Unification Modulo Nonnested Recursion Schemes via Anchored Semi-Unification

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Abstract

A recursion scheme is an orthogonal rewriting system with rules of the form $f(x_1, \ldots, x_n) \to s$. We consider terms to be equivalent if they rewrite to the same redex-free possibly infinite term after infinitary rewriting. For the restriction to the nonnested case, where nested redexes are forbidden, we prove the existence of principal unifiers modulo scheme equivalence. We give an algorithm computing principal unifiers by reducing the problem to a novel fragment of semi-unification we call anchored semi-unification. For anchored semi-unification, we develop a decision algorithm that returns a principal semi-unifier in the positive case.

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1 Introduction

Recursion schemes (in the following shortly called schemes) describe mutually recursive function definitions and go back to the 1970s [18, 2]. They can be understood as rewriting systems with rules of the form $f(x_1, \ldots, x_n) \to s$ such that there is exactly one rule per function symbol f and s is not a redex. Starting with a finite term, the limit of repeatedly rewriting with these rules is a possibly infinite term containing no redexes, see Dershowitz et al. [3]. We consider two terms to be equivalent if they rewrite to the same redex-free term. This equivalence is known as tree equivalence (introduced by Rosen [18]). Given a scheme $\mathcal{S}$, we will speak of $\mathcal{S}$-equivalence in the following. As shown by Courcelle [2], $\mathcal{S}$-equivalence is interreducible to the equivalence of deterministic pushdown automata (DPDA). DPDA equivalence was an open problem for 20 years and was finally shown to be decidable by Sénizergues [21]. Sénizergues' proof yields a non-elementary decision procedure. The best known upper bound for the complexity of the problem is primitive recursive (see Stirling [22]).

In this paper, we will consider schemes without nested redexes, which we call nonnested schemes following Courcelle [2]. Sabelfeld [19] gives a polynomial decision procedure for $\mathcal{S}$-equivalence where $\mathcal{S}$ is a nonnested scheme.

Our motivation to investigate nonnested schemes is compiler verification. Nonnested schemes suffice to encode control flow graphs (CFGs) where everything but register updates and jumps is left uninterpreted. See Figure 1 for an example of a CFG together with a corresponding scheme. If two CFGs result in equivalent recursion schemes, then they are observationally equivalent. So if a compiler optimization produces a transformed CFG whose recursion scheme is equivalent to the recursion scheme of the original CFG, then this optimization is sound. Thus in a verified compiler, a compilation phase that produces a scheme-equivalent CFG can be validated by running a recursion scheme equivalence checker.

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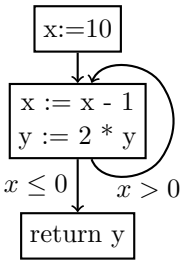


$$\begin{array}{lcl} f_1(x,y) & \to & f_2(10,y) \\ f_2(x,y) & \to & \text{if}(x-1>0, f_2(x-1,2*y), f_3(x-1,2*y)) \\ f_3(x,y) & \to & \text{return}(y) \end{array}$$

Figure 1 CFG with corresponding scheme.

We expect that compilation phases like global value numbering, code motion for assignments and register allocation (except for spilling) can be validated with this method.

We solve a more general problem than $\mathcal{S}$-equivalence, namely unification modulo $\mathcal{S}$-equivalence. This relies on our result that $\mathcal{S}$-unification is unitary (i.e., principal $\mathcal{S}$-unifiers exist). Given two terms s and t, we show how to compute a principal $\mathcal{S}$-unifier of s and t if one exists. For our results, we restrict substitutions such that variables are replaced with redex-free terms.

Our method is a certifying algorithm [15] in the following sense. Whenever we compute a principal substitution σ such that σs and σt are $\mathcal{S}$-equivalent, we also have a certificate proving that σs and σt are $\mathcal{S}$-equivalent.

Our method works by reducing the unification problem to a new decidable fragment of semi-unification we call *anchored semi-unification*. Semi-unification is a generalization of ordinary syntactic unification. Semi-unification was first identified by Lankford and Musser [10] in 1978 and rediscovered in different areas ten years later [5, 7, 17]. In its general form, semi-unification was shown to be undecidable by Kfoury et al. [8]. Several decidable fragments of semi-unification are known, but all of them differ significantly from our new fragment (see Section 11). Our semi-unification algorithm can be seen as a restriction of the diverging semi-unification rules of Leiß [12].

2 Overview

Our method is based on a coinductive definition of $\mathcal{S}$-equivalence. We seem to be the first to employ a coinductive definition of $\mathcal{S}$-equivalence. According to our definition, two terms s and t are $\mathcal{S}$-equivalent (i.e., $s \equiv t$) if there is a compliant relation relating s and t. Compared with process equivalence, compliant relations play the role of bisimulations. Our first main result is the existence of principal $\mathcal{S}$-unifiers. A substitution σ is a principal $\mathcal{S}$-unifier of two terms s and t if it is a principal (aka most general) substitution such that $\sigma s \equiv \sigma t$. From the existence of principal $\mathcal{S}$-unifiers, it follows that there are principal pairs. A pair of terms $(f\overline{s}, g\overline{t})$ is a principal pair for the function symbols f and g if for all tuples of terms $\overline{u}$ and $\overline{v}$, we have $f\overline{u} \equiv g\overline{v}$ iff $(f\overline{u}, g\overline{v})$ is a substitution instance of $(f\overline{s}, g\overline{t})$.

By weakening the conditions on compliant relations, we obtain a decidable criterion for finite relations that implies $\mathcal{S}$-equivalence for the pairs in the relation (and their substitution instances). We call finite relations that satisfy this criterion certificates. It turns out that every finite set of principal pairs can be extended to a certificate by adding more principal pairs. Thus there are certificates for all $\mathcal{S}$-equivalences between redexes. On the other hand, a principal certificate contains only principal pairs. A principal certificate is a relation σF where F is a frame and σ is a principal substitution such that σF is a certificate. A frame is roughly a relation on terms of the form $f\overline{x}$ that does not contain a variable twice. We will

show that if there is a principal pair for f and g, then it is possible to compute a frame F and a substitution σ such that σF is a principal certificate containing a principal pair for f and g.

