

# Solving the Canonical Representation and Star System Problems for Proper Circular-Arc Graphs in Logspace

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## Abstract

We present a logspace algorithm that constructs a canonical intersection model for a given proper circular-arc graph, where *canonical* means that isomorphic graphs receive identical models. This implies that the recognition and the isomorphism problems for these graphs are solvable in logspace. For the broader class of concave-round graphs, which still possess (not necessarily proper) circular-arc models, we show that a canonical circular-arc model can also be constructed in logspace. As a building block for these results, we design a logspace algorithm for computing canonical circular-arc models of circular-arc hypergraphs; this important class of hypergraphs corresponds to matrices with the *circular ones property*.

Furthermore, we consider the Star System Problem that consists in reconstructing a graph from its closed neighborhood hypergraph. We show that this problem is solvable in logarithmic space for the classes of proper circular-arc, concave-round, and co-convex graphs.

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## 1 Introduction

With a family of sets  $\mathcal{H}$  we associate the *intersection graph*  $\mathbb{I}(\mathcal{H})$  on vertex set  $\mathcal{H}$  where two sets  $A, B \in \mathcal{H}$  are adjacent if and only if they have a non-empty intersection. We call  $\mathcal{H}$  an *intersection model* of a graph  $G$  if  $G$  is isomorphic to  $\mathbb{I}(\mathcal{H})$ . Any isomorphism from  $G$  to  $\mathbb{I}(\mathcal{H})$  is called a *representation* of  $G$  by an intersection model. If  $\mathcal{H}$  consists of intervals (resp. arcs of a circle), it is also referred to as an *interval model* (resp. an *arc model*). An intersection model  $\mathcal{H}$  is *proper* if the sets in  $\mathcal{H}$  are pairwise incomparable by inclusion.  $G$  is called a (*proper*) *interval graph* if it has a (proper) interval model. The classes of *circular-arc* and *proper circular-arc* graphs are defined similarly. Throughout the paper we will use the shorthands *CA* and *PCA*, respectively.

We design a logspace algorithm that for a given PCA graph computes a canonical representation by a proper arc model, where *canonical* means that isomorphic graphs receive

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identical models. Note that this algorithm provides a simultaneous solution in logspace of both the recognition and the isomorphism problems for the class of PCA graphs.

In [18], along with Bastian Laubner we gave a logspace solution for the canonical representation problem of proper interval graphs. Though PCA graphs may at first glance appear close relatives of proper interval graphs, the extension of the result of [18] achieved here is far from being straightforward. Differences between the two classes of graphs are well known and have led to different algorithmic approaches also in the past; e.g. [11, 17, 22]. One difference, very important in our context, lies in the relationship of these graph classes to interval and circular-arc hypergraphs that we will explain shortly.

An *interval hypergraph* is a hypergraph isomorphic to a system of intervals of integers. A *circular-arc (CA) hypergraph* is defined similarly if, instead of integer intervals, we consider arcs in a discrete cycle. With any graph  $G$ , we associate its *closed neighborhood hypergraph*  $\mathcal{N}[G] = \{N[v]\}_{v \in V(G)}$  on the vertex set of  $G$ , where for each vertex  $v$  we have the hyperedge  $N[v]$  consisting of  $v$  and all vertices adjacent to  $v$ . Roberts [27] discovered that  $G$  is a proper interval graph if and only if  $\mathcal{N}[G]$  is an interval hypergraph. The circular-arc world is more complex. While  $\mathcal{N}[G]$  is a CA hypergraph whenever  $G$  is a PCA graph, the converse is not always true. PCA graphs are properly contained in the class of those graphs whose neighborhood hypergraphs are CA. Graphs with this property are called *concave-round* by Bang-Jensen, Huang, and Yeo [3] and *Tucker graphs* by Chen [8]. The latter name is justified by Tucker's result [29] saying that all these graphs are CA (although not necessarily proper CA). Hence, it is natural to consider the problem of constructing arc representations for concave-round graphs. We solve this problem in logspace and also in a canonical way.

Our working tool is a logspace algorithm for computing a canonical representation of CA hypergraphs. This algorithm can also be used to test in logspace whether a given Boolean matrix has the *circular ones property*, that is, whether the columns can be permuted so that the 1-entries in each row form a segment up to a cyclic shift. Note that a matrix has this property if and only if it is the incidence matrix of a CA hypergraph. The recognition problem of the circular ones property arises in computational biology, namely in analysis of circular genomes [13, 25].

Our techniques are also applicable to the *Star System Problem* where, for a given hypergraph  $\mathcal{H}$ , we have to find a graph  $G$  such that  $\mathcal{H} = \mathcal{N}[G]$ , if such a graph exists. In the restriction of the problem to a class of graphs  $\mathcal{C}$ , we seek for  $G$  only in  $\mathcal{C}$ . We give logspace algorithms solving the Star System Problem for PCA and for concave-round graphs.

**Comparison with previous work.** The recognition problem for PCA graphs, along with model construction, was solved in linear time by Deng, Hell, and Huang [11] and by Kaplan and Nussbaum [17]; and in  $\text{AC}^2$  by Chen [7]. Note that linear-time and logspace results are in general incomparable, while the existence of a logspace algorithm for a problem implies that it is solvable in  $\text{AC}^1$ . The isomorphism problem for PCA graphs was solved in linear time by Lin, Souignac, and Szwarcfiter [22]. In a very recent paper [10], Curtis et al. extend this result to concave-round graphs.

The isomorphism problem for concave-round graphs was solved in  $\text{AC}^2$  by Chen [8]. Circular-arc models for concave-round graphs were known to be constructible also in  $\text{AC}^2$  (Chen [6]).

Extending these complexity upper bounds to the class of all CA graphs remains a challenging problem. While this class can be recognized in linear time by McConnell's algorithm [24] (along with constructing an intersection model), no polynomial-time isomorphism test for CA graphs is currently known (see the discussion in [10], where a counterexample to the

correctness of Hsu's algorithm [14] is given). This provides further evidence that CA graphs are algorithmically harder than interval graphs. For the latter class we have linear-time algorithms for both recognition and isomorphism due to the seminal work by Booth and Lueker [4, 23], and a canonical representation algorithm taking logarithmic space is designed in [18].

The aforementioned circular ones property and the related *consecutive ones property* were studied in [4, 15, 16], where linear-time algorithms are given; parallel  $AC^2$  algorithms were suggested in [9, 2].

