)$, that is $(\mathsf{D}(E))^{\perp(\langle E\rangle)\perp(\langle E\rangle^{\mathsf{op}})} = \mathsf{D}(E)$. If E is a pop, we set $E^\perp = (\langle E\rangle^{\mathsf{op}}, (\mathsf{D}(E))^{\perp(\langle E\rangle)})$. Let E and F be pops. One defines $E \otimes F$ by $\langle E \otimes F\rangle = \langle E\rangle \times \langle F\rangle$ and $\mathsf{D}(E \otimes F) = \{u \times v \mid u \in \mathsf{D}(E) \text{ and } v \in \mathsf{D}(F)\}^{\perp(\langle E\rangle\times\langle F\rangle)\perp(\langle E^\perp\rangle\times\langle F^\perp\rangle)}$. Let $E \multimap F = (E \otimes F^\perp)^\perp$.

▶ **Lemma 3.** *Let $R \subseteq |\langle E \multimap F\rangle|$. One has $R \in \mathsf{D}(E \multimap F)$ iff any of the following equivalent conditions holds.*

- *If $u \in \mathsf{D}(E)$ and $v' \in \mathsf{D}(F^\perp)$, then $R \cap (u \times v') = \emptyset \Rightarrow R \cap (\downarrow_{\langle E\rangle} u \times \uparrow_{\langle F\rangle} v') = \emptyset$.*
- *If $u \in \mathsf{D}(E)$, then $R\,u \in \mathsf{D}(F)$ and $R{\downarrow}u \subseteq {\downarrow}(R\,u)$.*
- *If $u \in \mathsf{D}(E)$, then $R\,u \in \mathsf{D}(F)$ and ${\downarrow}(R\,u) = (\downarrow_{\langle E\rangle\multimap\langle F\rangle} R)\,({\downarrow}u)$.*

Proof. See [6]. ◻

To define the category **Pop** of pops, we set $\mathbf{Pop}(E, F) = \mathsf{D}(E \multimap F)$. By Lemma 3 $\mathrm{Id}_E = \{(a, a) \mid a \in |\langle E\rangle|\} \in \mathbf{Pop}(E, E)$, and if $Q \in \mathbf{Pop}(E, F)$ and $P \in \mathbf{Pop}(F, G)$, then $P\,Q \in \mathbf{Pop}(E, G)$ and so identities and composition of **Pop** are defined as in **Rel**. This category is $*$-autonomous: we have already defined the tensor product on objects. On morphisms, it is defined just as in **Rel**. The internal hom object $E \multimap F$ has also been defined above, and the linear evaluation relation is defined as in **Rel** again. Of course, one has to check carefully that all these relations are **Pop** morphisms, this is done in [6]. Notice that, as shown in that paper, this category is not compact closed. The category **Pop** is countably cartesian, $E = \&_{i\in I} E_i$ is defined by $\langle E\rangle = \&_{i\in I}\langle E_i\rangle$ and $w \subseteq \mathsf{D}(E)$ iff $\pi_i\, w \in \mathsf{D}(E_i)$ for each $i \in I$ (where π_i is the ith projection in the relational model). The projections morphism in **Pop** are those of the relational model. The category **Pop** is therefore also countably cocartesian, and one checks easily that it is countably additive.

We define now the exponential $!E$ of a pop E. One sets $\langle !E\rangle = !\langle E\rangle$ and therefore, we have $|\langle !E\rangle| = !|\langle E\rangle| = \mathcal{M}_{\mathrm{fin}}(|\langle E\rangle|)$ by our definition of $!E$ based on multisets and not on sets, see the remark in Section 3.2. We set $\mathsf{D}(!E) = \{u^! \mid u \in \mathsf{D}(E)\}^{\perp(\langle !E\rangle)\perp(\langle !E\rangle^{\mathsf{op}})}$, where $u^! = \mathcal{M}_{\mathrm{fin}}(u)$. Here is the main tool for dealing with this construction. See [6] for the proof.

▶ Proposition 4. Let E and F be pops and let $R \in \mathbf{Rel}(|\langle !E\rangle|, |\langle F\rangle|)$. One has $R \in \mathbf{Pop}(!E, F)$ iff, for any $u \in \mathsf{D}(E)$

- $R\,u^! \in \mathsf{D}(F)$
- $R\,(\downarrow_{\langle E\rangle} u)^! \subseteq \downarrow_{\langle F\rangle}(R\,u^!)$.

The Seely isomorphisms, and the dereliction and digging natural transformations are defined exactly as in **Rel**.