We solve the $\mathcal{S}$-unification problem (ScUP) by reducing it to the anchored semi-unification problem (AnSUP). This is done using three reduction steps.

$$\text{ScUP} \to \text{FIP} \to \text{SUP}^* \to \text{AnSUP}$$

ScUP is the initial problem, asking for a principal $\mathcal{S}$-unifier of two terms $f\overline{s}$ and $g\overline{t}$. To find an $\mathcal{S}$-unifier of $f\overline{s}$ and $g\overline{t}$, it suffices to determine a principal pair for f and g. If there is no principal pair for f and g, then $f\overline{s}$ and $f\overline{g}$ are not $\mathcal{S}$-unifiable. To find a principal pair for f and g, we compute a frame F for f and g such that F can be instantiated to a principal certificate for f and g iff there exists a principal pair for f and g. We call the new problem frame instantiation problem (FIP).

We reduce FIP to the standard semi-unification problem SUP. Since SUP is undecidable in general, we employ a further reduction to the anchored semi-unification problem AnSUP. We show that every instance of SUP obtained by our reduction from FIP translates to an instance of AnSUP. In the diagram above, SUP^* indicates the fragment of SUP reachable by our reduction from FIP. While SUP employs inequalities $s \dot{\preceq} t$, AnSUP employs equations $s \doteq t$ where s and t can contain instance variables αx consisting of a simple variable x and a substitution variable α that represents a substitution. The anchoredness constraint ensures that for every relevant instance variable αx, there is an equation $\alpha x \doteq s$ where s contains no instance variables. We solve instances of AnSUP with terminating semi-unification rules that are a restriction of the rules in [12]. The anchoredness constraint is preserved by applications of the semi-unification rules and allows us to always eliminate instance variables.

The paper is organized as follows. First, we define nonnested schemes and $\mathcal{S}$-equivalence. Then we formulate ScUP and prove the existence of principal $\mathcal{S}$-unifiers and principal pairs. Afterwards, we define FIP with frames and certificates and present the reductions from ScUP to FIP and from FIP to SUP. Next, we define AnSUP and present the reduction from FIP to AnSUP. Finally, we define a solved form and present terminating semi-unification rules for AnSUP. The sections on AnSUP and the semi-unification rules can be read independently of the rest of the paper.

3 Equivalence Modulo Nonnested Schemes

We assume an alphabet of *constants* (ranged over by a, b, c), an alphabet of *function symbols* (ranged over by f, g, h) and an alphabet of *variables* (ranged over by x, y, z). We assume that there are infinitely many variables and finitely many function symbols. We define *terms* (ranged over by s, t, u, v) using the grammar

$$s, t ::= a \mid x \mid s \cdot t \mid f\overline{s}$$

where $\overline{s}$ is a tuple of terms not containing a term of the form $f\overline{t}$.

Note that although we restrict ourselves to a single binary operator $(\cdot)$, we do not loose expressive power compared to full first-order terms since they can be encoded, for example with $a(s_1, s_2, \ldots, s_n) \rightsquigarrow (\cdots((a \cdot s_1) \cdot s_2) \cdot \cdots \cdot s_n)$. We impose the usual discipline that every function symbol has a *fixed arity* and that for $f\overline{s}$, the length of $\overline{s}$ is the arity of f. We omit parentheses such that $s \cdot t \cdot u = s \cdot (t \cdot u)$.

Since we will have a rewrite rule for every function symbol, we call every term of the form $f\overline{s}$ a *redex*. A term is *simple* if it contains no redex. A term is *plain* if it is simple or a redex.

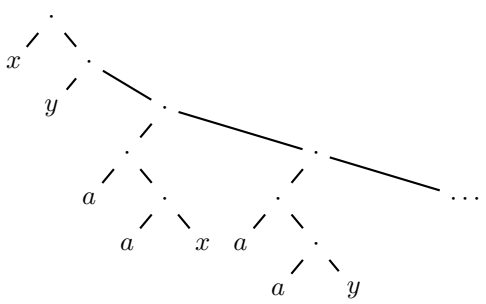
A *substitution* (ranged over by σ, τ, θ) is a function from variables to simple terms. Note that this is a nonstandard restriction and affects our results for unification modulo nonnested schemes. We lift substitutions to range over terms, tuples and sets in the usual way.

A *declaration* of f is a rewrite rule $f\overline{x} \to s_1 \cdot s_2$ with distinct variables $\overline{x}$ and terms s_1, s_2 containing only variables from $\overline{x}$. A *nonnested scheme* (in the following shortly *scheme*) $\mathcal{S}$ is a set of declarations that contains exactly one declaration for every function symbol. We assume that a scheme $\mathcal{S}$ is given. For technical reasons, we require that $\mathcal{S}$ is *reduced*, that is, for every declaration $f\overline{x} \to s_1 \cdot s_2$, both s_1 and s_2 are plain and at least one of them is a redex. Sabelfeld [19] uses a similar restriction of the same name. Given a redex $f\overline{s}$, its *unfolding* $\mathcal{S}(f\overline{s})$ is the term σt where $f\overline{x} \to t$ is the unique declaration of f in $\mathcal{S}$ and σ is a substitution with $\sigma\overline{x} = \overline{s}$. We write $\mathcal{S}_i s$ for t_i where $\mathcal{S}s = t_1 \cdot t_2$ and $i \in \{1, 2\}$.

▶ **Example 1.** For the reduced scheme

$$f(x, y) \to x \cdot g(a \cdot x, y)$$
$$g(x, y) \to y \cdot f(a \cdot x, a \cdot a \cdot y)$$

infinitary rewriting of the redex $f(x, y)$ results in the infinite tree



General Convention. From now on until Section 7, s, t, u and v will always denote plain terms and R will always denote a binary relation on plain terms.

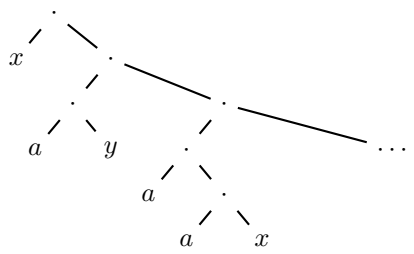
We now give a coinductive definition of $\mathcal{S}$-equivalence. We call a relation R on plain terms *closed* if $R(\mathcal{S}_i s)(\mathcal{S}_i t)$ for all pairs of redexes $(s, t) \in R$ and $i \in \{1, 2\}$. The *kernel* of a relation R is $\mathcal{K}R := \{(s, t) \in R \mid s \text{ or } t \text{ simple}\}$. A relation R is *coreflexive* if $s = t$ whenever Rst. We call a relation R *compliant* if it is closed and $\mathcal{K}R$ is coreflexive. Two plain terms s and t are *$\mathcal{S}$-equivalent* iff there is a compliant relation R with Rst. Since $\mathcal{S}$ is fixed, we simply write $s \equiv t$ to say that s and t are $\mathcal{S}$-equivalent and use $(\equiv)$ to denote the set of all $\mathcal{S}$-equivalent pairs. Note that a compliant relation is essentially a bisimulation between the trees generated by infinitary rewriting.