The decision version of the Star System Problem is in general NP-complete (Lalonde [21]). It stays NP-complete if restricted to non-co-bipartite graphs (Aigner and Triesch [1]) or to  $H$ -free graphs for  $H$  being a cycle or a path on at least 5 vertices (Fomin et al. [12]). The restriction to co-bipartite graphs has the same complexity as the general graph isomorphism problem [1]. Polynomial-time algorithms are known for  $H$ -free graphs for  $H$  being a cycle or a path on at most 4 vertices [12] and for bipartite graphs (Boros et al. [5]). An analysis of the algorithms in [12] for  $C_3$ - and  $C_4$ -free graphs shows that the Star System Problem for these classes is solvable even in logspace, and the same holds true for the class of bipartite graphs; see [20]. Moreover, the problem is solvable in logspace for any logspace-recognizable class of  $C_4$ -free graphs, in particular, for chordal, interval, and proper interval graphs; see [20].

Due to space limitations some proofs are only sketched; full details can be found in an e-print [19].

## 2 Basic definitions

The vertex set of a graph  $G$  is denoted by  $V(G)$ . The *complement of a graph  $G$*  is the graph  $\overline{G}$  with  $V(\overline{G}) = V(G)$  such that two vertices are adjacent in  $\overline{G}$  if and only if they are not adjacent in  $G$ . The set of all vertices at distance at most (resp. exactly) 1 from a vertex  $v \in V(G)$  is called the *closed* (resp. *open*) *neighborhood* of  $v$  and denoted by  $N[v]$  (resp.  $N(v)$ ). Note that  $N[v] = N(v) \cup \{v\}$ . We call vertices  $u$  and  $v$  *twins* if  $N[u] = N[v]$  and *fraternal vertices* if  $N(u) = N(v)$ . A vertex  $u$  is *universal* if  $N[u] = V(G)$ .

The *canonical labeling problem* for a class of graphs  $\mathcal{C}$  is, given a graph  $G \in \mathcal{C}$  with  $n$  vertices, to compute a map  $\lambda_G: V(G) \rightarrow \{1, \dots, n\}$  so that the graph  $\lambda_G(G)$ , the image of  $G$  under  $\lambda_G$  on the vertex set  $\{1, \dots, n\}$ , is the same for isomorphic input graphs. We say that  $\lambda_G$  is a *canonical labeling* and that  $\lambda_G(G)$  is a *canonical form* of  $G$ .

Recall that a *hypergraph* is a pair  $(X, \mathcal{H})$ , where  $X$  is a set of vertices and  $\mathcal{H}$  is a family of subsets of  $X$ , called *hyperedges*. We will use the same notation  $\mathcal{H}$  to denote a hypergraph and its hyperedge set and, similarly to graphs, we will write  $V(\mathcal{H})$  referring to the vertex set  $X$  of the hypergraph  $\mathcal{H}$ . We will allow *multiple hyperedges*; in this case an isomorphism has to respect multiplicities.

The *complement of a hypergraph  $\mathcal{H}$*  is the hypergraph  $\overline{\mathcal{H}} = \{\overline{H}\}_{H \in \mathcal{H}}$  on the same vertex set, where  $\overline{H} = V(\mathcal{H}) \setminus H$ . Each hyperedge  $\overline{H}$  of  $\overline{\mathcal{H}}$  inherits the multiplicity of  $H$  in  $\mathcal{H}$ . We associate with a graph  $G$  two hypergraphs defined on the vertex set  $V(G)$ . The *closed* (resp. *open*) *neighborhood hypergraph* of  $G$  is defined by  $\mathcal{N}[G] = \{N[v]\}_{v \in V(G)}$  (resp. by  $\mathcal{N}(G) = \{N(v)\}_{v \in V(G)}$ ). *Twins in a hypergraph* are two vertices such that every hyperedge contains either both or none of them. Note that two vertices are twins in  $\mathcal{N}[G]$  if and only if they are twins in  $G$ .

By  $\mathbb{C}_n$  we denote the directed cycle on the vertex set  $\{1, \dots, n\}$  with arrows from  $i$  to  $i + 1$  and from  $n$  to 1. An *arc*  $A$  is either empty ( $A = \emptyset$ ), complete ( $A = \{1, \dots, n\}$ ), or a segment  $A = [a^-, a^+]$  with *extreme points*  $a^-$  and  $a^+$  that consists of the points appearing in the

directed path from  $a^-$  to  $a^+$  in the cycle  $\mathbb{C}_n$ . An *arc system*  $\mathcal{A}$  is a hypergraph whose vertex set is  $\{1, \dots, n\}$  and whose hyperedges are arcs of  $\mathbb{C}_n$ .  $\mathcal{A}$  is *tight* if any two arcs  $A = [a^-, a^+]$  and  $B = [b^-, b^+]$  in  $\mathcal{A}$  have the following property: if  $\emptyset \neq A \subseteq B \neq \mathbb{C}_n$ , then  $a^- = b^-$  or  $a^+ = b^+$ .

An *arc representation* of a hypergraph  $\mathcal{H}$  is an isomorphism  $\rho$  from  $\mathcal{H}$  to an arc system  $\mathcal{A}$ . The arc system  $\mathcal{A}$  is referred to as an *arc model* of  $\mathcal{H}$ . The notions of an *interval representation* and an *interval model* of a hypergraph are introduced similarly, where *interval* means an interval of integers. Hypergraphs having arc representations are called *circular-arc (CA) hypergraphs*, and those having interval representations are called *interval hypergraphs*.

We call a CA hypergraph *tight*, if it admits a tight arc model. Recognition of tight CA hypergraphs reduces to recognition of CA hypergraphs. To see this, given a hypergraph  $\mathcal{H}$ , define its *tightened hypergraph*  $\mathcal{H}^\ominus$  by  $\mathcal{H}^\ominus = \mathcal{H} \cup \{A \setminus B : A, B \in \mathcal{H}\}$ . Then  $\mathcal{H}$  is a tight CA hypergraph if and only if  $\mathcal{H}^\ominus$  is a CA hypergraph (for if  $A, B \in \mathcal{H}$  and  $\emptyset \neq B \subsetneq A$ , then  $B$  cannot be an inner part of  $A$  in any arc model of  $\mathcal{H}^\ominus$ ).

