# Beyond Max-Cut: $\lambda$-Extendible Properties Parameterized Above the Poljak-Turzík Bound


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## Abstract

Poljak and Turzík (Discrete Math. 1986) introduced the notion of $\lambda$-extendible properties of graphs as a generalization of the property of being bipartite. They showed that for any $0 < \lambda < 1$ and $\lambda$-extendible property $\Pi$, any connected graph $G$ on $n$ vertices and $m$ edges contains a spanning subgraph $H \in \Pi$ with at least $\lambda m + \frac{1-\lambda}{2}(n-1)$ edges. The property of being bipartite is $\lambda$-extendible for $\lambda = 1/2$, and thus the Poljak-Turzík bound generalizes the well-known Edwards-Erdős bound for Max-Cut.

We define a variant, namely *strong* $\lambda$-extendibility, to which the Poljak-Turzík bound applies. For a strongly $\lambda$-extendible graph property $\Pi$, we define the parameterized Above Poljak-Turzík ($\Pi$) problem as follows: Given a connected graph $G$ on $n$ vertices and $m$ edges and an integer parameter $k$, does there exist a spanning subgraph $H$ of $G$ such that $H \in \Pi$ and $H$ has at least $\lambda m + \frac{1-\lambda}{2}(n-1) + k$ edges? The parameter is $k$, the surplus over the number of edges guaranteed by the Poljak-Turzík bound.

We consider properties $\Pi$ for which the Above Poljak-Turzík ($\Pi$) problem is fixed-parameter tractable (FPT) on graphs which are $O(k)$ vertices away from being a graph in which each block is a clique. We show that for all such properties, Above Poljak-Turzík ($\Pi$) is FPT for all $0 < \lambda < 1$. Our results hold for properties of oriented graphs and graphs with edge labels.

Our results generalize the recent result of Crowston et al. (ICALP 2012) on Max-Cut parameterized above the Edwards-Erdős bound, and yield FPT algorithms for several graph problems parameterized above lower bounds. For instance, we get that the above-guarantee Max $q$-Colorable Subgraph problem is FPT. Our results also imply that the parameterized above-guarantee Oriented Max Acyclic Digraph problem is FPT, thus solving an open question of Raman and Saurabh (Theor. Comput. Sci. 2006).



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## 1 Introduction

A number of interesting graph problems can be phrased as follows: Given a graph $G$ as input, find a subgraph $H$ of $G$ with the largest number of edges such that $H$ satisfies a specified property $\Pi$. Prominent among these is the MAX-CUT problem, which asks for a *bipartite* subgraph with the maximum number of edges. A *cut* of a graph $G$ is a partition of the vertex set of $G$ into two parts, and the *size* of the cut is the number of edges which *cross the cut*; that is, those which have their end points in distinct parts of the partition.

| MAX-CUT | |
|---|---|
| *Input:* | A graph $G$ and an integer $k$. |
| *Question:* | Does $G$ have a cut of size at least $k$? |

The MAX-CUT problem is among Karp's original list of 21 NP-complete problems [15], and it has been extensively investigated from the point of view of various algorithmic paradigms. Thus, for example, Goemans and Williamson showed [13] that the problem can be approximated in polynomial-time within a multiplicative factor of roughly 0.878, and Khot et al. showed that this is the best possible assuming the Unique Games Conjecture [16].

Our focus in this work is on the *parameterized* complexity of a generalization of the MAX-CUT problem. The central idea in the parameterized complexity analysis [8, 12] of NP-hard problems is to associate a *parameter* $k$ with each input instance of size $n$, and then to ask whether the resulting *parameterized problem* can be solved in time $f(k) \cdot n^c$ where $c$ is a constant and $f$ is some computable function. Parameterized problems which can be solved within such time bounds are said to be fixed-parameter tractable (FPT).

The *standard parameterization* of the MAX-CUT problem sets the parameter to be the size $k$ of the cut being sought. This turns out to be not very interesting for the following reason: Let $m$ be the number of edges in the input graph $G$. By an early result of Erdős [11], we know that every graph with $m$ edges contains a cut of size at least $m/2$. Therefore, if $k \leq m/2$ then we can immediately answer YES. In the remaining case $k > m/2$, and there are less than $2k$ edges in the input graph. It follows from this bound on the size of the input that any algorithm—even a brute-force method—which solves the problem runs in FPT time on this instance.

The best lower bound known on the size of a largest cut for connected loop-less graphs on $n$ vertices and $m$ edges is $\frac{m}{2} + \frac{n-1}{4}$, as proved by Edwards [9, 10]. This is called the *Edwards-Erdős bound*, and it is the best possible in the sense that it is tight for an infinite family of graphs, for example, the class of cliques of odd order $n$. A more interesting parameterization of MAX-CUT is, therefore, the following:

| MAX-CUT ABOVE TIGHT LOWER BOUND (MAX-CUT ATLB) | |
|---|---|
| *Input:* | A connected graph $G$, and an integer $k$. |
| *Parameter:* | $k$ |
| *Question:* | Does $G$ have a cut of size at least $\frac{m}{2} + \frac{n-1}{4} + k$? |

In the work which introduced the notion of "above-guarantee" parameterization, Mahajan and Raman [17] showed that the problem of asking for a cut of size at least $\frac{m}{2} + k$ is FPT parameterized by $k$, and stated the fixed-parameter tractability of MAX-CUT ATLB as an open problem. This question was resolved quite recently by Crowston et al. [6], who showed that the problem is in fact FPT.