▶ **Example 2.** For the scheme from Example 1,

$$\{(f(x, a \cdot y), g(y, x)),$$
$$(x, x),\ (g(a \cdot x, a \cdot y), f(a \cdot y, a \cdot a \cdot x)),$$
$$(a \cdot y, a \cdot y),\ (f(a \cdot a \cdot x, a \cdot a \cdot a \cdot y), g(a \cdot a \cdot y, a \cdot a \cdot x)), \ldots\}$$

is a compliant relation containing $(f(x, a \cdot y), g(y, x))$. Observe how this relation corresponds to a walk through the following tree, which can be obtained by infinitary rewriting of either

$f(x, a \cdot y)$ or $g(y, x)$.



▶ **Proposition 3.** *$\mathcal{S}$-equivalence ($\equiv$) is an equivalence relation on plain terms. Moreover, it is the largest compliant relation.*

Proof. Reflexivity follows from the fact that $\{(s, s) \mid s \text{ plain}\}$ is compliant. The other properties follow from the fact that the inverse, union, intersection and composition of compliant relations are compliant. ◀

Our coinductive definition of $\mathcal{S}$-equivalence is equivalent to the usual approach using infinite trees. For every term s, infinitary rewriting with the rules of the scheme yields a possibly infinite redex-free term that can be seen as a possibly infinite binary tree $\mathcal{T}s$ whose internal nodes are labeled with $\cdot$ and whose leaves are labeled with constants and variables. Two terms s and t are tree-equivalent if $\mathcal{T}s = \mathcal{T}t$. We write $(\mathcal{T}s) \cdot (\mathcal{T}t)$ for the tree whose root has exactly $\mathcal{T}s$ and $\mathcal{T}t$ as children. For a formal definition of tree equivalence, see e.g. Courcelle [2]. We will not define $\mathcal{T}s$, but instead we use the following two properties.

1. $\mathcal{T}(s \cdot t) = (\mathcal{T}s) \cdot (\mathcal{T}t)$
2. $\mathcal{T}s = (\mathcal{T}(\mathcal{S}_1 s)) \cdot (\mathcal{T}(\mathcal{S}_2 s))$ if s is a redex

In the following, we write $(\sim_{\mathcal{T}})$ for the relation on plain terms such that $s \sim_{\mathcal{T}} t$ iff $\mathcal{T}s = \mathcal{T}t$. Using property (1.), it is clear that for simple terms s and t, we have $s \sim_{\mathcal{T}} t$ iff $s = t$. For a simple term s, $\mathcal{T}s$ is finite. In contrast to this, $\mathcal{T}s$ is infinite if s is a redex because $\mathcal{S}$ is reduced. Taking these properties together, we obtain that $\mathcal{K}(\sim_{\mathcal{T}})$ is coreflexive. We also have that $(\sim_{\mathcal{T}})$ is closed because if $s \sim_{\mathcal{T}} t$ for redexes s and t, then, by property (2.), $(\mathcal{T}(\mathcal{S}_1 s)) \cdot (\mathcal{T}(\mathcal{S}_2 s)) = (\mathcal{T}(\mathcal{S}_1 t)) \cdot (\mathcal{T}(\mathcal{S}_2 t))$ and hence $\mathcal{S}_i s \sim_{\mathcal{T}} \mathcal{S}_i t$ for $i \in \{1, 2\}$. Thus $(\sim_{\mathcal{T}})$ is a compliant relation and therefore $(\sim_{\mathcal{T}}) \subseteq (\equiv)$.

It remains to show that $(\sim_{\mathcal{T}}) \supseteq (\equiv)$. Assume, for contradiction, that there are plain terms s and t with $s \equiv t$ that are not tree-equivalent. When two trees are different, then they differ at some finite level. Select s and t such that the level l at which $\mathcal{T}s$ and $\mathcal{T}t$ differ is minimal. Since $s \equiv t$, we have that s and t are both redexes because otherwise $s = t$. Since $\mathcal{T}s = (\mathcal{T}(\mathcal{S}_1 s)) \cdot (\mathcal{T}(\mathcal{S}_2 s))$ and $\mathcal{T}t = (\mathcal{T}(\mathcal{S}_1 t)) \cdot (\mathcal{T}(\mathcal{S}_2 t))$ differ at level l, there is $i \in \{1, 2\}$ such that $\mathcal{T}(\mathcal{S}_i s)$ and $\mathcal{T}(\mathcal{S}_i t)$ differ at level $l - 1$. So we have a contradiction because $\mathcal{S}_i s \equiv \mathcal{S}_i t$.

▶ Remark. We can restrict ourselves to plain terms because other terms can be transformed into plain terms by adding additional declarations to the scheme.

Also, the restriction to reduced schemes is inessential because every scheme can be transformed into an equivalent reduced one.

- A declaration $f\overline{x} \to s$ with s simple can be eliminated by replacing every redex $f\overline{t}$ in $\mathcal{S}$ with the simple term $\mathcal{S}(f\overline{t})$.
- A declaration with deep redexes can be split into several declarations, e.g.

$$f(x) \to a \cdot g() \cdot x \quad \rightsquigarrow \quad \begin{array}{l} f(x) \to a \cdot f'(x) \\ f'(x) \to g() \cdot x \end{array}$$

4 Scheme Unification Problem (ScUP)

A substitution σ is an *$\mathcal{S}$-unifier* of two terms s and t if $\sigma s \equiv \sigma t$. We define the usual *instantiation pre-order* ($\preceq$) on tuples of terms and substitutions. For two tuples of terms $\overline{s}$ and $\overline{t}$, we have $\overline{s} \preceq \overline{t}$ if $\sigma\overline{s} = \overline{t}$ for some substitution σ. For two substitutions σ and τ, we have $\sigma \preceq \tau$ if $\tau = \theta \circ \sigma$ for some substitution θ (as usual, $(\theta \circ \sigma)(x) := \theta(\sigma x)$). We call a set Σ of substitutions *quasi-principal* if Σ is either empty or contains a substitution σ such that $\Sigma = \{\tau \mid \sigma \preceq \tau\}$. In this case we call σ a *principal element* of Σ. A *principal $\mathcal{S}$-unifier* of s and t is a principal element of $\{\sigma \mid \sigma s \equiv \sigma t\}$.