A *circular order* on a finite set  $S = \{s_1, \dots, s_n\}$  is described by a circular successor relation  $\prec$  on  $S$  (meaning that the digraph  $(S, \prec)$  is a cycle). Each arc representation  $\rho$  of a CA hypergraph  $\mathcal{H}$  induces a circular order  $\prec$  on  $V(\mathcal{H})$  such that the hyperedges in  $\mathcal{H}$  are arcs w.r.t.  $\prec$ ; a circular order on  $V(\mathcal{H})$  with this property is called a *CA order* of  $\mathcal{H}$ . Conversely, each CA order of  $\mathcal{H}$  specifies an arc representation of  $\mathcal{H}$  up to rotation. Where rotations are not important, we will describe arc representations by CA orders. We call a CA order of  $\mathcal{H}$  *tight*, if it makes  $\mathcal{H}$  a tight arc system. Similarly, we will use the notion of (*tight*) *interval orders* in the case of interval hypergraphs.

Given a CA order  $\prec$  of a hypergraph  $\mathcal{H}$ , consider the set of all arcs  $A \subset V(\mathcal{H})$  w.r.t.  $\prec$  excepting the empty arc  $\emptyset$  and the complete arc  $V(\mathcal{H})$ . The CA order  $\prec$  induces a (lexicographic) circular order  $\prec^*$  on this set, where  $A \prec^* B$  if  $a^- = b^-$  and  $a^+ \prec b^+$  or if  $a^- \prec b^-$ ,  $|A| = n - 1$ , and  $|B| = 1$ . By “restricting”  $\prec^*$  to the set  $\mathcal{H}$  (assuming that  $\emptyset, V(\mathcal{H}) \notin \mathcal{H}$ ) we obtain a circular order  $\prec_{\mathcal{H}}$  on  $\mathcal{H}$ : For  $A, B \in \mathcal{H}$  we define  $A \prec_{\mathcal{H}} B$  if either  $A \prec^* B$  or there exist arcs  $X_1, \dots, X_k \notin \mathcal{H}$  such that  $A \prec^* X_1 \prec^* \dots \prec^* X_k \prec^* B$ . We say that the circular order  $\prec_{\mathcal{H}}$  on  $\mathcal{H}$  is *lifted from* the CA order  $\prec$  on  $V(\mathcal{H})$ .

An *arc representation of a graph*  $G$  is an isomorphism  $\alpha: V(G) \rightarrow \mathcal{A}$  from  $G$  to the intersection graph  $\mathbb{I}(\mathcal{A})$  of an arbitrary arc system  $\mathcal{A}$ . If  $\emptyset, V(\mathcal{A}) \notin \mathcal{A}$  (this always holds when  $G$  has neither an isolated nor a universal vertex), we use the lifted circular order  $\prec_{\mathcal{A}}$  on  $\mathcal{A}$  to define a circular order  $\prec_{\alpha}$  on  $V(G)$ , where  $u \prec_{\alpha} v$  if and only if  $\alpha(v) \prec_{\mathcal{A}} \alpha(u)$ . We call  $\prec_{\alpha}$  the *geometric order* on  $V(G)$  associated with  $\alpha$ .

**Roadmap.** In Section 3 we describe a logspace algorithm for computing a canonical arc representation of a given CA hypergraph. In Section 4 we show that for non-co-bipartite PCA graphs  $G$ , the neighborhood hypergraph  $\mathcal{N}[G]$  admits a unique CA order, which coincides with the geometric order  $\prec_{\alpha}$  for any proper arc representation  $\alpha$  of  $G$ . In Section 5, we employ these facts to compute canonical representations of non-co-bipartite PCA graphs in logspace. To achieve the same for co-bipartite PCA graphs  $G$  (and all concave-round graphs), we use the fact that  $\mathcal{N}(\overline{G})$  is in this case an interval hypergraph and show how to convert an interval representation of  $\mathcal{N}(\overline{G})$  into an arc representation of  $G$ . Finally, in Section 6 we apply the techniques of Sections 3 and 4 to the Star System Problem.

### 3 Canonical arc representations of hypergraphs

► **Theorem 1.** *The canonical representation problem for CA hypergraphs is solvable in logspace.*

**Proof sketch.** We prove this result by a logspace reduction to the canonical representation problem for edge-colored interval hypergraphs, which is already known to be in logspace [18]. Let  $\mathcal{H}$  be an input CA hypergraph with  $n$  vertices. For each vertex  $x \in V(\mathcal{H})$  we construct the hypergraph  $\mathcal{H}_x = \{H_x\}_{H \in \mathcal{H}}$  on the same vertex set, where  $H_x = H$  if  $x \notin H$  and  $H_x = \overline{H}$  otherwise. Observe that every  $\mathcal{H}_x$  is an interval hypergraph; cf. [29, Theorem 1]. Canonizing each  $\mathcal{H}_x$  using the algorithm from [18], we obtain  $n$  interval representations  $\rho_x: V(\mathcal{H}) \rightarrow \{1, \dots, n\}$ ; recall that  $V(\mathcal{H}_x) = V(\mathcal{H})$ . Each  $\rho_x$  gives us an arc model  $\rho_x(\mathcal{H})$  of  $\mathcal{H}$ , which is obtained from the corresponding canonical interval model  $\rho_x(\mathcal{H}_x)$  of  $\mathcal{H}_x$  by complementing the intervals corresponding to complemented hyperedges. Among these  $n$  candidates, we choose the lexicographically least arc model as canonical and output the corresponding arc representation  $\rho_x$ .

There is a subtle point in this procedure: We need to distinguish between complemented and non-complemented hyperedges when canonizing  $\mathcal{H}_x$ ; otherwise reversing the complementation could lead to non-equal models for isomorphic CA hypergraphs. For this reason we endow each interval hypergraph  $\mathcal{H}_x$  with the edge-coloring  $c_x: \mathcal{H}_x \rightarrow \{0, 1\}$ , where  $c_x(H_x) = 1$  if  $x \in H$  and  $c_x(H_x) = 0$  otherwise. If both  $H$  and  $\overline{H}$  are present in  $\mathcal{H}$ , this results in “multi-hyperedges” that have different colors. ◀

The *canonical labeling problem* for a class of hypergraphs  $\mathcal{C}$  is defined exactly as for graphs. The canonical representation algorithm given by Theorem 1 also solves the canonical labeling problem for CA hypergraphs in logarithmic space. We conclude this section with noting that it can also be used to compute a canonical labeling for the duals of CA hypergraphs; this will be needed in Section 6.