We generalize the result of Crowston et al. by extending it to apply to a special case of the so-called $\lambda$-*extendible properties*. Roughly stated[1], for a fixed $0 < \lambda < 1$ a graph property $\Pi$ is said to be $\lambda$-extendible if: Given a graph $G = (V, E) \in \Pi$, an "extra" edge $uv$ not in $G$, and *any set* $F$ of "extra" edges each of which has one end point in $\{u, v\}$ and the other in $V$, there exists a graph $H \in \Pi$ which contains (i) all of $G$, (ii) the edge $uv$, and (iii) at least a $\lambda$ fraction of the edges in $F$. The notion was introduced by Poljak and Turzík who showed [20] that for any $\lambda$-extendible property $\Pi$ and edge-weighting function $c : E \to \mathbb{R}^+$, any connected graph $G = (V, E)$ contains a spanning subgraph $H = (V, F) \in \Pi$ such that $c(F) \geq \lambda \cdot c(E) + \frac{1-\lambda}{2} c(T)$. Here $c(X)$ denotes the total weight of all the edges in $X$, and $T$ is the set of edges in a minimum-weight spanning tree of $G$. It is not difficult to see that the property of being bipartite is $\lambda$-extendible for $\lambda = 1/2$, and so—once we assign unit weights to all edges—the Poljak and Turzík result implies the Edwards-Erdős bound. Other examples of $\lambda$-extendible properties—with different values of $\lambda$—include $q$-colorability and acyclicity in oriented graphs.

In this work we study the natural above-guarantee parameterized problem for $\lambda$-extendible properties $\Pi$, which is: given a connected graph $G = (V, E)$ and an integer $k$ as input, does $G$ contain a spanning subgraph $H = (V, F) \in \Pi$ such that $c(F) = \lambda \cdot c(E) + \frac{1-\lambda}{2} c(T) + k$? To derive a generic FPT algorithm for this class of problems, we use the "reduction" rules of Crowston et al. To make these rules work, however, we need to make a couple of concessions. Firstly, we slightly modify the notion of lambda extendibility; we define a (potentially) stronger notion which we name *strong* $\lambda$-*extendibility*. Every strongly $\lambda$-extendible property is also $\lambda$-extendible by definition, and so the Poljak-Turzík bound applies to strongly $\lambda$-extendible properties as well. Observe that for each way of assigning edge-weights, the Poljak and Turzík result yields a (potentially) different lower bound on the weight of the subgraph. Following the spirit of the question posed by Mahajan and Raman and solved by Crowston et al., we choose from among these the lower bound implied by the unit-edge-weighted case. This is our second simplification, and for this "unweighted" case the Poljak and Turzík result becomes: for any strongly $\lambda$-extendible property $\Pi$, any connected graph $G = (V, E)$ contains a spanning subgraph $H = (V, F) \in \Pi$ such that $|F| = \lambda \cdot |E| + \frac{1-\lambda}{2}(|V| - 1)$.

The central problem which we discuss in this work is thus the following; here $0 < \lambda < 1$, and $\Pi$ is an arbitrary—but fixed—strongly $\lambda$-extendible property:

| | |
|---|---|
| Above Poljak-Turzík ($\Pi$) (APT($\Pi$)) | |
| *Input:* | A connected graph $G = (V, E)$ and an integer $k$. |
| *Parameter:* | $k$ |
| *Question:* | Is there a spanning subgraph $H = (V, F) \in \Pi$ of $G$ such that $\lvert F\rvert \geq \lambda\lvert E\rvert + \frac{1-\lambda}{2}(\lvert V\rvert - 1) + k$? |

## 1.1 Our Results and their Implications

We show that the Above Poljak-Turzík ($\Pi$) problem is FPT for every strongly $\lambda$-extendible property $\Pi$ for which APT($\Pi$) is FPT on a class of "almost-forests of cliques". Informally[1], this is a class of graphs which are a small number ($O(k)$) of vertices away from being a graph in which each block is a clique. This requirement is satisfied by the properties underlying a number of interesting problems, including Max-Cut, Max $q$-Colorable Subgraph, and Oriented Max Acyclic Digraph. The main result of this paper is the following.

[1] See subsection 1.2 and section 2 for the definitions of various terms used in this section.

▶ **Theorem 1.** *The* Above Poljak-Turzík (Π) *problem is fixed-parameter tractable for a* $\lambda$*-extendible property* Π *of graphs if*

- Π *is strongly* $\lambda$*-extendible and*
- Above Poljak-Turzík (Π) *is* FPT *on almost-forests of cliques.*

*This also holds for such properties of oriented and/or labelled graphs.*

We prove Theorem 1 using the classical "Win/Win" approach of parameterized complexity. To wit: given an instance $(G, k)$ of a strongly $\lambda$-extendible property Π, in polynomial time we either (i) show that $(G, k)$ is a yes instance, or (ii) find a vertex subset $S$ of $G$ of size at most $6k/(1 - \lambda)$ such that deleting $S$ from $G$ leaves a "forest of cliques". To do this we use the "reduction" rules derived by Crowston et al. [6] to solve Max-Cut. Our main technical contribution is a proof that these rules are sufficient to show that *every* NO *instance* of APT(Π) is at a vertex-deletion distance of $O(k)$ from a forest of cliques. This proof requires several new ideas: a result which holds for *all* strongly $\lambda$-extendible properties Π is a significant step forward from Max-Cut. Our main result unifies and generalizes several known FPT results, and implies new ones. These include such algorithms for (i) Max-Cut, (ii) finding a $q$-colorable subgraph of the maximum size, and (ii) finding a maximum-size acyclic subdigraph in an oriented graph. We also obtain a linear *vertex* kernel for the latter problem, complementing the quadratic *arc* kernel by Gutin et al. [14].