▶ **Problem 1** (ScUP)**.** Given two plain terms s and t, find a principal $\mathcal{S}$-unifier of s and t if one exists.

In the remainder of this section, we prove that two terms have a principal $\mathcal{S}$-unifier if they are $\mathcal{S}$-unifiable. The principal $\mathcal{S}$-unifiers will turn out to be essential for our reduction, because they allow us to characterize all $\mathcal{S}$-equivalences between terms with the finite set of principal $\mathcal{S}$-unifiers between terms of the form $f\overline{x}$.

The *closure* $\mathcal{C}(s,t)$ of s and t is the smallest closed relation containing (s,t).

▶ **Proposition 4.** *$s \equiv t$ iff $\mathcal{K}(\mathcal{C}(s,t))$ is coreflexive.*

Proof. If $s \equiv t$, then there is a compliant relation R containing (s,t). Thus $\mathcal{C}(s,t) \subseteq R$. Hence $\mathcal{K}(\mathcal{C}(s,t)) \subseteq \mathcal{K}R$ is coreflexive.

If $\mathcal{K}(\mathcal{C}(s,t))$ is coreflexive, then $\mathcal{C}(s,t)$ is a compliant relation containing (s,t). Thus $s \equiv t$. ◀

▶ **Proposition 5.** *For every redex s, we have $\mathcal{S}(\sigma s) = \sigma(\mathcal{S}s)$.*

▶ **Lemma 6.** $\mathcal{C}(\sigma s, \sigma t) = \sigma\,\mathcal{C}(s,t)$

Proof. Using Proposition 5, we obtain that $\sigma\,\mathcal{C}(s,t)$ is closed. Since also $(\sigma\,\mathcal{C}(s,t))(\sigma s)(\sigma t)$, we have $\mathcal{C}(\sigma s, \sigma t) \subseteq \sigma\,\mathcal{C}(s,t)$. It remains to show that $\mathcal{C}(\sigma s, \sigma t) \supseteq \sigma\,\mathcal{C}(s,t)$. This follows from a simple induction on the derivations of the elements in $\mathcal{C}(s,t)$. ◀

The following well-known result generalizes the existence of principal unifiers for ordinary unification to infinite systems of equations. Note that σR is coreflexive iff σ is an (ordinary) unifier of R.

▶ **Proposition 7.** *For a relation R on terms with finitely many variables, $\{\sigma \mid \sigma R$ coreflexive$\}$ is quasi-principal.*

Proof. See Proposition 4.10 in Eder [4]. ◀

▶ **Theorem 8.** *$\{\sigma \mid \sigma s \equiv \sigma t\}$ is quasi-principal.*

Proof. We have

$$
\begin{aligned}
&\{\sigma \mid \sigma s \equiv \sigma t\} \\
&= \{\sigma \mid \mathcal{K}(\mathcal{C}(\sigma s, \sigma t)) \text{ coreflexive}\} && \text{by Proposition 4} \\
&= \{\sigma \mid \mathcal{K}(\sigma\,\mathcal{C}(s,t)) \text{ coreflexive}\} && \text{by Lemma 6} \\
&= \{\sigma \mid \sigma(\mathcal{K}(\mathcal{C}(s,t))) \text{ coreflexive}\}
\end{aligned}
$$

and $\{\sigma \mid \sigma(\mathcal{K}(\mathcal{C}(s,t))) \text{ coreflexive}\}$ is quasi-principal by Proposition 7. ◀

For a relation R on plain terms, we define $\uparrow R := \{(s',t') \mid \exists (s,t) \in R.\ (s,t) \preceq (s',t')\}$. Given function symbols f and g, we call $(f\overline{s}, g\overline{t})$ a *principal pair* for (f,g) if $\uparrow\{(f\overline{s}, g\overline{t})\} = \{(f\overline{u}, g\overline{v}) \mid f\overline{u} \equiv g\overline{v}\}$. Note that a principal pair for f and g completely characterizes $\mathcal{S}$-equivalence for terms of the form $f\overline{s}$ and $g\overline{t}$.

▶ **Corollary 9.** *If σ is a principal $\mathcal{S}$-unifier of $f\overline{x}$ and $g\overline{y}$, and $\overline{x}, \overline{y}$ are pairwise distinct variables, then $\sigma(f\overline{x}, g\overline{y})$ is a principal pair.*

5 Frame Instantiation Problem (FIP)

In this section, we introduce the frame instantiation problem, which will allow us to compute principal pairs. In Section 6, we will then show how to construct a frame that can be instantiated to a certificate consisting of principal pairs only.

For a relation R on redexes, we define $\Uparrow R := \uparrow R \cup \{(s,s) \mid s \text{ simple}\}$. A finite relation R on redexes is a *certificate* if $\Uparrow R(\mathcal{S}_i s)(\mathcal{S}_i t)$ for all redexes $(s,t) \in R$ and $i \in \{1,2\}$. Note that a certificate is essentially a bisimulation up-to instantiation in the sense of up-to techniques for bisimulations [20].

▶ **Example 10.** For the scheme from Example 1,

$$R := \{(f(x, a \cdot y), g(y,x)),\ (g(y,x), f(x, a \cdot y))\}$$

is a certificate. Observe that $\Uparrow R$ is a superset of the compliant relation from Example 2. Also note that R consists of two principal pairs.

▶ **Lemma 11.** *Let R be a certificate. Then $\Uparrow R \subseteq (\equiv)$.*

Proof. It suffices to show that $\Uparrow R$ is compliant. We have that $\mathcal{K}\Uparrow R$ is coreflexive because $\mathcal{K}\Uparrow R = \Uparrow(\mathcal{K}R) = \Uparrow\{\}$. It remains to show that $\Uparrow R$ is closed. Consider a pair of redexes $(s,t) \in \Uparrow R$. We have to show that $\Uparrow R(\mathcal{S}_i s)(\mathcal{S}_i t)$ for $i \in \{1,2\}$. By the definition of $\Uparrow R$, there are redexes $(u,v) \in R$ with $(u,v) \preceq (s,t)$. Since R is a certificate, we have $\Uparrow R(\mathcal{S}_i u)(\mathcal{S}_i v)$. As $(\mathcal{S}_i u, \mathcal{S}_i v) \preceq (\mathcal{S}_i s, \mathcal{S}_i t)$ by Proposition 5, it follows that $\Uparrow R(\mathcal{S}_i s)(\mathcal{S}_i t)$. ◀

We write $s \approx t$ if s and t are redexes with the same function symbol, and $(s,t) \approx (u,v)$ if $s \approx u$ and $t \approx v$. We call a relation R on redexes *unitary* if there are no distinct pairs $(s_1,s_2), (t_1,t_2) \in R$ with $(s_1,s_2) \approx (t_1,t_2)$. Note that every unitary relation is finite because there are only finitely many function symbols. We call a unitary relation F on redexes of the form $f\overline{x}$ a *frame* if no variable occurs twice in F.