Given a hypergraph  $\mathcal{H}$  and a vertex  $v \in V(\mathcal{H})$ , let  $v^* = \{H \in \mathcal{H} : v \in H\}$ . The hypergraph  $\mathcal{H}^* = \{v^* : v \in V(\mathcal{H})\}$  on the vertex set  $V(\mathcal{H}^*) = \mathcal{H}$  is called the *dual hypergraph* of  $\mathcal{H}$  (multiple hyperedges in  $\mathcal{H}$  become twin vertices in  $\mathcal{H}^*$ ). The map  $\varphi: v \mapsto v^*$  is an isomorphism from  $\mathcal{H}$  to  $(\mathcal{H}^*)^*$ . If  $\mathcal{H}^*$  is a CA hypergraph, this map can be combined with a canonical labeling  $\lambda$  of  $\mathcal{H}^*$  in order to obtain a canonical labeling  $\hat{\lambda}$  of  $\mathcal{H}$ . More precisely,  $\hat{\lambda}$  is obtained from the map  $\lambda'(v) = \{\lambda(H) : v \in H\}$  by sorting and renaming the values of  $\lambda'$ .

► **Corollary 2.** *The canonical labeling problem for hypergraphs whose duals are CA can be solved in logspace.*

### 4 Linking PCA graphs and tight CA hypergraphs

In this section, we establish the connections between some classes of CA graphs and CA hypergraphs that will be used in the design of our algorithms.

Bang-Jensen et al. [3] call a graph  $G$  *concave-round* (resp. *convex-round*) if  $\mathcal{N}[G]$  (resp.  $\mathcal{N}(G)$ ) is a CA hypergraph. Since  $\overline{\mathcal{N}[G]} = \mathcal{N}(\overline{G})$ , concave-round and convex-round graphs are co-classes. Using this terminology, a result of Tucker [29] says that PCA graphs are concave-round, and concave-round graphs are CA.

To connect the canonical representation problem for PCA and concave-round graphs to that of CA hypergraphs, we use the fact that the graph classes under consideration can be characterized in terms of neighborhood hypergraphs. For concave-round graphs, this

directly follows from their definition, and we can find accompanying hypergraphs also for PCA graphs.

► **Theorem 3.** *A graph  $G$  is PCA if and only if  $\mathcal{N}[G]$  is a tight CA hypergraph.*

The forward direction of Theorem 3 follows from Lemma 4 below. To prove the other direction, we distinguish two cases. If  $\overline{G}$  is bipartite, then any tight arc model for  $\mathcal{N}[G]$  can be transformed into a proper arc model for  $G$ , as described in Section 5. If  $\overline{G}$  is not bipartite, then a result of Tucker [29] says that  $G$  is a PCA graph whenever  $\mathcal{N}[G]$  is a CA hypergraph.

► **Lemma 4.** *The geometric order  $\prec_\alpha$  on  $V(G)$  associated with a proper arc representation  $\alpha$  of a PCA graph  $G$  is a tight CA order for the hypergraph  $\mathcal{N}[G]$ .*

**Proof.** Let  $G$  be a PCA graph and let  $\alpha: V(G) \rightarrow \mathcal{A}$  be a proper arc representation of  $G$ . We first show that the neighborhood  $N[u]$  of any vertex  $u \in V(G)$  is an arc w.r.t. to the order  $\prec_\alpha$ . If  $u$  is universal, the claim is trivial. Otherwise, let  $\alpha(u) = [a^-, a^+]$ . We split  $N(u)$  in two parts, namely  $N^-(u) = \{v \in N(u) : a^- \in \alpha(v)\}$  and  $N^+(u) = \{v \in N(u) : a^+ \in \alpha(v)\}$ . Indeed, no vertex  $v$  is contained in both  $N^-(u)$  and  $N^+(u)$ . Otherwise, since  $\mathcal{A}$  is proper, the arcs  $\alpha(v)$  and  $\alpha(u)$  would cover the whole cycle, both intersecting any other arc  $\alpha(w)$ , contradicting the assumption that  $u$  is non-universal.

Now let  $v \in N^+(u)$  and assume that  $u \prec_\alpha v_1 \prec_\alpha \dots \prec_\alpha v_k \prec_\alpha v$ . We claim that every vertex  $v_i$  is in  $N^+(u)$ . Indeed, by the definition of  $\prec_\alpha$ , we have  $\alpha(u) \prec_{\mathcal{A}} \alpha(v_1) \prec_{\mathcal{A}} \dots \prec_{\mathcal{A}} \alpha(v_k) \prec_{\mathcal{A}} \alpha(v)$ . If  $\alpha(v) = [c^-, c^+]$  and  $\alpha(v_i) = [b^-, b^+]$ , we see that  $b^- \in (a^-, c^-)$ ,  $b^+ \in (a^+, c^+)$  and, hence,  $a^+ \in [b^-, b^+]$ . It follows that  $N^+(u) \cup \{u\}$  is an arc starting at  $u$ . By a symmetric argument,  $N^-(u) \cup \{u\}$  is an arc ending at  $u$ . Hence, also  $N[u]$  is an arc, implying that  $\prec_\alpha$  is a CA order for  $\mathcal{N}[G]$ .

It remains to show that the CA order  $\prec_\alpha$  is tight. Suppose that  $N[u] = [u^-, u^+] \subseteq N[v] = [v^-, v^+]$  and  $v$  is non-universal with  $\alpha(v) = [c^-, c^+]$ . Let's first assume that  $u \in N^+(v) = (v, v^+]$ . Since  $u, v^+ \in N^+(v)$ , it follows that  $c^+ \in \alpha(u) \cap \alpha(v^+)$ . Hence,  $u$  and  $v^+$  are adjacent or equal, which implies that  $u^+ = v^+$ . If  $u \in [v^-, v)$ , a symmetric argument shows that  $u^- = v^-$ . ◀

Theorem 3 suggests that, given a tight CA order of  $\mathcal{N}[G]$ , we can use it to construct a proper arc model for  $G$ . For this we need the converse of Lemma 4. In the case that  $\overline{G}$  is not bipartite, the following lemma implies that indeed each CA order of  $\mathcal{N}[G]$  is the geometric order of some proper arc representation of  $G$ .