**Related Work.** The notion of parameterizing above (or below) some kind of "guaranteed" values—lower and upper bounds, respectively—was introduced by Mahajan and Raman [17]. It has proven to be a fertile area of research, and Max-Cut is now just one of a host of interesting problems for which we have FPT results for such questions [1, 2, 3, 4, 5, 6, 14, 18, 21]. Indeed, very recent work due to Crowston et al. [4], where they take up an above-guarantee parameterization of the problem of finding a large acyclic digraph in an oriented graph, appears as an article in these selfsame proceedings; see the discussion after Corollary 32.

## 1.2 Preliminaries

We use "graph" to denote simple graphs without self-loops, directions, or labels, and assume the graph terminology of Diestel [7]. $V(G)$ and $E(G)$ denote, respectively, the vertex and edge sets of a graph $G$. For $S \subseteq V(G)$, we use (i) $G[S]$ to denote the subgraph of $G$ induced by the set $S$, (ii) $G \setminus S$ to denote $G[V(G) \setminus S]$, (iii) $\delta(S)$ to denote the set of edges in $G$ which have exactly one end-point in $S$, and (iv) $e_G(S)$ to denote $|E(G[S])|$; we omit the subscript $G$ if it is clear from the context. A *clique* in a graph $G$ is a set of vertices $C$ such that between any pair of vertices in $C$ there is an edge in $E(G)$. A *block* of graph $G$ is a maximal 2-connected subgraph of $G$, and a graph $G$ is a *forest of cliques* if the vertex set of every block forms a clique. A *leaf clique* of a forest of cliques is a block of the graph which contains at most one cutvertex of the graph.

For $F \subseteq E(G)$, (i) we use $G \setminus F$ to denote the graph $(V(G), E(G) \setminus F)$, and (ii) for a weight function $c : E(G) \to \mathbb{R}^+$, we use $c(F)$ to denote the sum of the weights of all the edges in $F$. A *graph property* is a collection of graphs. For $i, j \in \mathbb{N}$ we use $K_i$ to denote the complete graph on $i$ vertices, and $K_{i,j}$ to denote the complete bipartite graph in which the two parts of vertices are of sizes $i, j$.

Our results also apply to graphs with oriented edges, and those with edge labels. Subgraphs of an oriented or labelled graph $G$ inherit the orientation or labelling—as is the case—of $G$ in the natural manner: each surviving edge keeps the same orientation/labelling as it had in $G$. For a graph $G$ of any kind, we use $G_S$ to denote the simple graph obtained by removing all orientations and labels from $G$; we say that $G$ is connected (or contains a clique, and so forth) if $G_S$ is connected (or contains a clique, and so forth).

## 2 Definitions

The following notion is a variation on the concept of $\lambda$-extendibility [20].

▶ **Definition 2** (Strong $\lambda$-extendibility)**.** Let $\mathcal{G}$ be the class of (possibly oriented and/or labelled) graphs, and let $0 < \lambda < 1$. A property $\Pi \subseteq \mathcal{G}$ is *strongly $\lambda$-extendible* if it satisfies the following:

**Inclusiveness** $\{G \in \mathcal{G} \,|\, G_S \in \{K_1, K_2\}\} \subseteq \Pi$

**Block additivity** $G \in \mathcal{G}$ belongs to $\Pi$ if and only if each block of $G$ belongs to $\Pi$.

**Strong $\lambda$-subgraph extension** Let $G \in \mathcal{G}$ and $S \subseteq V(G)$ be such that $G[S] \in \Pi$ and $G \setminus S \in \Pi$. For any weight function $c : E(G) \to \mathbb{R}^+$ there exists an $F \subseteq \delta(S)$ with $c(F) \geq \lambda \cdot c(\delta(S))$, such that $G \setminus (\delta(S) \setminus F) \in \Pi$.

The strong $\lambda$-subgraph extension requirement can be rephrased as follows: Let $V(G) = X \uplus Y$ be a cut of graph $G$ such that $G[X], G[Y] \in \Pi$, and let $F$ be the set of edges which cross the cut. For *any* weight function $c : F \to \mathbb{R}^+$, there exists a subset $F' \subseteq F$ such that (i) $c(F') \leq (1 - \lambda) \cdot c(F)$, and (ii) $(G \setminus F') \in \Pi$. Informally, one can pick a $\lambda$-fraction of the cut and delete the rest to obtain a graph which belongs to $\Pi$.

We recover Poljak and Turzík's definition of $\lambda$-extendibility from the above definition by replacing strong $\lambda$-subgraph extension with the following property:

**$\lambda$-edge extension** Let $G \in \mathcal{G}$ and $S \subseteq V(G)$ be such that $G_S[S]$ *is isomorphic to* $K_2$ and $G \setminus S \in \Pi$. For any weight function $c : E(G) \to \mathbb{R}^+$ there exists an $F \subseteq \delta(S)$ with $c(F) \geq \lambda \cdot c(\delta(S))$, such that $G \setminus (\delta(S) \setminus F) \in \Pi$.

Observe from the definitions that any graph property which is strongly $\lambda$-extendible is also $\lambda$-extendible. It follows that Poljak and Turzík's result for $\lambda$-extendible properties applies also to strongly $\lambda$-extendible properties.