▶ **Example 12.** Consider the frame $F := \{(f(x_1,x_2), g(x_3,x_4)),\ (g(x_5,x_6), f(x_7,x_8))\}$. There is a substitution σ such that σF is the certificate from Example 10. Moreover, σ is principal with this property if it is the identity on all variables not occurring in F.

▶ **Problem 2** (FIP)**.** Given a frame F, find a principal element of $\{\sigma \mid \sigma F \text{ is certificate}\}$ if it exists.

It is easy to decide if a finite relation is a certificate. Thus a certificate proves that its elements are $\mathcal{S}$-equivalent. So for every solution σ of a frame F, we obtain a certificate (in the sense of a certifying algorithm [15]) for the fact that all pairs of terms in σF are $\mathcal{S}$-equivalent.

6 ScUP to FIP

To solve ScUP, it suffices to compute principal pairs. Suppose we want to compute a principal $\mathcal{S}$-unifier of two plain terms s and t. We distinguish three cases.

1. If s and t are both redexes, then consider a principal pair (s', t') with $(s', t') \approx (s, t)$ and fresh variables. We obtain an ordinary unification problem because
$$\{\sigma \mid \sigma s \equiv \sigma t\} = \{\sigma \mid (s', t') \preceq \sigma(s, t)\} = \{\sigma \mid \exists \tau.\ \tau(s', t') = \sigma(s, t)\}$$
2. If s and t are both simple, then we are about to solve an ordinary unification problem since for simple terms, $\mathcal{S}$-equivalence is (syntactic) equality.
3. Otherwise, one of them is a redex and the other one is a simple term and hence they cannot be $\mathcal{S}$-unifiable.

So in the following, we only consider the first case and construct a frame F such that for its principal solution σ, a suitable principle pair is contained in σF.

We call a relation R on redexes *pointwise $\mathcal{S}$-unifiable* if s and t are $\mathcal{S}$-unifiable whenever Rst. We call a relation R on redexes *function-closed* if $\Uparrow R(\mathcal{S}_i s)(\mathcal{S}_i t)$ whenever Rst, $i \in \{1, 2\}$ and $\mathcal{S}_i s$ and $\mathcal{S}_i t$ are both redexes. For example, the frame from Example 12 is pointwise $\mathcal{S}$-unifiable and function-closed.

▶ **Lemma 13.** *For every function-closed and pointwise $\mathcal{S}$-unifiable frame F, there is a substitution σ such that σF is a certificate containing only principal pairs.*

Proof. For $(s, t) \in F$, let $\sigma_{s,t}$ be a principal $\mathcal{S}$-unifier of s and t. These $\mathcal{S}$-unifiers exist because F is pointwise $\mathcal{S}$-unifiable. Since the elements of F have disjoint variables, the following substitution is well-defined.

$$\tau x = \begin{cases} \sigma_{s,t} x & \text{if there is } (s, t) \in F \text{ s.t. } x \text{ occurs in } (s, t) \\ x & \text{otherwise} \end{cases}$$

For all $(s, t) \in F$, we have that $\sigma_{s,t}(s, t) = \tau(s, t)$ and hence $\tau(s, t)$ is a principal pair by Corollary 9. Thus τF consists only of principal pairs and $\Uparrow(\tau F) = (\equiv) \cap \Uparrow F$.

It remains to show that τF is a certificate. Consider a pair of redexes $(s, t) \in \tau F$. We have to show that $\Uparrow(\tau F)(\mathcal{S}_i s)(\mathcal{S}_i t)$ for $i \in \{1, 2\}$. There are $(s', t') \in F$ with $\tau(s', t') = (s, t)$. We have that $s \equiv t$ and hence $\mathcal{S}_i s \equiv \mathcal{S}_i t$. By Proposition 5, $\tau(\mathcal{S}_i s') = \mathcal{S}_i s \equiv \mathcal{S}_i t = \tau(\mathcal{S}_i t')$. Thus either both $\mathcal{S}_i s'$ and $\mathcal{S}_i t'$ are redexes or both are simple. Hence and because F is function-closed, we have $\Uparrow F(\mathcal{S}_i s')(\mathcal{S}_i t')$. Thus also $\Uparrow F(\tau(\mathcal{S}_i s'))(\tau(\mathcal{S}_i t'))$ and equivalently $\Uparrow F(\mathcal{S}_i s)(\mathcal{S}_i t)$. Since $\Uparrow(\tau F) = (\equiv) \cap \Uparrow F$, we conclude that $\Uparrow(\tau F)(\mathcal{S}_i s)(\mathcal{S}_i t)$. ◀

However, we are still missing a way to construct an appropriate function-closed and pointwise $\mathcal{S}$-unifiable frame. It turns out that we get pointwise $\mathcal{S}$-unifiability for free if we just make the frame as small as we can.

Given function symbols f and g, we write $\mathcal{F}(f, g)$ for a fixed function-closed frame with minimum cardinality containing a pair $(f\overline{x}, g\overline{y})$ for some pairwise distinct variables $\overline{x}, \overline{y}$. Since every maximum cardinality frame is function-closed, $\mathcal{F}(f, g)$ always exists. For example, we can take the frame from Example 12 for $\mathcal{F}(f, g)$ where f and g are the function symbols from Example 1. One can easily compute $\mathcal{F}(f, g)$ by incrementally adding elements $(h\overline{x}, h'\overline{y})$ with fresh and distinct variables $\overline{x}, \overline{y}$ as required by the function-closedness constraint.