Fixpoints of types. Let E and F be pops. We write $E \subseteq F$ if $\langle E\rangle \subseteq \langle F\rangle$, $\mathsf{D}(E) \subseteq \mathsf{D}(F)$ and, for any $v \in \mathsf{D}(F)$, one has $v \cap |\langle E\rangle| \in \mathsf{D}(E)$ and $\downarrow_{\langle F\rangle} v \cap |\langle E\rangle| \subseteq \downarrow_{|\langle E\rangle|}(v \cap |\langle E\rangle|)$. This is an order relation on the class of preorders with projections, and we write $\mathbf{Pop}^{\subseteq}$ for the corresponding partially ordered class. It is shown in [6] that this partially ordered class is complete (all directed lubs exist) and we define as usual the notion of continuous function $(\mathbf{Pop}^{\subseteq})^n \to \mathbf{Pop}^{\subseteq}$, one checks that all constructions of linear logic are continuous functions, and that any continuous function $\Phi : \mathbf{Pop}^{\subseteq} \to \mathbf{Pop}^{\subseteq}$ admits a least fixpoint $\bigcup_{n=0}^{\infty} \Phi^n(\emptyset)$.

Forgetful LL functors. There is an obvious functor $\rho : \mathbf{Pop} \to \mathbf{Rel}$ defined on objects by $\rho(E) = |\langle E\rangle|$ and which is the identity on morphisms. With a preorder with projection E, we can also associate a preorder $\sigma(E) = \langle E\rangle$. This operation is extended to morphisms as follows: let $R \in \mathbf{Pop}(E, F)$, we set $\sigma(R) = \downarrow_{\langle E\rangle \multimap \langle F\rangle} R$.

▶ **Lemma 5.** *Both ρ and σ are LL functors.*

The proof can be found in [6]. The statement concerning ρ is straightforward. Concerning σ, LL functoriality is made possible by the presence of the sets $\mathsf{D}(E)$. For instance functoriality results directly from Lemma 3 and Lemma 1. But notice that, given preorders S, S' and S'' and *arbitrary* relations $R \in \mathbf{Rel}(|S|, |S'|)$ and $R' \in \mathbf{Rel}(|S'|, |S''|)$, the inclusion $(\downarrow_{S' \multimap S''} R')\,(\downarrow_{S \multimap S'} R) \subseteq \downarrow_{S \multimap S''}(R'\, R)$ does not hold in general.

▶ **Lemma 6.** *When restricted to inclusions, ρ induces a continuous function $\mathbf{Pop}^{\subseteq} \to \mathbf{Rel}^{\subseteq}$ and σ induces a continuous function $\mathbf{Pop}^{\subseteq} \to \mathbf{Pol}^{\subseteq}$.*

4 The cbv λ-calculus

Our syntax for the cbv λ-calculus is a slight modification of the ordinary λ-calculus syntax.

- If V is a value, then $\langle V\rangle$ is a term;
- if M and N are terms, then $M\,N$ is a term;
- if x is a variable, then x is a value;
- if M is a term and x is a variable, then $\lambda x\, M$ is a value.

We use Λ_{t} for the set of terms, Λ_{v} for the set of values and Λ_{e} for the disjoint union of these two sets, whose elements will be called expressions and denoted with letters $P, Q, \dots$. As usual, expressions are considered up to α-equivalence.

Reduction relations. One can define a general reduction relation β_{V} for this calculus: it is the contextual closure of the basic reduction rule $\langle \lambda x\, M\rangle\,\langle V\rangle\ \beta_{\mathsf{V}}\ M\,[V/x]$. We use β_{V}^* for the transitive closure of β_{V}. We also define a *weak* reduction relation $\hat\beta_{\mathsf{V}}$ which is included in β_{V} and which consists in reducing only redexes not occurring inside a value (that is, under a λ). It is defined by the following rules.

$$\frac{}{\langle \lambda x\, M\rangle\,\langle V\rangle\ \hat\beta_{\mathsf{V}}\ M\,[V/x]} \qquad \frac{M\ \hat\beta_{\mathsf{V}}\ M'}{M\,N\ \hat\beta_{\mathsf{V}}\ M'\,N} \qquad \frac{N\ \hat\beta_{\mathsf{V}}\ N'}{M\,N\ \hat\beta_{\mathsf{V}}\ M\,N'}$$

5 Linear-logic based models

Let $\mathcal{C}$ be an LL model. We present here a general notion of model for the cbv λ-calculus $\mathcal{C}$, which corresponds to a translation of intuitionistic logic into LL alluded to by Girard in [11] and called by him "boring". It is compatible with the translation of the cbv λ-calculus into LL given in [15] and with other notions of model such as [20] and of course with [17] if one keeps in mind that the functor "!" defines a strong monad on the Kleisli category $\mathcal{C}_!$.