▶ **Theorem 3** (Poljak-Turzík bound)**.** [20] *Let $\mathcal{G}$ be a class of (possibly oriented and/or labelled) graphs. Let $0 < \lambda < 1$, and let $\Pi \subseteq \mathcal{G}$ be a strongly $\lambda$-extendible property. For any connected graph $G \in \mathcal{G}$ and weight function $c : E(G) \to \mathbb{R}^+$, there exists a spanning subgraph $H \in \Pi$ of $G$ such that $c(E(H)) \geq \lambda \cdot c(E(G)) + \frac{1-\lambda}{2} c(T)$, where $T$ is the set of edges in a minimum-weight spanning tree of $G_S$.*

When all edges are assigned weight 1, we get:

▶ **Corollary 4.** *Let $\mathcal{G}, \lambda, \Pi$ be as in Theorem 3. Any connected graph $G \in \mathcal{G}$ on $n$ vertices and $m$ edges has a spanning subgraph $H \in \Pi$ with at least $\lambda m + \frac{1-\lambda}{2}(n-1)$ edges.*

Our results apply to properties which satisfy the additional requirement of being FPT on almost-forests of cliques.

▶ **Definition 5** (FPT on almost-forests of cliques)**.** Let $0 < \lambda < 1$, and let $\Pi$ be a strongly $\lambda$-extendible property (of graphs with or without orientations/labels). The Structured Above Poljak-Turzík ($\Pi$) problem is a variant of the Above Poljak-Turzík ($\Pi$) problem in which, along with the graph $G$ and $k \in \mathbb{N}$, the input contains a set $S \subseteq V(G)$ such that $|S| = O(k)$ and $G \setminus S$ is a forest of cliques. We say that the property $\Pi$ is *FPT on almost-forests of cliques* if the Structured Above Poljak-Turzík ($\Pi$) problem is FPT.

In other words, a $\lambda$-extendible property $\Pi$ is FPT on almost-forests of cliques, if for any constant $q$ there is an algorithm that, given a connected graph $G, k$ and a set $S \subseteq V(G)$ of size at most $q \cdot k$ such that $G \setminus S$ is a forest of cliques, correctly decides whether $(G, k)$ is a yes-instance of APT($\Pi$) in $O(f(k) \cdot n^{O(1)})$ time, for some computable function $f$.

## 3 Fixed-Parameter Algorithms for Above Poljak-Turzík ($\Pi$)

We now prove Theorem 1 using Crowston et al.'s line of attack for solving Max-Cut [6]. The crux of their strategy is a polynomial-time procedure which takes the input $(G, k)$ of Max-Cut and finds a subset $S \subseteq V(G)$ such that (i) $G \setminus S$ is a forest of cliques, and (ii) if $(G, k)$ is a NO instance, then $|S| \leq 3k$. Thus if $|S| > 3k$, then one can immediately answer YES; otherwise one solves the problem in FPT time using the fact that Max-Cut is FPT on almost-forests of cliques (Definition 5).

The nontrivial part of our work consists of proving that the procedure for Max-Cut applies also to the much more general family of strongly $\lambda$-extendible problems, where the bound on the size of $S$ depends on $\lambda$. To do this, we show that each of the four rules used for Max-Cut is safe to apply for any strongly $\lambda$-extendible property. From this we get

▶ **Lemma 6.** *Let $0 < \lambda < 1$, and let $\Pi$ be a strongly $\lambda$-extendible graph property. Given a connected graph $G$ with $n$ vertices and $m$ edges and an integer $k$, in polynomial time we can do one of the following:*

1. *Decide that there is a spanning subgraph $H \in \Pi$ of $G$ with at least $\lambda m + \frac{1-\lambda}{2}(n-1) + k$ edges, or;*
2. *Find a set $S$ of at most $\frac{6}{1-\lambda}k$ vertices in $G$ such that $G \setminus S$ is a forest of cliques.*

*This also holds for strongly $\lambda$-extendible properties of oriented and/or labelled graphs.*

We give an algorithmic proof of Lemma 6. Let $(G, k)$ be an instance of Above Poljak-Turzík ($\Pi$). The algorithm initially sets $\tilde{G} := G$, $\tilde{S} := \emptyset$, $\tilde{k} := k$, and then applies a series of rules to the tuple $(\tilde{G}, \tilde{S}, \tilde{k})$. Each application of a rule to $(\tilde{G}, \tilde{S}, \tilde{k})$ produces a tuple $(G', S', k')$ such that (i) if $\tilde{G} \setminus \tilde{S}$ is connected then so is $G' \setminus S'$, and (ii) if $(\tilde{G} \setminus \tilde{S}, \tilde{k})$ is a NO instance of APT($\Pi$) then so is $(G' \setminus S', k')$; the converse may not hold. The algorithm then sets $\tilde{G} := G', \tilde{S} := S', \tilde{k} := k'$, and repeats the process, till none of the rules applies.

We now state the four rules—which, but for minor changes, are due to Crowston et al. [6]—and show that they suffice to prove Lemma 6. We assume throughout that $\lambda$ and $\Pi$ are as in Lemma 6. For brevity we assume that the empty graph is in $\Pi$, and we let $\lambda' = \frac{1}{2}(1-\lambda)$ so that $\lambda + 2\lambda' = 1$.

▶ **Rule 1.** Let $\tilde{G} \setminus \tilde{S}$ be connected. If $v \in (V(\tilde{G}) \setminus \tilde{S})$ and $X \subseteq (V(\tilde{G}) \setminus (\tilde{S} \cup \{v\}))$ are such that (i) $\tilde{G}[X]$ is a connected component of $\tilde{G} \setminus (\tilde{S} \cup \{v\})$, and (ii) $X \cup \{v\}$ is a clique in $\tilde{G}$, then delete $X$ from $\tilde{G}$ to get $G'$; set $S' = \tilde{S}, k' = \tilde{k}$.