▶ **Lemma 14.** *If $f\overline{s}$ and $g\overline{t}$ are $\mathcal{S}$-unifiable, then $\mathcal{F}(f, g)$ is pointwise $\mathcal{S}$-unifiable.*

Proof. Suppose, for contradiction, that $F := \{(u,v) \in \mathcal{F}(f,g) \mid u, v\ \mathcal{S}\text{-unifiable}\} \neq \mathcal{F}(f,g)$. Then F has strictly smaller cardinality. We have $\Uparrow F(f\overline{s})(g\overline{t})$ since $f\overline{s}$ and $g\overline{t}$ are $\mathcal{S}$-unifiable. Thus we can conclude a contradiction by showing that F is also function-closed. Let $(u,v) \in F$ and $i \in \{1,2\}$ such that $\mathcal{S}_i u$ and $\mathcal{S}_i v$ are both redexes. We need to show that $\Uparrow F(\mathcal{S}_i u)(\mathcal{S}_i v)$. Since $\mathcal{F}(f,g)$ is function-closed, we have $(\Uparrow \mathcal{F}(f,g))(\mathcal{S}_i u)(\mathcal{S}_i v)$. Select a substitution σ with $\sigma u \equiv \sigma v$. By Proposition 5, we have $\sigma(\mathcal{S}_i u) = \mathcal{S}_i(\sigma u) \equiv \mathcal{S}_i(\sigma v) = \sigma(\mathcal{S}_i v)$. Thus $\mathcal{S}_i u$ and $\mathcal{S}_i v$ are $\mathcal{S}$-unifiable and hence $\Uparrow F(\mathcal{S}_i u)(\mathcal{S}_i v)$. ◀

▶ **Corollary 15.** *If $f\overline{s}$ and $g\overline{t}$ are $\mathcal{S}$-unifiable, then $\{\sigma \mid \sigma\,\mathcal{F}(f,g)$ is certificate$\}$ is nonempty and for every principal element σ, we have that $\sigma\,\mathcal{F}(f,g)$ contains a principal pair for (f,g).*

We will reduce FIP to AnSUP in a way that a frame F is mapped to an anchored system of equations E with $\{\sigma \mid \sigma F$ is certificate$\} = \{\sigma \mid \sigma$ semi-unifies $E\}$ and we will prove that $\{\sigma \mid \sigma$ semi-unifies $E\}$ is quasi-principal. Thus if $f\overline{s}$ and $g\overline{t}$ are $\mathcal{S}$-unifiable, then $\{\sigma \mid \sigma\,\mathcal{F}(f,g)$ is certificate$\}$ contains a principal element σ and $\sigma\,\mathcal{F}(f,g)$ contains a principal pair for (f,g). Otherwise $\{\sigma \mid \sigma\,\mathcal{F}(f,g)$ is certificate$\}$ is empty. This concludes the reduction from ScUP to FIP.

7 FIP to SUP

In this section, we define the semi-unification problem (SUP) using systems of inequalities and give a reduction from FIP to SUP.

Let D be a relation on tuples of plain terms. We write the elements $(\overline{s},\overline{t}) \in D$ as *inequalities* $\overline{s} \dot{\preceq} \overline{t}$ and call D a *system of inequalities*. We call D *directed* if $\overline{s} \preceq \overline{t}$ for all $(\overline{s} \dot{\preceq} \overline{t}) \in D$. If σD is directed, then σ is a *semi-unifier* of D.

▶ **Problem 3** (SUP)**.** Given a system of inequalities D, find a principal semi-unifier of D.

In this section, we will show how to construct a system of inequalities with the same principal solution as a given instance of FIP.

Note that our definition of SUP corresponds roughly to the usual definition of the semi-unification problem (e.g. in Henglein [5]). Since semi-unification is undecidable, we cannot hope to solve the problem in general. Instead, we only consider the image of the reduction from FIP. In Section 9, we will show how to reduce this image to the anchored fragment.

For a unitary relation R, we define the *projection* of a pair (s,t) of plain terms as follows.

$$\pi_R(s,t) := \begin{cases} (s',t') & \text{if } Rs't' \text{ and } (s',t') \approx (s,t) \\ (s,s) & \text{if } s,t \text{ simple} \\ () & \text{otherwise} \end{cases}$$

Note that $\pi_R(s,t)$ is well-defined because R is unitary. The following proposition holds due to the fact that $(s,s) \preceq (s,t)$ iff $s = t$.

▶ **Proposition 16.** *Let R be unitary. Then $\Uparrow Rst$ iff $\pi_R(s,t) \preceq (s,t)$.*

For a unitary relation R, we construct the system of inequalities

$$\mathcal{D}\,R := \{\pi_R(\mathcal{S}_i s, \mathcal{S}_i t) \dot{\preceq} (\mathcal{S}_i s, \mathcal{S}_i t) \mid Rst, i \in \{1,2\}\}$$

▶ **Example 17.** For the frame F from Example 12, we construct the system of inequalities

$$\begin{aligned}\mathcal{D}\,F = \{ \quad & (x_1, x_1) \dot{\preceq} (x_1, x_4),\\ & (g(x_5, x_6), f(x_7, x_8)) \dot{\preceq} (g(a \cdot x_1, x_2), f(a \cdot x_3, a \cdot a \cdot x_4)),\\ & (x_6, x_6) \dot{\preceq} (x_6, x_7),\\ & (f(x_1, x_2), g(x_3, x_4)) \dot{\preceq} (f(a \cdot x_5, a \cdot a \cdot x_6), g(a \cdot x_7, x_8)) \quad \}\end{aligned}$$

▶ **Proposition 18.** *Let R be unitary. Then $\mathcal{D}\,R$ is directed iff R is a certificate.*

Proof. Follows from the construction of $\mathcal{D}\,R$ using Proposition 16. ◀

▶ **Proposition 19.** *Let R be unitary. Then $\sigma(\pi_R(s,t)) = \pi_{(\sigma R)}(\sigma s, \sigma t)$ and $\sigma(\mathcal{D}\,R) = \mathcal{D}(\sigma R)$.*

▶ **Lemma 20.** *Let R be unitary. Then σ semi-unifies $\mathcal{D}\,R$ iff σR is a certificate.*

Proof. Follows immediately from Proposition 18 and Proposition 19. ◀

Thus $\mathcal{D}$ is a reduction from FIP to SUP.

8 Anchored Semi-Unification Problem (AnSUP)

The definition as well as the algorithm for AnSUP rely on a formulation of semi-unification that uses systems of equations between terms with explicit substitution variables instead of inequalities. A similar formulation has been used by Leiß [12]. For example, the inequalities $(x_1, x_2) \dot{\preceq} (x_3, x_4)$ and $(x_5) \dot{\preceq} (x_6)$ corresponds to the equations $\alpha x_1 \doteq x_3$, $\alpha x_2 \doteq x_4$ and $\beta x_5 \doteq x_6$ with the substitution variables α and β. The substitution variables make the additional substitution on the left-hand side of inequalities explicit.