A $\mathcal{C}$-model of cbv as a triple $(U, \mathsf{app}, \mathsf{lam})$ where U is an object of $\mathcal{C}$, $\mathsf{app} \in \mathcal{C}(U, !U \multimap !U)$ and $\mathsf{lam} \in \mathcal{C}(!U \multimap !U, U)$ are such that $\mathsf{app}\,\mathsf{lam} = \mathrm{Id}_{!U \multimap !U}$.

Given an expression P and a sequence of variables $\vec{x} = (x_1, \dots, x_n)$ adapted to P (this means that the sequence is repetition-free and contains all the free variables of P), we define $[P]^{\vec{x}} \in \mathcal{C}((!U)^{\otimes n}, X)$ where $X = U$ if P is a value and $X = !U$ if P is a term. The definition is by induction on P, and we consider first the cases where P is a term.

Assume first that $P = \langle V \rangle$. By inductive hypothesis we have $[V]^{\vec{x}} : (!U)^{\otimes(n)} \to U$, and we set $[P]^{\vec{x}} = ([V]^{\vec{x}})^! : (!U)^{\otimes(n)} \to !U$. Assume next that $P = M\,N$. By inductive hypothesis, we have $[M]^{\vec{x}}, [N]^{\vec{x}} \in \mathcal{C}((!U)^{\otimes n}, !U)$. Therefore $\mathsf{app}\,\mathsf{d}_U\,[M]^{\vec{x}} \in \mathcal{C}((!U)^{\otimes n}, !U \multimap !U)$. So we set $[P]^{\vec{x}} = \mathsf{ev}\,((\mathsf{app}\,\mathsf{d}_U\,[M]^{\vec{x}}) \otimes [N]^{\vec{x}})\,\mathsf{c}_U^n \in \mathcal{C}((!U)^{\otimes n}, !U)$.

Now we interpret values. Assume first that P is a variable, so that $P = x_i$ for an uniquely determined $i \in \{1, \dots, n\}$. Then we set $[M]^{\vec{x}} = \mathsf{w}_U^{\otimes(i-1)} \otimes \mathsf{d}_U \otimes \mathsf{w}_U^{\otimes(n-i)} : (!U)^{\otimes n} \to 1^{\otimes(i-1)} \otimes U \otimes (1)^{\otimes(n-i)} \simeq U$ (we keep this isomorphism implicit). Assume last that $P = \lambda x\,M$. We can assume that x does not occur in $\vec{x}$. By inductive hypothesis, we have $[M]^{\vec{x},x} \in \mathcal{C}((!U)^{\otimes n} \otimes !U, !U)$ and hence $\lambda([M]^{\vec{x},x}) \in \mathcal{C}((!U)^{\otimes n}, !U \multimap !U)$ and we set $[P]^{\vec{x}} = (\mathsf{lam}\,\lambda([P]^{\vec{x},x})) \in \mathcal{C}((!U)^{\otimes n}, U)$.

▶ **Lemma 7** (Substitution Lemma). *Let P be an expression, x a variable and V a value. Let $\vec{x}$ which does not contain x, is adapted to V and such that $\vec{x}, x$ is adapted to E. We have $[P\,[V/x]]^{\vec{x}} = [P]^{\vec{x},x}\,((!U)^{\otimes n} \otimes ([V]^{\vec{x}})^!)\,\mathsf{c}_U^n$ where n is the length of $\vec{x}$.*

▶ **Theorem 8.** *Let $\vec{x}$ be adapted to the expressions P and P' and assume that $P\ \beta_{\mathsf{V}}\ P'$. Then $[P]^{\vec{x}} = [P']^{\vec{x}}$.*

6 A relational model and the associated type system

Let $\Phi_{\mathsf{R}} : \mathbf{Rel}^{\subseteq} \to \mathbf{Rel}^{\subseteq}$ be the continuous function defined by $\Phi_{\mathsf{R}}(X) = !X \multimap !X$. Let $\mathcal{U}_{\mathsf{R}}$ be its least fixpoint, then we have $\mathcal{U}_{\mathsf{R}} = !\mathcal{U}_{\mathsf{R}} \multimap !\mathcal{U}_{\mathsf{R}}$ so that $\mathcal{U}_{\mathsf{R}}$ is a $\mathbf{Rel}$-model of cbv with $\mathsf{app} = \mathsf{lam} = \mathrm{Id}$. An element of $\mathcal{U}_{\mathsf{R}}$ is a pair (p, q) where p and q are finite multisets of elements of $\mathcal{U}_{\mathsf{R}}$. The simplest of these elements is $\varepsilon = ([], [])$, here is another one: $([\varepsilon, \varepsilon, ([\varepsilon], [])], [\varepsilon])$.