► **Lemma 5.** *If  $G$  is a connected twin-free PCA graph and  $\overline{G}$  is not bipartite, then  $\mathcal{N}[G]$  has a unique CA order up to reversing.*

The proof of Lemma 5 is based on a result of Deng, Hell, and Huang [11, Corollary 2.9]. We need some special properties of CA orders  $\prec$  of  $\mathcal{N}[G]$  when  $G$  is a PCA graph with non-bipartite complement. We keep using the notation  $N[u] = [u^-, u^+]$  w.r.t.  $\prec$ . Parts 3 and 4 of the next proposition will be needed for proving Lemma 11.

► **Proposition 6.** *Let  $\prec$  be any CA order of  $\mathcal{N}[G]$ , where  $G$  is a PCA graph with non-bipartite complement, and let  $u, v \in V(G)$ .*

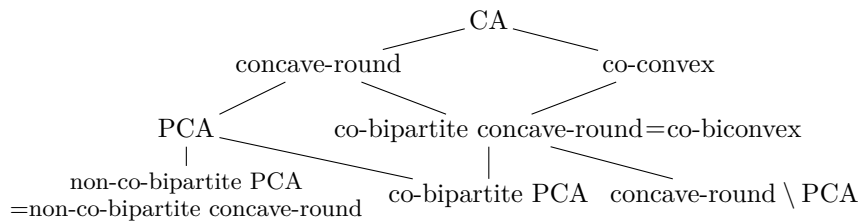
1.  $N[u] \neq V(G)$  and  $u$  divides  $N[u] = [u^-, u^+]$  into two cliques  $[u^-, u]$  and  $[u, u^+]$  of  $G$ .
2.  $v \in [u, u^+]$  if and only if  $u \in [v^-, v]$ .
3. If  $v \in [u, u^+]$ , then  $v^- \in [u^-, u]$  and  $u^+ \in [v, v^+]$ .
4. If  $v \in [u, u^+]$  and  $u \prec v$ , then  $u^-, v^-, u, v, u^+,$  and  $v^+$  occur under the order  $\prec$  exactly in this circular sequence, where some of the neighboring vertices except  $u^-$  and  $v^+$  may coincide.

**Proof of Lemma 5.** Call an orientation of a graph  $G$  *round* if there is a CA order  $\prec$  of  $\mathcal{N}[G]$ , such that each vertex  $v$  separates the arc  $N[v] = [v^-, v^+]$  of  $(V(G), \prec)$  into two arcs  $[v^-, v)$  and  $(v, v^+]$  such that the former consists of the in-neighbors of  $v$  and the latter consists of the out-neighbors of  $v$ . We also call such an orientation *compatible* with  $\prec$ .

As stated in [11, Corollary 2.9], any twin-free connected PCA graph with non-bipartite complement has a unique round orientation (up to reversal). Therefore, in order to prove that  $\mathcal{N}[G]$  has a unique CA order (up to reversal), it suffices to show that any CA order  $\prec$  of  $\mathcal{N}[G]$  determines a compatible round orientations of  $G$  and that this correspondence is injective. Indeed, orienting an edge  $\{u, v\}$  of  $G$  as  $(u, v)$  if  $u \in [v^-, v)$  and as  $(v, u)$  if  $u \in (v, v^+]$  is well-defined, since by Proposition 6,  $G$  has no universal vertex and  $v \in [u, u^+]$  if and only if  $u \in [v^-, v]$ . Further, it is not hard to see that different CA orders produce different orientations. ◀

We close this section by giving a characterization of concave-round graphs  $G$  with bipartite complement using properties of  $\mathcal{N}(\overline{G})$ . Given a bipartite graph  $H$  and a bipartition  $V(H) = U \cup W$  of its vertices into two independent sets, by  $\mathcal{N}_U(H)$  we denote the hypergraph  $\{N(w)\}_{w \in W}$  on the vertex set  $U$ . Note that  $(\mathcal{N}_U(H))^* \cong \mathcal{N}_W(H)$ . A bipartite graph  $H$  is called *convex* if its vertex set admits splitting into two independent sets  $U$  and  $W$ , such that  $\mathcal{N}_U(H)$  is an interval hypergraph. If both  $\mathcal{N}_U(H)$  and  $\mathcal{N}_W(H)$  are interval hypergraphs,  $H$  is called *biconvex*; see, e.g., [28]. As  $G$  is co-bipartite concave-round if and only if its complement  $H = \overline{G}$  is bipartite convex-round, the following fact gives the desired characterization.

► **Proposition 7** (implicitly in Lemma 3 of [29]). *A graph  $H$  is bipartite convex-round if and only if it is biconvex and if and only if  $\mathcal{N}(H)$  is an interval hypergraph.*



■ **Figure 1** Inclusion structure of the classes of graphs under consideration.

## 5 Canonical arc representations of concave-round and PCA graphs

We are now ready to present our canonical representation algorithm for concave-round and PCA graphs.

► **Theorem 8.** *There is a logspace algorithm that solves the canonical arc representation problem for the class of concave-round graphs. Moreover, this algorithm outputs a proper arc representation whenever the input graph is PCA.*

For any class of intersection graphs, a canonical representation algorithm readily implies a canonical labeling algorithm of the same complexity. Vice versa, a canonical representation algorithm readily follows from a canonical labeling algorithm *and* a representation algorithm (not necessarily a canonical one). Proving Theorem 8 according to this scheme, we split our task in two parts: We first compute a canonical labeling  $\lambda$  of the input graph  $G$  and then we compute an arc representation  $\alpha$  of the canonical form  $\lambda(G)$ . Then the composition  $\alpha \circ \lambda$

is a canonical arc representation of  $G$ . As twins can be easily re-inserted in a (proper) arc representation, it suffices to compute  $\alpha$  for the twin-free version of  $\lambda(G)$ , where in each twin-class we only keep one vertex.

We distinguish two cases depending on whether  $\overline{G}$  is bipartite; see Fig. 1 for an overview of the involved graph classes.

**Non-co-bipartite concave-round graphs.** As mentioned above, any concave-round graph  $G$  whose complement is not bipartite is actually a PCA graph [29]. Hence, we have to compute a proper arc representation in this case.

*Canonical labeling.* We first transform  $G$  into its twin-free version  $G'$ , where we only keep one vertex in each twin-class. Let  $n$  be the number of vertices in  $G'$ . We use the algorithm given by Theorem 1 to compute an arc representation  $\rho'$  of  $\mathcal{N}[G']$ . By Lemma 5,  $\mathcal{N}[G']$  has a CA order which is unique up to reversing. Hence, in order to determine a canonical labeling of  $G$ , it suffices to consider the  $2n$  arc representations  $\rho_1, \dots, \rho_{2n}$  of  $\mathcal{N}[G]$  that can be obtained from  $\rho'$  by cyclic shifts and reversing and by re-inserting all the removed twins. As a canonical labeling  $\rho_i$  of  $G$ , we appoint one of these  $2n$  variants that gives the lexicographically least canonical form  $\rho_i(G)$  of  $G$ .