We call the variables we have used so far *simple variables* and assume an additional alphabet of *substitution variables* (ranged over by α, β, γ). An *instance variable* αx is a pair of a substitution variable and a simple variable. We define a new kind of *terms* that extend the simple terms we defined before with instance variables.

$$s, t ::= a \mid x \mid s \cdot t \mid \alpha x$$

Given a simple term s and a substitution variable α, we write $\hat{\alpha} s$ for the term obtained from s by replacing every variable x by αx. As before, substitutions are functions from variables to simple terms. In the semi-unification context [12, 5, 7], this is the standard notion of substitution. We lift substitutions to terms according to the equations $\sigma(\alpha x) = \hat{\alpha}(\sigma x)$, $\sigma(s \cdot t) = \sigma s \cdot \sigma t$ and $\sigma a = a$.

We say a term s *occurs* in a term t if s is a subterm of t. We do not consider x to be a subterm of αx.

An *assignment* A is a function from substitution variables to substitutions. Assignments are lifted to terms such that As is the term obtained from s by replacing every occurrence of an instance variable αx with the simple term $(A\alpha)x$.

A *system of equations* E is a finite set of pairs of terms. We take the freedom to write $s \doteq t$ for a pair (s, t). A substitution σ is a *semi-unifier* of E if there is an assignment A such that for all $s \doteq t \in E$, we have $A(\sigma s) = A(\sigma t)$.

The formulation of semi-unification as just presented has the same expressivity as the formulation with inequalities. We will now restrict the systems of equations we consider to obtain the anchored fragment.

An *atom* (ranged over by X, Y, Z) is either a simple variable or an instance variable. A *partial equivalence relation* (*PER*) is a symmetric and transitive relation. Note that a PER ($\sim$) is an equivalence relation on $\{X \mid X \sim X\}$.

The anchoredness condition ensures that there is an equation $\alpha x \doteq s$ with s simple for every instance variable αx that might occur when the semi-unification rules to be defined later are applied. A system of equations E is *anchored* with a PER ($\sim$) on atoms if the following conditions hold.

1. If $s \doteq t \in E$ with X and Y occurring in $s \doteq t$, then $X \sim Y$.
2. If $x \sim y$ and $\alpha x \sim \alpha x$, then $\alpha x \sim \alpha y$.
3. If $\alpha x \sim \alpha x$, then there is a simple term s with $\alpha x \doteq s \in E$ or $x \doteq s \in E$.

For example, $\{x \doteq z \cdot \beta y,\ \alpha x = a \cdot b\}$ is not anchored, but $\{x \doteq z \cdot \beta y,\ \alpha x = a \cdot b,\ \beta y = z,\ \alpha z = z\}$ is anchored with the symmetric and transitive closure of $\{(x, z), (z, \beta y), (\alpha x, \alpha z), (\alpha z, z)\}$.

▶ **Problem 4** (AnSUP)**.** Given an anchored system of equations E, find a principal semi-unifier.

For comparison to other fragments of semi-unification, we now give a definition of the anchoredness condition for systems of inequalities. A system of inequalities is anchored if there is a partition X of the variables such that for every inequality $\overline{s} \dot{\preceq} \overline{t}$, the following two conditions hold.

1. All variables in $\overline{t}$ are in a single class of the partition X.
2. All variables in $\overline{s}$ are in a single class A of the partition X and every variable in A occurs at least once at a position in $\overline{s}$ that also exists in $\overline{t}$. Expressed formally, there is a tuple $\overline{u}$ and substitutions σ, τ such that $\sigma\overline{u} = \overline{s}$, $\tau\overline{u} = \overline{t}$ and for all variables $x \in A$, we have that $\sigma x = x$ and x occurs in u.

9 FIP to AnSUP

Let F be a frame. We will translate the system of inequalities $\mathcal{D}\,F$ into an equivalent anchored system of equations E such that σ semi-unifies E iff σ semi-unifies $\mathcal{D}\,F$.

The case where $\mathcal{D}\,F$ contains an element of the form $() \dot{\preceq} (s, t)$ is trivial, since in this case, there is no substitution that semi-unifies $\mathcal{D}\,F$. So in the following, we assume that $\mathcal{D}\,F$ contains only inequalities between pairs.

We construct E by translating every element of $\mathcal{D}\,F$ into equations according to the rules

$$(f\overline{x}, g\overline{y}) \dot{\preceq} (f\overline{s}, g\overline{t}) \quad \rightsquigarrow \quad \alpha\overline{x} \doteq \overline{s}, \alpha\overline{y} \doteq \overline{t}$$
$$(s, s) \dot{\preceq} (s, t) \quad \rightsquigarrow \quad s \doteq t$$

where α is a fresh substitution variable for every inequality and $\alpha\overline{x} \doteq \overline{s}$ stands for $\alpha x_1 \doteq s_1, \ldots, \alpha x_n \doteq s_n$ with $\overline{x} = (x_1, \ldots, x_n)$ being a tuple of variables and $\overline{s} = (s_1, \ldots, s_n)$ being a tuple of simple terms of the same length. Note that the rules cover all elements of $\mathcal{D}\,F$ and that in the second rule, s and t are always simple terms.

▶ **Example 21.** The system of inequalities from Example 17 is translated into the system of equations

$$x_1 \doteq x_4,$$
$$\alpha x_5 \doteq a \cdot x_1,\ \alpha x_6 \doteq x_2, \qquad \alpha x_7 \doteq a \cdot x_3,\ \alpha x_8 \doteq a \cdot a \cdot x_4,$$
$$x_6 \doteq x_7,$$
$$\beta x_1 \doteq a \cdot x_5,\ \beta x_2 \doteq a \cdot a \cdot x_6,\ \beta x_3 \doteq a \cdot x_7,\ \beta x_4 \doteq x_8$$

▶ **Lemma 22.** *Let E be a translation of $\mathcal{D}\,F$ for some frame F. Then σ semi-unifies $\mathcal{D}\,F$ iff σ semi-unifies E.*

Proof. Since distinct inequalities in $\mathcal{D}\,F$ are translated into equations with disjoint substitution variables, it suffices to consider the translation E' of a singleton subset $D' \subseteq \mathcal{D}\,F$.