6.1 Non-idempotent intersection types

We introduce a typing system for deriving judgments of shape $\Gamma \vdash M : m$ where M is a term, $m \in !\mathcal{U}_{\mathsf{R}}$ and Γ is a context (that is, a finite function from variables to $!\mathcal{U}_{\mathsf{R}}$) and judgments of shape $\Gamma \vdash V : a$ where V is a value and $a \in \mathcal{U}_{\mathsf{R}}$. The sum of contexts $\Gamma + \Delta$ is defined pointwise (using the sum of multisets), when Γ and Δ have the same domain. A context Γ is often written $\Gamma = (x_1 : m_1, \dots, x_n : m_n)$ where the x_i's are pairwise distinct variables and $m_1, \dots, m_n \in !\mathcal{U}_{\mathsf{R}}$. The typing rules for terms are

$$\frac{\Gamma \vdash M : [(p, q)] \qquad \Delta \vdash N : p}{\Gamma + \Delta \vdash M\,N : q} \qquad \frac{\Gamma_1 \vdash V : a_1 \qquad \cdots \qquad \Gamma_k \vdash V : a_k}{\Gamma_1 + \cdots + \Gamma_k \vdash \langle V \rangle : [a_1, \dots, a_k]}$$

The second rule conveys the intuition that $[a_1, \dots, a_k]$ represents the intersection of types $a_1, \dots, a_n$. The typing rules for values are

$$\frac{}{x_1 : [], \dots, x_n : [], x : [a] \vdash x : a} \qquad \frac{\Gamma, x : p \vdash M : q}{\Gamma \vdash \lambda x\, M : (p, q)}$$

▶ Proposition 9. Let P be an expression and let $\vec{x} = (x_1, \dots, x_n)$ be a list of variables adapted to P. Let $\vec{p} \in (!\mathcal{U}_\mathsf{R})^n$ and let $\alpha \in X$ (where $X = \mathcal{U}_\mathsf{R}$ if P is a value and $X = !\mathcal{U}_\mathsf{R}$ if P is a term). Then one has $(\vec{p}, \alpha) \in [P]^{\vec{x}}_\mathsf{R}$ iff the typing judgment $x_1 : p_1, \dots, x_n : p_n \vdash P : \alpha$ is derivable.

The proof is a simple verification, by induction on the structure of P.

6.2 A CBV resource calculus

We introduce a resource calculus whose terms can be used to denote typing derivations in the typing system described above.

6.2.1 Notation. Given a finite family $(a_i)_{i \in I}$ and a predicate P on I, we use $[a_i \mid P(i)]$ for the multiset whose elements are the a_i's such that $P(i)$ holds, taking multiplicities into account.

6.2.2 Syntax. We describe first the syntax of our resource calculus.

- If $v_1, \dots, v_n$ are simple values, then $\langle v_1, \dots, v_n \rangle$ is a simple term;
- if s and t are simple terms, then $s\, t$ is a simple term;
- if x is a variable, then x is a simple value;
- if x is a variable and s is a simple term, then $\lambda x\, s$ is a simple value.

Terms are sets of simple terms, and values are defined similarly. We speak of (simple) expressions when we don't want to be specific. We write these sets as a sums to insist on the algebraic flavor of the semantical background. The above syntactic constructs are extended to non simple expressions, by linearity. For instance, if $v = \sum_{i \in I} v_i$ and $w = \sum_{j \in J} w_j$ are values (the summands being simple), the expression $\langle v, w \rangle$ denotes $\sum_{i \in I, j \in J} \langle v_i, w_j \rangle$. And if $s = \sum_{i \in I} s_i$ is a term, then $\lambda x\, s$ denotes $\sum_{i \in I} \lambda x\, s_i$, which is a value.

Given a simple expression e and simple values $v_1, \dots, v_n$, we define the linear substitution $\partial_x(e; v_1, \dots, v_n)$, which is an expression of the same kind as e, by

$$\partial_x(e; v_1, \dots, v_n) = \begin{cases} \sum_{f \in \mathfrak{S}_n} & e\left[v_1/x_{f(1)}, \dots, v_n/x_{f(n)}\right] \\ & \text{if } n = \deg_x e \\ 0 & \text{otherwise} \end{cases}$$

where $\deg_x e$ is the number of free occurrences of x in e and $x_1, \dots, x_n$ are these occurrences (in the case $n = \deg_x e$).

6.2.3 Reduction rules. We can give now the reduction rules of the calculus. We define a reduction relation denoted as δ from simple expressions to *generally non simple* expressions by the following rules.

$$\frac{}{\langle \lambda x\, s \rangle \, \langle v_1, \dots, v_n \rangle \; \delta \; \partial_x(s; v_1, \dots, v_n)}$$

$$\frac{}{\langle v_1, \dots, v_n \rangle \, t \; \delta \; 0} \text{ if } n \neq 1 \qquad \frac{s \; \delta \; s'}{\lambda x\, s \; \delta \; \lambda x\, s'}$$

$$\frac{s \; \delta \; s'}{s\, t \; \delta \; s'\, t} \qquad \frac{t \; \delta \; t'}{s\, t \; \delta \; s\, t'} \qquad \frac{v \; \delta \; v'}{\langle v, v_1, \dots, v_n \rangle \; \delta \; \langle v', v_1, \dots, v_n \rangle}$$

This reduction can be extended to non simple expressions, but this is not needed here and is postponed to a longer version of this paper.