▶ **Rule 2.** Let $\tilde{G} \setminus \tilde{S}$ be connected. Suppose Rule 1 does not apply, and let $X_1, \ldots, X_\ell$ be the connected components of $\tilde{G} \setminus (\tilde{S} \cup \{v\})$ for some $v \in V(\tilde{G}) \setminus \tilde{S}$. If at least one of the $X_i$s is a clique, and at most one of them is *not* a clique, then

- Delete all the $X_i$s which are cliques—let these be $d$ in number—to get $G'$, and
- Set $S' := \tilde{S} \cup \{v\}$ and $k' := \tilde{k} - d\lambda'$.

▶ **Rule 3.** Let $\tilde{G} \setminus \tilde{S}$ be connected. If $a, b, c \in V(\tilde{G}) \setminus \tilde{S}$ are such that $\{a, b\}, \{b, c\} \in E(\tilde{G})$, $\{a, c\} \notin E(\tilde{G})$, and $\tilde{G} \setminus (\tilde{S} \cup \{a, b, c\})$ is connected, then

- Set $S' := \tilde{S} \cup \{a, b, c\}$ and $k' := \tilde{k} - \lambda'$.

▶ **Rule 4.** Let $\tilde{G} \setminus \tilde{S}$ be connected. Suppose Rule 3 does not apply, and let $x, y \in V(\tilde{G}) \setminus \tilde{S}$ be such that $\{x, y\} \notin E(\tilde{G})$. Let $C_1, \ldots, C_\ell$ be the connected components of $\tilde{G} \setminus (\tilde{S} \cup \{x, y\})$. If there is at least one $C_i$ such that (i) both $V(C_i) \cup \{x\}$ and $V(C_i) \cup \{y\}$ are cliques in $\tilde{G} \setminus \tilde{S}$, and (ii) there is at most one $C_i$ for which (i) does *not* hold, then

- Delete all the $C_i$s which satisfy condition (i) above to get $G'$, and,
- Set $S' := \tilde{S} \cup \{x, y\}$, $k' := \tilde{k} - \lambda'$.

Let $(G^\star, S, k^\star)$ be the tuple which we get by applying these rules exhaustively to the input tuple $(G, \emptyset, k)$. To prove Lemma 6, it is sufficient to prove the following claims: (i) the rules can be exhaustively applied in polynomial time; (ii) $G \setminus S$ is a forest of cliques; (iii) the rules transform NO-instances to NO-instances; and, (iv) if $(G, k)$ is a NO instance, then $|S| \leq \frac{6}{1-\lambda} k$. We now proceed to prove these over several lemmata. Our rules are identical to those of Crowston et al. in how the rules modify the graph; the only difference is in how we change the parameter $k$. The first two claims thus follow directly from their work.

▶ **Lemma 7.** [$\star$][2] *Rules 1 to 4 can be exhaustively applied to an instance* $(G, k)$ *of* Above Poljak-Turzík (Π) *in polynomial time. The resulting tuple* $(G^\star, S, k^\star)$ *has* $|V(G^\star) \setminus S| \leq 1$.

▶ **Lemma 8.** **[6, Lemma 8]** *Let* $(G^\star, S, k^\star)$ *be the tuple obtained by applying Rules 1 to 4 exhaustively to an instance* $(G, k)$ *of* Above Poljak-Turzík (Π). *Then* $G \setminus S$ *is a forest of cliques.*

The correctness of the remaining two claims is a consequence of the λ-extendibility of property Π, and we make critical use of this fact in building the rest of our proof. This is the one place where this work is significantly different from the work of Crowston et al.; they could take advantage of the special characteristics of *one specific* property, namely bipartitedness, to prove the analogous claims for Max-Cut.

We say that a rule is *safe* if it preserves NO instances.

▶ **Definition 9.** Let $(\tilde{G}, \tilde{S}, \tilde{k})$ be an arbitrary tuple to which one of the rules 1, 2, 3, or 4 applies, and let $(G', S', k')$ be the resulting tuple. We say that the rule is *safe* if, whenever $(G' \setminus S', k')$ is a YES instance of Above Poljak-Turzík (Π), then so is $(\tilde{G} \setminus \tilde{S}, \tilde{k})$.

We now prove that each of the four rules is safe. For a graph $G$ we use $val(G)$ to denote the maximum number of edges in a subgraph $H \in \Pi$ of $G$, and $pt(G)$ to denote the Poljak-Turzík bound for $G$. Thus if $G$ is connected and has $n$ vertices and $m$ edges then $pt(G) = \lambda m + \lambda'(n-1)$, and Corollary 4 can be written as $val(G) \geq pt(G)$. For each rule we assume that $G' \setminus S'$ has a spanning subgraph $H' \in \Pi$ with at least $pt(G' \setminus S') + k'$ edges, and show that $\tilde{G} \setminus \tilde{S}$ has a spanning subgraph $\tilde{H} \in \Pi$ with at least $pt(\tilde{G} \setminus \tilde{S}) + \tilde{k}$ edges.

We first derive a couple of lemmas which describe how contributions from subgraphs of a graph $G$ add up to yield lower bounds on $val(G)$.