*Proper arc representation.* As mentioned above, it suffices to find such a representation for the twin-free graph  $G'$ . The arc representation  $\rho'$  of  $\mathcal{N}[G']$  that we have already computed provides us with a CA order  $\prec$  for  $\mathcal{N}[G']$ . By Lemmas 4 and 5, there is a proper arc representation  $\alpha: V(G') \rightarrow \mathcal{A}$  of  $G'$  such that  $\prec$  coincides with the associated geometric order  $\prec_\alpha$ . In order to construct  $\alpha$  from  $\prec$ , we can assume that no two arcs  $\alpha(v) = [a_v^-, a_v^+]$  and  $\alpha(u) = [a_u^-, a_u^+]$  in  $\mathcal{A}$  share an extreme point and that  $V(\mathcal{A})$  consists of exactly  $2n$  points. To determine a suitable circular order on  $V(\mathcal{A})$ , order the start points  $a_v^-$  according to  $\prec$ . Then there is a unique way to place the end points  $a_v^+$  to make  $\alpha$  a proper arc representation of  $G'$ . A careful implementation shows that this computation can be done in logspace.

**Co-bipartite concave-round graphs.** By Proposition 7, co-bipartite concave-round graphs are precisely the co-biconvex graphs. In fact, all co-convex graphs are circular-arc (this is implicit in [29]) and we can actually compute a canonical arc representation for these graphs in logspace.

*Canonical labeling.* A canonical labeling algorithm for convex graphs, and hence also for co-convex graphs, is designed in [18].

*(Proper) arc representation.* We first recall Tucker's argument [29] showing that, if the complement of  $G$  is a convex graph, then  $G$  is CA. We can assume that  $\overline{G}$  has no fraternal vertices as those would correspond to twins in  $G$ .

Let  $V(G) = U \cup W$  be a partition of  $\overline{G}$  into independent sets such that  $\mathcal{N}_U(\overline{G})$  is an interval hypergraph. Let  $u_1, \dots, u_k$  be an interval order on  $U$  for  $\mathcal{N}_U(\overline{G})$ . We construct an arc representation  $\alpha$  for  $G$  on the cycle  $\mathbb{Z}_{2k+2}$  by setting  $\alpha(u_i) = [i, i+k]$  for each  $u_i \in U$  and  $\alpha(w) = [j+k+1, i-1]$  for each  $w \in W$ , where  $N_{\overline{G}}(w) = [u_i, u_j]$  and the subscript  $\overline{G}$  means that the vertex neighborhood is considered in the complement of  $G$ . Note that  $\alpha(w) = \mathbb{Z}_{2k+2} \setminus \bigcup_{u \in N_{\overline{G}}(w)} \alpha(u)$ . In the case that  $N_{\overline{G}}(w) = \emptyset$ , we set  $\alpha(w) = [0, k]$ . By construction, all arcs  $\alpha(u)$  for  $u \in U$  share a point (even two,  $k$  and  $k+1$ ), the same holds true for all  $\alpha(w)$  for  $w \in W$  (they share the point 0), and any pair  $\alpha(u)$  and  $\alpha(w)$  is intersecting if and only if  $u$  and  $w$  are adjacent in  $G$ . Thus,  $\alpha$  is indeed an arc representation for  $G$ .

In order to compute  $\alpha$  in logspace, it suffices to compute a suitable bipartition  $\{U, W\}$  of  $\overline{G}$  and an interval order of the hypergraph  $\mathcal{N}_U(\overline{G})$  in logspace. Finding a bipartition



$\{U, W\}$  such that  $\mathcal{N}_U(\overline{G})$  is an interval hypergraph can be done by splitting  $\overline{G}$  into connected components  $H_1, \dots, H_k$  (using Reingold’s algorithm [26]) and finding such a bipartition  $\{U_i, W_i\}$  for each component  $H_i$ . By using the logspace algorithm of [18] we can actually compute interval orders of the hypergraphs  $\mathcal{N}_{U_i}(H_i)$  which can be easily pasted together to give an interval order of  $\mathcal{N}_U(\overline{G})$ . Together with the canonical labeling algorithm this implies that the canonical arc representation problem for co-convex graphs and, in particular, for co-bipartite concave-round graphs is solvable in logspace.

It remains to show that for co-bipartite PCA graphs we can actually compute a proper arc representation in logspace. As above, we assume that  $G$  is twin-free. By Lemma 4, the hypergraph  $\mathcal{N}[G]$  has a tight CA order  $\prec$ . We can compute  $\prec$  in logspace by running the algorithm given by Theorem 1 on the tightened hypergraph  $(\mathcal{N}[G])^\ominus$ . Any tight CA order of  $\mathcal{N}[G]$  is also a tight CA order of  $\mathcal{N}(\overline{G})$ . Let  $V(G) = U \cup W$  be a bipartition of  $\overline{G}$  into two independent sets. Note that the restriction of a tight CA order of  $\mathcal{N}(\overline{G})$  to  $\mathcal{N}_U(\overline{G})$  is a tight interval order of the interval hypergraph  $\mathcal{N}_U(\overline{G})$ . Retracing Tucker’s construction of an arc representation  $\alpha$  for a co-convex graph  $G$  in the case that the interval order of  $\mathcal{N}_U(\overline{G})$  is tight, we see that  $\alpha$  now gives us a tight arc model for  $G$ . Thus, we only have to convert  $\alpha$  into a proper arc representation  $\alpha'$ . Tucker [29] described such a transformation, and Chen [7] showed that it can be implemented in  $\text{AC}^1$ . It is not hard to see that it can even be done in logspace. This completes the proof of Theorem 8 and we have additionally proved the following corollary.