One defines a size function on simple expressions by $\|x\| = 0$, $\|\lambda x\, s\| = 1 + \|s\|$, $\|s\, t\| = \|s\| + \|t\|$ and $\|\langle v_1, \dots, v_k\rangle\| = \sum_{j=1}^{k} \|v_j\|$. In other words $\|e\|$ is the number of of λ's in e.

▶ **Lemma 10.** *There is no infinite sequence $(e_i)_{i\in\mathbb{N}^+}$ of simple expressions such that, for each i, $e_i\ \delta\ e'$ with $e_{i+1} \in e'$.*

Proof. $\|e_{i+1}\| < \|e_i\|$. ◻

6.3 Categorical denotational semantics

Let U be a $\mathcal{C}$-model of cbv, where we assume moreover that $\mathcal{C}$ is a weak differential LL model which is countably additive and where hom-sets have idempotent sums. We show how to interpret the cbv resource calculus in such a structure.

We introduce a convenient notation. Let $g_1, \dots, g_k \in \mathcal{C}((!U)^{\otimes n}, U)$. We set $\langle g_1, \dots, g_k\rangle = \bar{\mathsf{d}}_U^k\, (g_1 \otimes \cdots \otimes g_k)\, \mathsf{c}_U^{n,k}$, where $\mathsf{c}_U^{n,k} \in \mathcal{C}((!U)^{\otimes n}, ((!U)^{\otimes n})^{\otimes k})$ is an obvious generalization of c_U^n.

Given a simple expression e and an adapted sequence of variables $\vec{x}$, we define $[e]^{\vec{x}} \in \mathcal{C}((!U)^{\otimes n}, X)$ where $X = U$ if e is a value and $X = !U$ if e is a term. The definition is by induction on e. For the syntactical constructs which are similar to those of the cbv λ-calculus (namely: variables, application and abstraction), the interpretation is the same as in Section 5. To complete the definition we have just to define the semantics of $\langle v_1, \dots, v_k\rangle$. By inductive hypothesis we have defined $g_j = [v_j]^{\vec{x}} \in \mathcal{C}((!U)^{\otimes n}, U)$ and we set $[\langle v_1, \dots, v_k\rangle]^{\vec{x}} = \langle g_1, \dots, g_k\rangle$. If e is an expression, that is a set of simple expressions $e = \sum_{i\in I} e_i$ and a list of variables $\vec{x}$ adapted to all x_i's, we set $[e]^{\vec{x}} = \sum_{i\in I} [e_i]^{\vec{x}}$ which is well defined because we have assumed that the sum of morphisms is idempotent in $\mathcal{C}$.

▶ **Lemma 11.** *If $e\ \delta\ e'$ and $\vec{x}$ is adapted to e and e', then $[e]^{\vec{x}} = [e']^{\vec{x}}$.*

Proof. It suffices to prove the result in the case where e is simple, by induction on e. The proof uses the following property of linear substitution wrt. the interpretation (substitution lemma). Let e be a simple expression and $v_1, \dots, v_k$ be simple values. Let $\vec{x}, x$ a sequence of variable adapted to e and to all v_j's. Let n be the length of $\vec{x}$. Then we have

$$[\partial_x(e; v_1, \dots, v_k)]^{\vec{x}} = [e]^{\vec{x},x}\, (!U^{\otimes n} \otimes \left\langle [v_1]^{\vec{x}}, \dots, [v_k]^{\vec{x}} \right\rangle)\, \mathsf{c}_U^n$$

and this is proved by a simple induction on e. ◻

For any expression P of the cbv λ-calculus, we define a set $\mathcal{T}(P)$ of simple expressions by induction.

$$\mathcal{T}(x) = \{x\} \qquad \mathcal{T}(\lambda x\, M) = \{\lambda x\, s \mid s \in \mathcal{T}(M)\}$$
$$\mathcal{T}(M\, N) = \{s\, t \mid s \in \mathcal{T}(M) \text{ and } t \in \mathcal{T}(N)\}$$
$$\mathcal{T}(\langle V\rangle) = \{\langle v_1, \dots, v_k\rangle \mid k \in \mathbb{N} \text{ and } \forall i\, v_i \in \mathcal{T}(V)\}\,.$$

Observe that the set $\mathcal{T}(P)$ is infinite as soon as P has a subterm of shape $\langle V\rangle$.