▶ **Lemma 10.** [$\star$] *Let $v$ be a cutvertex of a connected graph $G$, and let $\mathcal{C} = C_1, C_2, \ldots C_r; r \geq 2$ be sets of vertices of $G$ such that (i) for every $i \neq j$ we have $C_i \cap C_j = \{v\}$, (ii) there is no edge between $C_i \setminus \{v\}$ and $C_j \setminus \{v\}$, and (iii) $\bigcup_{1 \leq i \leq r} C_i = V(G)$. For $1 \leq i \leq r$, let $H_i \in \Pi$ be a subgraph of $G[C_i]$ with $pt(G[C_i]) + k_i$ edges, and let $H = (V(G), \bigcup_{i=1}^{r} E(H_i))$. Then $H$ is a subgraph of $G$, $H \in \Pi$, and $|E(H)| \geq pt(G) + \sum_{i=1}^{r} k_i$.*

[2] Proofs of results marked with a $\star$ have been deferred to the full version of the paper, a preprint of which is available on arXiv [19].

▶ **Lemma 11.** [⋆] *Let $G$ be a graph, and let $S \subseteq V(G)$ be such that there exist (i) a subgraph $H_S \in \Pi$ of $G[S]$ with at least $pt(G[S]) + \lambda' + k_S$ edges, and (ii) a subgraph $\overline{H} \in \Pi$ of $G \setminus S$ with at least $pt(G \setminus S) + \lambda' + \overline{k}$ edges. Then there is a subgraph $H \in \Pi$ of $G$ with at least $pt(G) + \lambda' + k_S + \overline{k}$ edges.*

This lemma has a useful special case which we state as a corollary:

▶ **Corollary 12.** *Let $G$ be a graph, and let $S \subseteq V(G)$ be such that (i) there exists a subgraph $H_S \in \Pi$ of $G[S]$ with at least $pt(G[S]) + \lambda' + k_S$ edges, and (ii) the subgraph $G \setminus S$ has a perfect matching. Then there is a subgraph $H \in \Pi$ of $G$ with at least $pt(G) + \lambda' + k_S$ edges.*

**Proof.** Recall that the graph $K_2$ is in $\Pi$ by definition, and observe that $pt(K_2) = \lambda + \lambda'$. Thus $K_2$ has $pt(K_2) + \lambda'$ edges. The corollary now follows by repeated application of Lemma 11, each time considering a new edge of the matching as the graph $\overline{H}$. ◀

The safeness of Rule 1 is now a consequence of the block additivity property.

▶ **Lemma 13.** [⋆] *Rule 1 is safe.*

We now prove some useful facts about certain simple graphs, in the context of strongly $\lambda$-extendible properties. Observe that every block of a forest is one of $\{K_1, K_2\}$, which are both in $\Pi$. From this and the block additivity property of $\Pi$ we get

▶ **Observation 14.** Every forest (with every orientation and labeling) is in $\Pi$.

The graph $K_{2,1}$ is a useful special case.

▶ **Observation 15.** The graph $K_{2,1}$—also with any kind of orientation or labelling—is in $\Pi$, and it has $pt(K_{2,1}) + \lambda' + \lambda'$ edges.

The graph obtained by removing one edge from $K_4$ is another useful object, since it always has more edges than its Poljak-Turzík bound.

▶ **Lemma 16.** [⋆] *Let $G$ be a graph formed from the graph $K_4$—also with any kind of orientation or labelling—by removing one edge. Then (i) $val(G) \geq 3$, (ii) $val(G) \geq 4$ if $\lambda > 1/3$, and (iii) $val(G) = 5$ if $\lambda > 1/2$. As a consequence,*

$$val(G) \geq pt(G) + \lambda' + \begin{cases} (1 - 3\lambda) & \text{if } \lambda \leq 1/3, \\ (2 - 3\lambda) & \text{if } 1/3 < \lambda \leq 1/2, \text{ and,} \\ (3 - 3\lambda) & \text{if } \lambda > 1/2. \end{cases} \quad (1)$$

The above lemmata help us prove that Rules 2 and 3 are safe.

▶ **Lemma 17.** [⋆] *Rule 2 is safe.*

Following the notation of Rule 3, observe that for the vertex subset $T = \{a, b, c\} \subseteq V(\tilde{G} \setminus \tilde{S})$ we have—from Observation 15—that $\tilde{G}[T] \in \Pi$ and $val(T) \geq pt(T) + \lambda' + \lambda'$. Since $G' \setminus S' = (\tilde{G} \setminus \tilde{S}) \setminus T$, if $val(G' \setminus S') \geq pt(G' \setminus S') + k'$ then applying Lemma 11 we get that $val(\tilde{G} \setminus \tilde{S}) \geq pt(\tilde{G} \setminus \tilde{S}) + \lambda' + k' = pt(\tilde{G} \setminus \tilde{S}) + \tilde{k}$. Hence we get

▶ **Lemma 18.** *Rule 3 is safe.*

To show that Rule 4 is safe, we need a number of preliminary results. We first observe that—while the rule is stated in a general form—the rule only ever applies when it can delete exactly one component.

▶ **Observation 19.** [⋆] Whenever Rule 4 applies, there is exactly one component to be deleted, and this component has at least 2 vertices.

Our next few lemmas help us further restrict the structure of the subgraph to which Rule 4 applies. We start with a result culled from Crowston et al.'s analysis of the four rules.

▶ **Lemma 20.** [**6**][⋆] *If none of Rules 1, 2, and 3 applies to $(\tilde{G}, \tilde{S}, \tilde{k})$, and Rule 4* does *apply, then one can find*

- *A vertex $r \in V(\tilde{G} \setminus \tilde{S})$ and a set $X \subseteq V(\tilde{G} \setminus \tilde{S})$ such that $X$ is a connected component of $\tilde{G} \setminus (\tilde{S} \cup \{r\})$, and the graph $(\tilde{G} \setminus \tilde{S})[X \cup \{r\}]$ is 2-connected;*
- *Vertices $x, y \in X$ such that $\{x, y\} \notin E(\tilde{G})$ and*
  - *$(\tilde{G} \setminus \tilde{S}) \setminus \{x, y\}$ has exactly two components $G', C$,*
  - *$r \in G'$; $C \cup \{x\}, C \cup \{y\}$ are cliques, and each of $x, y$ is adjacent to some vertex in $G'$*

From this we get the following.