► **Corollary 9.** *The canonical arc representation problem for co-convex graphs is solvable in logspace.*

## 6 Solving the Star System Problem

In this section, we present logspace algorithms for the Star System Problem: Given a hypergraph  $\mathcal{H}$ , find a graph  $G$  in a specified class of graphs  $\mathcal{C}$  such that  $\mathcal{N}[G] = \mathcal{H}$  (if such a graph exists). The term *star* refers to the closed neighborhood of a vertex in  $G$ . In this terminology, the problem is to identify the center of each star  $H$  in the star system  $\mathcal{H}$ . To denote this problem, we use the abbreviation *SSP*. Note that a logspace algorithm  $\mathcal{A}$  solving the SSP for a class  $\mathcal{C}$  cannot be directly used for solving the SSP for a subclass  $\mathcal{C}'$  of  $\mathcal{C}$ . For example, if  $\mathcal{A}$  on input  $\mathcal{H}$  outputs a solution  $G$  in  $\mathcal{C} \setminus \mathcal{C}'$ , then we don’t know whether there is another solution  $G'$  in  $\mathcal{C}'$ . However, if the SSP for  $\mathcal{C}$  has unique solutions and if membership in  $\mathcal{C}'$  is decidable in logspace, then it is easy to convert  $\mathcal{A}$  into a logspace algorithm  $\mathcal{A}'$  solving the SSP for  $\mathcal{C}'$ .

► **Theorem 10.**

1. *The SSP for PCA and for co-convex graphs is solvable in logspace.*
2. *If  $G$  is a co-convex graph, then  $\mathcal{N}[G] \cong \mathcal{N}[H]$  implies  $G \cong H$ .*

The implication stated in Theorem 10.2 is known to be true also for concave-round graphs (Chen [8]). As a consequence, since concave-round graphs form a logspace decidable subclass of the union of PCA and co-convex graphs, we can also solve the SSP for concave-round graphs in logspace.

The proof of Theorem 10 is outlined in the rest of this section. We design logspace algorithms  $\mathcal{A}_1$  and  $\mathcal{A}_2$  solving the SSP for non-co-bipartite PCA graphs and for co-convex graphs, respectively. Since by Theorem 10.2, the output of  $\mathcal{A}_2$  is unique up to isomorphism, we can easily combine the two algorithms to obtain a logspace algorithm  $\mathcal{A}_3$  solving the SSP

for all PCA graphs: On input  $\mathcal{H}$  run  $\mathcal{A}_1$  and  $\mathcal{A}_2$  and check if one of the resulting graphs is PCA (recall that co-bipartite PCA graphs are co-convex; see Fig. 1).

Clearly, it suffices to consider the case that the input hypergraph  $\mathcal{H}$  is connected.

**Non-co-bipartite PCA graphs.** Let  $\mathcal{H}$  be the given input hypergraph and assume that  $\mathcal{H} = \mathcal{N}[G]$  for a PCA graph  $G$ . By Theorem 3,  $\mathcal{H}$  has to be a tight CA hypergraph, a condition that can be checked by testing if the tightened hypergraph  $\mathcal{H}^\ominus$  is CA. Since  $G$  is concave-round, Proposition 7 implies that  $G$  is co-bipartite if and only if  $\mathcal{N}(\overline{G}) = \overline{\mathcal{H}}$  is an interval hypergraph. It follows that the SSP on  $\mathcal{H}$  can only have a non-co-bipartite PCA graph as solution if  $\mathcal{H}^\ominus$  is CA and  $\overline{\mathcal{H}}$  is not interval. Both conditions can be checked in logspace using the algorithms given by Theorem 1 and [18]. Further, it follows by Theorem 3 and Proposition 7 that in this case any SSP solution for  $\mathcal{H}$  is a non-co-bipartite PCA graph (which is also connected because  $\mathcal{H}$  is assumed to be connected).

By considering the quotient hypergraph with respect to twin-classes, we can additionally assume that  $\mathcal{H}$  is twin-free.

In order to reconstruct  $G$  from  $\mathcal{H}$ , we have to choose the center in each star  $H \in \mathcal{H}$ . The following lemma considerably restricts this choice.

► **Lemma 11.** *Let  $G$  be a connected, non-co-bipartite and twin-free PCA graph and let  $\prec$  be a circular order on  $V(G)$  that is a CA order of  $\mathcal{N}[G]$ . Then  $u \prec v$  holds exactly when  $N[u] \prec_{\mathcal{N}[G]} N[v]$ , where  $\prec_{\mathcal{N}[G]}$  is the circular order on  $\mathcal{N}[G]$  lifted from  $\prec$ .*

**Proof.** It suffices to show that  $u \prec v$  implies  $N[u] = [u^-, u^+] \prec_{\mathcal{N}[G]} N[v] = [v^-, v^+]$ . To this end we show that there is no third vertex  $w$  such that the arcs  $N[u]$ ,  $N[w]$ , and  $N[v]$  appear in this sequence under the circular order  $\prec_{\mathcal{N}[G]}$ .

Suppose first that  $u$  and  $v$  are adjacent. Then it follows from Proposition 6.4, that the vertices  $u^-, v^-, u, v, u^+$ , and  $v^+$  appear in this circular sequence; see Fig. 2(a). We split our analysis into three cases, depending on the position of  $w$  on the cycle  $(V(G), \prec)$ . If  $w \in (v, v^+]$ , then Proposition 6.3 implies that  $w^- \in [v^-, v]$  and  $v^+ \in [w, w^+]$ . If  $w^- \neq v^-$ , then  $N[u]$ ,  $N[v]$ , and  $N[w]$  appear in this sequence under  $\prec_{\mathcal{N}[G]}$ . The same holds true if  $w^- = v^-$  because then the arc  $[w^-, w^+]$  has to be longer than the arc  $[v^-, v^+]$  (note that, if also  $u^- = v^-$ , then  $[u^-, u^+]$  is shorter than  $[v^-, v^+]$ ). The case that  $w \in [u^-, u)$  is similar. If  $w \in (v^+, u^-)$ , then  $w^- \in (v, u^-)$ , and again  $N[w]$  cannot be intermediate.



■ **Figure 2** The two cases in the proof of Lemma 11.