▶ **Lemma 12.** *Let P be an expression and let V be a value. Let $e \in \mathcal{T}(P)$ and $v_1, \dots, v_k \in \mathcal{T}(V)$. Then $\partial_x(e; v_1, \dots, v_k) \subseteq \mathcal{T}(P\,[V/x])$.*

Proof. Easy induction on P. ◻

▶ **Lemma 13.** *If the Taylor formula holds in $\mathcal{C}$ then for any expression P and any $\vec{x}$ adapted to P we have $[P]^{\vec{x}} = \sum_{e\in\mathcal{T}(P)}[e]^{\vec{x}}$.*

Proof. Easy induction on P. ◻

6.4 Adequacy in Rel

▶ **Lemma 14.** *Let P, P' be expressions and let $e \in \mathcal{T}(P)$. If $P\ \hat{\beta}_{\mathsf{V}}\ P'$ then there exists $e' \subseteq \mathcal{T}(P')$ such that $e\ \delta\ e'$.*

Proof. Simple inspection, using Lemma 12. ◻

One has to be careful when using this lemma because, using the notations of the lemma, nothing prevents the expression e' – which is not simple in general – from being empty.

▶ **Theorem 15.** *Let P be an expression. Let $\vec{x}$ be adapted to P and let n be the length of $\vec{x}$. Let $m_1, \dots, m_n \in\, !\mathcal{U}_{\mathsf{R}}$ and let $\alpha \in X$ where $X = \mathcal{U}_{\mathsf{R}}$ if P is a value and $X = !\mathcal{U}_{\mathsf{R}}$ if P is a term. If $x_1 : m_1, \dots, x_n : m_n \vdash P : \alpha$ then P is $\hat{\beta}_{\mathsf{V}}$ strongly normalizing.*

Proof. By Proposition 9, our hypothesis means that $(\vec{m}, \alpha) \in [P]^{\vec{x}}$. By Lemma 13 there exists $e \in \mathcal{T}(P)$ such that $(\vec{m}, \alpha) \in [e]^{\vec{x}}$. If $P\ \hat{\beta}_{\mathsf{V}}\ P'$ there then there exists $e' \subseteq \mathcal{T}(P')$ such that $e\ \delta\ e'$ by Lemma 14. By Lemma 11 we have $(\vec{m}, \alpha) \in [e']^{\vec{x}}$ and hence there exists $f \in e'$ such that $(\vec{m}, \alpha) \in [f]^{\vec{x}}$ (so e' is not empty!). Therefore, for any reduction $P = P_1\ \hat{\beta}_{\mathsf{V}}\ P_2 \dots \hat{\beta}_{\mathsf{V}}\ P_l$ we can find $e_1 \in \mathcal{T}(P_1), \dots, e_l \in \mathcal{T}(P_l)$ with $\|e_1\| > \|e_2\| > \dots > \|e_l\|$ and $(\vec{m}, \alpha) \in [e_i]^{\vec{x}}$ for each i. ◻

7 A Scott model and the associated type system

Let $\Phi_{\mathsf{S}} : \mathbf{Pol}^{\subseteq} \to \mathbf{Pol}^{\subseteq}$ be the continuous functions defined by $\Phi_{\mathsf{S}}(S) = !S \multimap !S$. Let $\mathcal{U}_{\mathsf{S}}$ be the least fixpoint of Φ_{S}, then $\mathcal{U}_S$ (equipped with two identity morphisms) is a $\mathbf{Pol}^{\subseteq}$-model of the cbv λ-calculus. We use $\leq$ for the both preorder relations $\leq_{\mathcal{U}_{\mathsf{S}}}$ and $\leq_{!\mathcal{U}_{\mathsf{S}}}$.

It is clear that $|\mathcal{U}_{\mathsf{S}}| = \mathcal{U}_{\mathsf{R}}$, so that an element of $|\mathcal{U}_{\mathsf{S}}|$ is a pair (p, q) where p and q are finite multisets of elements of $|\mathcal{U}_{\mathsf{S}}|$. On finite multisets, the preorder is given by $p \leq_{!\mathcal{U}_{\mathsf{S}}} p'$ if $\forall a \in |p|\ \exists a' \in |p'|\ a \leq_{\mathcal{U}_{\mathsf{S}}} a'$ and on pairs, the $\mathcal{U}_{\mathsf{S}}$-preorder is given by $(p, q) \leq_{\mathcal{U}_{\mathsf{S}}} (p', q')$ if $p' \leq_{!\mathcal{U}_{\mathsf{S}}} p$ and $q \leq_{!\mathcal{U}_{\mathsf{S}}} q'$.