▶ **Lemma 21.** [⋆] *Suppose Rules 1, 2, and 3 do not apply, and Rule 4 applies. Then we can apply Rule 4 in such a way that if $x, y$ are the vertices to be added to $\tilde{S}$ and $C$ the clique to be deleted, then $N(x) \cup N(y) \setminus (C \cup \tilde{S})$ contains at most one vertex $z$ such that $\tilde{G} \setminus (\tilde{S} \cup \{z\})$ is disconnected.*

We now show that in such a case $N(x) \setminus (C \cup \tilde{S}) = N(y) \setminus (C \cup \tilde{S}) = \{r\}$, and so the graph $\tilde{G} \setminus (\tilde{S} \cup \{r\})$ is not connected. First we need the following simple lemma.

▶ **Lemma 22.** [⋆] *Whenever Rule 4 applies, with $x, y$ the vertices to be added to $\tilde{S}$ and $C$ the clique to be deleted, every $u$ in $N(x) \setminus (C \cup \tilde{S})$ is a cutvertex in $\tilde{G} \setminus (\tilde{S} \cup \{x\})$ and every $u$ in $N(y) \setminus (C \cup \tilde{S})$ is a cutvertex in $\tilde{G} \setminus (\tilde{S} \cup \{y\})$.*

This allows us to enforce a very special way of applying Rule 4.

▶ **Lemma 23.** [⋆] *Suppose Rules 1, 2, and 3 do not apply, and Rule 4 applies. Then we can apply Rule 4 in such a way that if $x, y$ are the vertices to be added to $\tilde{S}$ and $C$ the clique to be deleted, then $N(x) \setminus (C \cup \tilde{S}) = N(y) \setminus (C \cup \tilde{S}) = \{z\}$, and $\tilde{G} \setminus (\tilde{S} \cup \{z\})$ is disconnected.*

These lemmas help us prove that Rule 4 is safe.

▶ **Lemma 24.** [⋆] *Rule 4 is safe.*

The next lemma gives us a bound on the size of the set $S$ which we compute.

▶ **Lemma 25.** [⋆] *Let $\tilde{G}$ be a connected graph, $\tilde{S} \subseteq V(\tilde{G})$, and $\tilde{k} \in \mathbb{N}$, and let one application of Rule 1, 2, 3, or 4 to $(\tilde{G}, \tilde{S}, \tilde{k})$ result in the tuple $(G', S', k')$. Then $|S' \setminus \tilde{S}| \leq 3(\tilde{k} - k')/\lambda'$.*

Now we are ready to prove Lemma 6, from which our main result (Theorem 1) easily follows.

**Proof (of Lemma 6).** Let $(G, k)$ be an input instance of Above Poljak-Turzík (Π), and let $(G^\star, S, k^\star)$ be the tuple which we get by applying the four rules exhaustively to the tuple $(G, \emptyset, k)$. From Lemma 7 we know that this can be done in polynomial time, and that the resulting graph satisfies $|V(G^\star) \setminus S| \leq 1$.

Thus $G^\star \setminus S$ is either $K_1$ or the empty graph, and so $G^\star \setminus S \in \Pi$ and $pt(G^\star \setminus S) = 0, |E(G^\star \setminus S)| = 0$. Hence if $k^\star \leq 0$ then $(G^\star \setminus S, k^\star)$ is a YES instance of Above Poljak-Turzík (Π). Since all the four rules are safe—Lemmas 13, 17, 18, and 24—we get that in this case $(G, k)$ is a YES instance, and we can return YES. On the other hand if $k^\star > 0$ then we know—using Lemma 25—that $|S| < 3k/\lambda' = 6k/(1 - \lambda)$, and—from Lemma 8—that $G \setminus S$ is a forest of cliques. This completes the proof. ◀

## 4 Applications

In this section we use Theorem 1 to show that Above Poljak-Turzík ($\Pi$) is FPT for almost all natural examples of $\lambda$-extendible properties listed by Poljak and Turzík [20]. For want of space, we defer the definitions and all proofs to the full version of the paper.

### 4.1 Application to Partitioning Problems

First we focus on properties specified by a homomorphism to a vertex transitive graph. As a graph is $h$-colorable if and only if it has a homomorphism to $K_h$, searching for a maximal $h$-colorable subgraph is one of the problems resolved in this section. In particular, a maximum cut equals a maximum bipartite subgraph and, hence, is also one of the properties studied in this section. We use $\mathcal{G}$ to denote the class of graphs—oriented or edge-labelled—to which the property in question belongs.

It is not difficult to see that every vertex-transitive graph $G$ is a regular graph. In particular, if $\mathcal{G}$ allows labels and/or orientations, then for every label $l$ and every orientation $r$ each vertex of a vertex transitive graph $G$ is incident to the same number of edges of label $l$ and orientation $r$. Let us denote this number $d_{l,r}(G)$. Let us also denote by $d_{\mathcal{G}}(G)$ the minimum of $d_{l,r}(G)$ over all labels and orientations allowed in $\mathcal{G}$.