Suppose now that  $u$  and  $v$  are not adjacent. It follows that  $N[u] = [u^-, u]$  and  $N[v] = [v, v^+]$ ; see Fig. 2(b). By Proposition 6.1, both  $N[u]$  and  $N[v]$  are cliques. Again we have to show that for no third vertex  $w$ , the arcs  $N[u]$ ,  $N[w]$ , and  $N[v]$  appear in this sequence under  $\prec_{\mathcal{N}[G]}$ . This is clear if  $w^- \in (v, u^-)$ . This is also so if  $w^- = v$ , because then the arc  $[v, v^+]$ , being a clique in  $G$ , must be shorter than the arc  $[w^-, w^+]$ . Finally we show that the remaining case  $w^- \in [u^-, v)$  is not possible. Indeed, in this case  $v \notin N[w]$ , for else the non-adjacent vertices  $u$  and  $v$  would belong to the clique  $[w, w^+]$ . Hence, it would follow that  $N[w] = [w^-, w^+] \subsetneq [u^-, u^+] = N[u]$ , contradicting the fact that  $N[u]$  is a clique. ◀

Lemma 11 states that the mapping  $v \mapsto N[v]$  is an isomorphism between the two directed cycles  $(V(G), \prec)$  and  $(\mathcal{N}[G], \prec_{\mathcal{N}[G]})$ . Since there are exactly  $n$  such isomorphisms, we get exactly  $n$  candidates  $f_1, \dots, f_n$  for the mapping  $v \mapsto N[v]$ . Hence, all we have to do is to use the algorithm given by Theorem 1 to compute a CA order  $\prec$  of  $\mathcal{H}$  and the corresponding lifted order  $\prec_{\mathcal{H}}$  in logspace. Now for each isomorphism  $f$  between  $(V(G), \prec)$  and  $(\mathcal{H}, \prec_{\mathcal{H}})$  we have to check if selecting  $v$  as the center of the star  $f(v)$  results in a graph  $G$ , that is, if for all  $v, u \in V(G)$  it holds that  $v \in f(v)$  and  $v \in f(u) \Leftrightarrow u \in f(v)$ .

**Co-convex graphs.** Let  $\mathcal{H}$  be the given hypergraph and assume that  $\mathcal{H} = \mathcal{N}[G]$  for a co-convex graph  $G$ . To facilitate the exposition, we also assume that the bipartite complement  $\overline{G}$  is connected, with vertex partition  $U, W$ . Then  $\overline{\mathcal{H}} = \mathcal{N}(\overline{G}) = \mathcal{N}_U(\overline{G}) \cup \mathcal{N}_W(\overline{G})$ , where the vertex-disjoint hypergraphs  $\mathcal{U} = \mathcal{N}_U(\overline{G})$  and  $\mathcal{W} = \mathcal{N}_W(\overline{G})$  are dual (i.e.,  $\mathcal{U}^* \cong \mathcal{W}$ ), both connected, and at least one of them is interval, say,  $\mathcal{U}$ . We need a simple auxiliary fact.

► **Proposition 12.** *Let  $H$  be a graph without isolated vertices and let  $\mathcal{L}$  be a connected component of  $\mathcal{N}(H)$ . Denote  $U = V(\mathcal{L})$ . Then either  $U$  is an independent set in  $H$  or  $U$  spans a connected component of  $H$ . Moreover, if  $U$  is independent, then there is a connected component of  $H$  that is a bipartite graph with  $U$  being one of its vertex classes.*

Denote  $\mathcal{K} = \overline{\mathcal{H}}$  and assume that  $\mathcal{K} = \mathcal{N}(H)$  for some graph  $H$ . Note that  $H$  cannot have an isolated vertex. Proposition 12 implies that either  $H$  is a connected bipartite graph with partition  $U, W$  or  $H$  has two connected components  $H_1$  and  $H_2$  with  $V(H_1) = U$  and  $V(H_2) = W$ . However, the second possibility leads to a contradiction. Indeed, since the hypergraph  $\mathcal{N}(H_1) = \mathcal{U}$  is interval, Proposition 7 implies that  $H_1$  is bipartite, contradicting the connectedness of  $\mathcal{U}$ . Therefore,  $H$  must be connected and bipartite with vertex partition  $U, W$ .

Recall that the *incidence graph* of a hypergraph  $\mathcal{X}$  is the bipartite graph with vertex classes  $V(\mathcal{X})$  and  $\mathcal{X}$  where  $x \in V(\mathcal{X})$  and  $X \in \mathcal{X}$  are adjacent if  $x \in X$  (if  $X$  has multiplicity  $k$  in  $\mathcal{X}$ , it contributes  $k$  fraternal vertices in the incidence graph). Since  $H$  is isomorphic to the incidence graph of the hypergraph  $\mathcal{U}$  (as well as  $\mathcal{W}$ ),  $H$  is logspace reconstructible from  $\mathcal{K}$  up to isomorphism and, in particular,  $H \cong \overline{G}$ . Thus, the solution to the SSP on  $\mathcal{H}$  is unique up to isomorphism.

After these considerations we are ready to describe our logspace algorithm for solving the SSP for the class of co-convex co-connected graphs. Given a hypergraph  $\mathcal{H}$ , we first check if  $\overline{\mathcal{H}}$  has exactly two connected components, say  $\mathcal{U}$  and  $\mathcal{W}$ . This can be done by running Reingold’s algorithm for the connectivity problem [26] on the intersection graph  $\mathbb{I}(\overline{\mathcal{H}})$ . If this is not the case, there is no solution in the desired class. Otherwise, we construct the incidence graph  $H$  of the hypergraph  $\mathcal{U}$  (or of  $\mathcal{W}$ , which should give the same result up to isomorphism) and take its complement  $\overline{H}$ . Note that this works well even if  $\overline{H}$  has twins: the twins in  $V(\mathcal{U})$  are explicitly present, while the twins in  $V(\mathcal{W})$  are represented by multiple hyperedges in  $\mathcal{U}$ .

As argued above, if the SSP on  $\mathcal{H}$  has a co-convex co-connected solution, then the closed neighborhood hypergraph  $\mathcal{H}' = \mathcal{N}[\overline{H}]$  of  $\overline{H}$  is isomorphic to  $\mathcal{H}$ . However, it may not be equal to  $\mathcal{H}$ . In this case we compute an isomorphism  $\varphi$  from  $\mathcal{H}'$  to  $\mathcal{H}$  or, the same task, from  $\overline{\mathcal{H}'}$  to  $\overline{\mathcal{H}}$ . This can be done by the algorithms of [18] and Corollary 2, because at least one of the connected components of  $\overline{\mathcal{H}'} \cong \overline{\mathcal{H}}$  is an interval hypergraph and the other component is isomorphic to the dual of an interval hypergraph. Now, the isomorphic image  $G = \varphi(\overline{H})$  of  $\overline{H}$  is the desired solution to the SSP on  $\mathcal{H}$  as  $\mathcal{N}[\varphi(\overline{H})] = \varphi(\mathcal{N}[\overline{H}]) = \mathcal{H}$ .

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