7.1 Idempotent intersection types

We introduce a typing system for deriving judgments of shape $\Gamma \vdash_{\mathsf{S}} M : m$ and $\Gamma \vdash_{\mathsf{S}} V : a$ with the same notations as in Section 6.1: just as in that section, the types are the elements of $|\mathcal{U}_{\mathsf{S}}| = \mathcal{U}_{\mathsf{R}}$ and the contexts associate finite multisets of types with variables. But the typing rules are different. For terms, they are given by

$$\frac{\Gamma \vdash_{\mathsf{S}} M : [(p, q)] \qquad \Gamma \vdash_{\mathsf{S}} N : p}{\Gamma \vdash_{\mathsf{S}} M\, N : q} \qquad \frac{\Gamma \vdash_{\mathsf{S}} V : a_1 \qquad \cdots \qquad \Gamma \vdash_{\mathsf{S}} V : a_k}{\Gamma \vdash_{\mathsf{S}} \langle V \rangle : [a_1, \dots, a_k]}$$

and for values, they are given by

$$\frac{[a] \leq m}{\Gamma, x : m \vdash_{\mathsf{S}} x : a} \qquad \frac{\Gamma, x : p \vdash_{\mathsf{S}} M : q}{\Gamma \vdash_{\mathsf{S}} \lambda x\, M : (p, q)}$$

Similar typing systems for cbv have already been proposed, see [5] in particular.

▶ Proposition 16. Let P be an expression and let $\vec{x} = (x_1, \dots, x_n)$ be a list of variables adapted to P. Let $\vec{m} \in (|!\mathcal{U}_\mathsf{S}|)^n$ and let $\alpha \in |S|$ (where $S = \mathcal{U}_\mathsf{S}$ if P is a value and $S = !\mathcal{U}_\mathsf{S}$ if P is a term). Then one has $(\vec{m}, \alpha) \in [P]^{\vec{x}}_\mathsf{S}$ iff the typing judgment $x_1 : m_1, \dots, x_n : m_n \vdash P : \alpha$ is derivable.

The proof is a simple verification, by induction on the structure of P.

▶ Remark. We can define a model $\mathcal{U}'_\mathsf{S}$ of cbv λ-calculus using the exponential $!_\mathsf{s}\_$ mentioned in the remark of Section 3.2, and by this remark, the models $\mathcal{U}_\mathsf{S}$ and $\mathcal{U}'_\mathsf{S}$ *are isomorphic* in **Pol**. Now the typing system associated with $\mathcal{U}'_\mathsf{S}$ is exactly the same as the system presented above, up to the fact that all multisets occurring in types should be replaced by the corresponding sets and, up to this transformation, it is equivalent to the system above. In that sense, the typing system of this section is actually an idempotent intersection typing system. And indeed, if say $\vdash_\mathsf{S} M : p$ is derivable (where M is a closed term to simplify the notations) and if p' is equivalent to p in the preorder $!\mathcal{U}_\mathsf{S}$ (in other words $p \leq p'$ and $p' \leq p$), then one can infer $\vdash_\mathsf{S} M : p'$ by an isomorphic typing derivation. This is in particular the case if p and p' differ only by the multiplicities of their subtypes, that is if $p^- = p'^-$ where $[a_1, \dots, a_n]^- = \{a_1^-, \dots, a_n^-\} \in |!_\mathsf{s}\mathcal{U}'_\mathsf{S}|$ and $(p, q)^- \in (p^-, q^-) \in |\mathcal{U}'_\mathsf{S}|$.

7.2 Adequacy in the idempotent case

One can prove an analog of Theorem 15 for this idempotent typing system, but the same technique does not apply because, as we have seen with Proposition 2, **Pol** is not a model of the cbv resource calculus. The standard method to prove adequacy in this model is by reducibility, an example of such a proof will be given in a longer version of this paper.

▶ **Theorem 17.** *Let P be an expression. Let $\vec{x}$ be adapted to P and let n be the length of $\vec{x}$. Let $m_1, \dots, m_n \in |!\mathcal{U}_\mathsf{S}|$ and let $\alpha \in |X|$ where $X = \mathcal{U}_\mathsf{S}$ if P is a value and $X = !\mathcal{U}_\mathsf{S}$ if P is a term. If $x_1 : m_1, \dots, x_n : m_n \vdash_\mathsf{S} P : \alpha$ then P is $\hat{\beta}_\mathsf{V}$ strongly normalizing.*

We show now how to use the model **Pop** and the non-idempotent adequacy result to prove this theorem.

7.3 Adequacy in the idempotent case, using preorders with projections

Let $\Phi_\mathsf{P} : \mathbf{Pop}^\subseteq \to \mathbf{Pop}^\subseteq$ be the continuous function defined by $\Phi_\mathsf{P}(E) = !E \multimap !E$. Let $\mathcal{U}_\mathsf{P}$ be the least fixpoint of Φ_P, then $\mathcal{U}_\mathsf{P}$ (equipped with two identity morphisms) is a **Pop**-model of the cbv λ-calculus.