▶ **Lemma 26.** [$\star$] *Let $G_0$ be a vertex-transitive graph with at least one edge of every label and orientation allowed in $\mathcal{G}$. Then the property "to have a homomorphism to $G_0$" is strongly $d/n_0$-extendible in $\mathcal{G}$, where $n_0$ is the number of vertices of $G_0$ and $d = d_{\mathcal{G}}(G_0)$.*

Note that while the above lemma poses no restrictions on the graphs considered, we can prove the following only for simple graphs.

▶ **Lemma 27.** [$\star$] *If $G_0$ is an unoriented unlabeled graph, then the problem APT("to have a homomorphism into $G_0$") is FPT on almost-forests of cliques.*

Lemma 26 and Lemma 27 together with Theorem 1 imply the following corollary.

▶ **Corollary 28.** *The problem APT("to have a homomorphism into $G_0$") is fixed-parameter tractable for every unoriented unlabeled vertex transitive graph $G_0$.*

In particular, by setting $G_0 = K_q$ we get the following result.

▶ **Corollary 29.** *Given a graph $G$ with $m$ edges and $n$ vertices and an integer $k$, it is FPT to decide whether $G$ has an $q$-colorable subgraph with at least $m \cdot (q-1)/q + (n-1)/(2q) + k$ edges.*

This shows that the Max $q$-Colorable Subgraph problem is FPT when parameterized above the Poljak and Turzík bound [20].

### 4.2 Finding Acyclic Subgraphs of Oriented Graphs

In this section we show how to apply our result to the problem of finding a maximum-size directed acyclic subgraph of an oriented graph, where the size of the subgraph is defined as the number of arcs in the subgraph. Recall that an oriented graph is a directed graph where between any two vertices there is at most one arc. We show that Theorem 1 applies to this problem. To this end we need the following two lemmata.

▶ **Lemma 30.** [$\star$] *The property "acyclic oriented graphs" is strongly $1/2$-extendible in the class of oriented graphs.*

▶ **Lemma 31.** [⋆] *The problem APT("acyclic oriented graphs") is* FPT *on almost-forests of cliques.*

Combining Lemmata 30 and 31 with Theorem 1 we get the following corollary.

▶ **Corollary 32.** *The problem APT("acyclic oriented graphs") is fixed-parameter tractable.*

To put this result in some context, we recall a couple of open problems posed by Raman and Saurabh [21]: Are the following questions FPT parameterized by $k$?

- Given an oriented directed graph on $n$ vertices and $m$ arcs, does it have a subset of at least $\frac{m}{2} + \frac{1}{2}(\lceil \frac{n-1}{2} \rceil) + k$ arcs that induces an acyclic subgraph?
- Given a directed graph on $n$ vertices and $m$ arcs, does it have a subset of at least $m/2 + k$ arcs that induces an acyclic subgraph?

In the first question, a "more correct" lower bound is the one of Poljak and Turzík, i.e., $\frac{m}{2} + \frac{1}{2}\frac{(n-1)}{2}$, and the lower bound is true only for connected graphs. Without the connectivity requirement, the problem is NP-hard already for $k = 0$: The reduction is from the NP-hard problem of finding if a connected oriented graph has an acyclic subgraph with $\frac{m}{2} + \frac{1}{2}\frac{(n-1)}{2} + k$ arcs, and consists of adding $4k$ disjoint oriented 3-cycles. Corollary 32 answers the corrected question. Crowston et al. take up this problem in their recent independent work [4], a version of which appears as another article in these proceedings. In addition to showing that the problem is FPT, they also derive an $O(k^2)$ kernel for the problem.

For the second question, observe that each maximal acyclic subgraph contains exactly one arc from every pair of opposite arcs. Hence we can remove these pairs from the digraph without changing the relative solution size, as exactly half of the removed arcs can be added to any solution to the modified instance. Thus, we can restrict ourselves to oriented graphs.

Now suppose that the oriented graph which we have as input is disconnected. It is easy to check that picking two vertices from different connected components and identifying them does not change the solution size, as this way we never create a cycle from an acyclic graph. After applying this reduction rule exhaustively, the digraph becomes an oriented connected graph, and the parameter is unchanged. But then if $k \le (n-1)/4$ then $m/2 + k \le m/2 + (n-1)/4$ and we can answer YES due to Corollary 4. Otherwise $n \le 4k$, we have a linear vertex kernel, and we can solve the problem by the well known dynamic programming on the kernel [22]. The total running time of this algorithm is $O(2^{4k} \cdot k^2 + m)$. The smallest kernel previously known for this problem is by Gutin et al., and has a quadratic number of arcs [14].

## 5 Conclusion and Open Problems

In this paper we studied a generalization of the graph property of being bipartite, from the point of view of parameterized algorithms. We showed that for every strongly $\lambda$-extendible property $\Pi$ which satisfies an additional "solvability" constraint, the ABOVE POLJAK-TURZÍK ($\Pi$) problem is FPT. As an illustration of the usefulness of this result, we obtained FPT algorithms for the above-guarantee versions of three graph problems.

Note that for each of the three problems—MAX-CUT, MAX $q$-COLORABLE SUBGRAPH, and ORIENTED MAX ACYCLIC DIGRAPH—for which we used Theorem 1 to derive FPT algorithms for the above-guarantee question, we needed to devise a separate FPT algorithm which works for graphs that are at a vertex deletion distance of $O(k)$ from forests of cliques. We leave open the important question of finding a right logic that captures these examples, and of showing that any problem expressible in this logic is FPT parameterized by deletion distance to forests of cliques. We also leave open the kernelization complexity question for $\lambda$-extendible properties.

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