By Lemma 6, we have $\rho(\mathcal{U}_\mathsf{P}) = \mathcal{U}_\mathsf{R}$ and $\sigma(\mathcal{U}_\mathsf{P}) = \mathcal{U}_\mathsf{S}$.

Let P be an expression and let $\vec{x}$ be a sequence of variables adapted to P, let n be the length of $\vec{x}$. Because ρ is an LL functor, we have $[P]^{\vec{x}}_\mathsf{P} = \rho([P]^{\vec{x}}_\mathsf{P}) = [P]^{\vec{x}}_\mathsf{R}$, and similarly we have $\sigma([P]^{\vec{x}}_\mathsf{P}) = [P]^{\vec{x}}_\mathsf{S}$. These properties are proved by a straightforward induction on P. As a consequence, using the definition of the functor σ, we get the following result, which relates the relational semantics of an expression to its Scott semantics.

▶ **Theorem 18.** *Let P be an expression and let $\vec{x}$ be a sequence of variables of length n, adapted to P. Then $[P]^{\vec{x}}_\mathsf{S} = \downarrow [P]^{\vec{x}}_\mathsf{R}$ where the downwards closure is taken in $(!\mathcal{U}_\mathsf{S})^{\otimes n} \multimap S$ (with $S = \mathcal{U}_\mathsf{S}$ if P is a value and $S = !\mathcal{U}_\mathsf{S}$ if P is a term).*

We prove Theorem 17. We deal with the case of a term, but the proof is of course similar for values. So assume that $x_1 : m_1, \dots, x_k : m_k \vdash_\mathsf{S} M : m$ which means that $(\vec{m}, m) \in [M]^{\vec{x}}_\mathsf{S}$ where $m_1, \dots, m_n, m \in |!\mathcal{U}_\mathsf{S}|$. Then by Theorem 18, we can find $m'_1, \dots, m'_n, m' \in |!\mathcal{U}_\mathsf{S}| = !\mathcal{U}_\mathsf{R}$

with $\forall i\ m'_i \leq_{!\mathcal{U}_S} m_i$ and $m \leq_{!\mathcal{U}_S} m'$ and such that $(m'_1, \dots, m'_n, m') \in [M]^{\vec{x}}_{\mathsf{R}}$. By Theorem 15, M is $\hat{\beta}_{\mathsf{V}}$ strongly normalizing.

▶ Remark. It is quite instructive to try to prove Theorem 18 by a direct induction on derivations in the idempotent typing system of Section 7.1 and to observe that it is not as easy as one could think. Given a derivation of $\Gamma \vdash_{\mathsf{S}} P : \alpha$ we want to find a derivation $\Gamma' \vdash P : \alpha'$ such that $\alpha \leq \alpha'$ and $\Gamma' \leq \Gamma$ (this means that Γ and Γ' have same domain and that $\Gamma'(x) \leq \Gamma(x)$ for each x). Observe first that we cannot hope to have $\Gamma' = \Gamma$ and $\alpha' = \alpha$ in general, because our proof would fail on its base case (variables). Assume that the derivation ends with an application rule: $P = M\,N$, $\alpha = q$, and we have (shorter) derivations of $\Gamma \vdash_{\mathsf{S}} M : [(p, q)]$ and $\Gamma \vdash_{\mathsf{S}} N : p$. By inductive hypothesis, we can find Γ'_1, p'_1, q'_1 with $\Gamma'_1 \leq \Gamma$, $p'_1 \leq p$ and $q \leq q'_1$ such that $\Gamma'_1 \vdash M : [(p'_1, q'_1)]$ is derivable and also Γ'_2, p'_2 such that $\Gamma'_2 \leq \Gamma$, $p \leq p'_2$ and $\Gamma'_2 \vdash N : p'_2$ is derivable. To build a typing derivation for $M\,N$ in this non-idempotent system, we would need to force $p'_2 = p'_1$ but nothing in our inductive hypothesis guarantees that this is possible. It is precisely the point of the model $\mathcal{U}_{\mathsf{P}}$ in **Pop** to show that, for "well behaved" sets of the relational model (those belonging to $\mathsf{D}(\mathcal{U}_{\mathsf{S}})$) downwards closure commutes with application – this is the main content of Lemma 3 – and that the relational interpretation of cbv λ-calculus expressions are precisely such well-behaved sets.

Conclusion

We have shown how to use a purely semantical construction (the model **Pop**) to reduce the proof of an adequacy theorem usually proved by reducibility to a purely combinatorial argument and we have illustrated this approach in the cbv λ-calculus. In further work, we'll apply this approach to other languages and other notions of normalization to understand better how the reducibility structure is encoded in the model. We'll also explore the probable connections between our work and [1].

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