Let $D' = \{(f\overline{x}, g\overline{y}) \dot{\preceq} (f\overline{s}, g\overline{t})\}$. Then $E' = \{\alpha\overline{x} \doteq \overline{s}, \alpha\overline{y} \doteq \overline{t}\}$. If σ semi-unifies D', then there is a substitution τ with $\tau(\sigma(f\overline{x}, g\overline{y})) = \sigma(f\overline{s}, g\overline{t})$. Thus σ semi-unifies E' with the assignment A where $A\alpha := \tau$. If σ semi-unifies E', then σ semi-unifies D' because $(A\alpha)(\sigma\overline{x}) = A(\sigma(\alpha\overline{x})) = A(\sigma\overline{s}) = \sigma\overline{s}$ and the same for $\alpha\overline{y}$ and $\overline{t}$.

Let $D' = \{(s, s) \dot{\preceq} (s, t)\}$ with s and t being simple. Then $E' = \{s \doteq t\}$. If σ semi-unifies D', then there is a substitution τ with $\tau(\sigma s) = \sigma s$ and $\tau(\sigma s) = \sigma t$. Thus $\sigma s = \sigma t$ and hence σ semi-unifies E'. If σ semi-unifies E', then $\sigma s = \sigma t$ and hence σ semi-unifies D'. ◀

We need the following technical lemma to show that the translation is anchored.

▶ **Lemma 23.** *Let $(\sim)$ be an equivalence relation on simple variables and let E be a system of equations containing only equations of the form $\alpha x \doteq s$ or $s \doteq t$ where s and t are simple terms. Then E is anchored if the following conditions are satisfied.*

1. *If αx occurs in E and $x \sim y$, then αy occurs in E.*
2. *If $\alpha x_1 \doteq s \in E$ and $\alpha x_2 \doteq t \in E$, then $y_1 \sim y_2$ for all simple variables y_1, y_2 that occur in s or t.*
3. *If $s \doteq t \in E$, then $x \sim y$ for all simple variables x, y that occur in s or t.*

Proof. Let $\hat{\sim}$ be the smallest symmetric relation on atoms such that

- If $x \sim y$, then $x \mathrel{\hat{\sim}} y$.
- If αx and αy occur in E, then $\alpha x \mathrel{\hat{\sim}} \alpha y$.
- If $\alpha x \doteq s \in E$ and y occurs in s, then $\alpha x \mathrel{\hat{\sim}} y$.

The transitive closure $\hat{\sim}^*$ of $\hat{\sim}$ is a PER. We will show that E is anchored with $\hat{\sim}^*$. But first, we prove that $x \sim y$ whenever $x \mathrel{\hat{\sim}^*} y$. We do this by induction on the length of the shortest sequence $x \mathrel{\hat{\sim}} X_1 \mathrel{\hat{\sim}} \cdots \mathrel{\hat{\sim}} X_n \mathrel{\hat{\sim}} y$. If $n = 0$, then the claim follows immediately from the definition of $\hat{\sim}$. For the inductive case, we distinguish two cases. If there is some simple variable X_i with $i \in \{1, \ldots, n\}$, then the inductive hypothesis for $x \mathrel{\hat{\sim}} X_1 \mathrel{\hat{\sim}} \cdots \mathrel{\hat{\sim}} X_i$ and $X_i \mathrel{\hat{\sim}} \cdots \mathrel{\hat{\sim}} X_n \mathrel{\hat{\sim}} y$ yield $x \sim X_i$ and $X_i \sim y$ and thus $x \sim y$. Otherwise, by the definition of $\hat{\sim}$, there are simple variables $z_1, \ldots, z_n$ and a single substitution variable α such that $X_i = \alpha z_i$ for all $i \in \{1, \ldots, n\}$. By the definition of $\hat{\sim}$, there are simple terms s, t such that $\alpha z_1 \doteq s \in E$ and $\alpha z_n \doteq t \in E$ where x occurs in s and y occurs in t. Thus $x \sim y$ by requirement (2) on $(\sim)$.

Now, we show that E is anchored with $\hat{\sim}^*$.

1. Let $s \doteq t \in E$ with X and Y occurring in $s \doteq t$. We need to show that $X \mathrel{\hat{\sim}^*} Y$. If s and t are simple, then this follows immediately from requirement (3) on $(\sim)$. Otherwise $s = \alpha x$ for some instance variable αx and the claim follows from the definition of $\hat{\sim}$.
2. Let $x \mathrel{\hat{\sim}^*} y$ and $\alpha x \mathrel{\hat{\sim}^*} \alpha x$. We need to show that $\alpha x \mathrel{\hat{\sim}^*} \alpha y$. Because of requirement (1) on $(\sim)$, it suffices to show that $x \sim y$. As we proved before, this follows from $x \mathrel{\hat{\sim}^*} y$.
3. Let $\alpha x \mathrel{\hat{\sim}^*} \alpha x$. It suffices to show that there is a simple term s with $\alpha x \doteq s \in E$. This follows from the conditions on E as $\hat{\sim}^*$ only contains instance variables from E. ◀

▶ **Lemma 24.** *Let E be a translation of $\mathcal{D}\,F$ for some frame F. Then E is anchored.*

Proof. Consider the equivalence relation $(\sim)$ on simple variables such that $x \sim y$ iff x and y occur in the same element of F or $x = y$. It is easy to check that $(\sim)$ and E satisfy the conditions of Lemma 23. ◀

10 Solving AnSUP

In this section, we present rules to solve the anchored semi-unification problem. Our rules are a restriction of the rules presented by Leiß [12].

We first define the solved form computed by our algorithm. We write $E, s \doteq t$ for the disjoint union $E \dot\cup \{s \doteq t\}$. Given a system of equations E, we call an atom X *eliminated* if E has the form $E', X \doteq s$ such that neither X nor (in case X is a simple variable) an instance variable αX occur in E' or in s. A system of equations E is in *solved form* if all equations have the form $X \doteq s$ where X is eliminated and s is simple.

▶ **Lemma 25.** *If a system of equations E is in solved form, then it has a principal semi-unifier.*

Proof. Since E is in solved form, we can unambiguously define a substitution σ and an assignment A as follows.

$$\sigma x = \begin{cases} s & \text{if } x \doteq s \in E \\ x & \text{else} \end{cases}$$

$$(A\alpha)x = \begin{cases} s & \text{if } \alpha x \doteq s \in E \\ x & \text{else} \end{cases}$$

For σ to be a semi-unifier of E, we have to show that for every equation $X \doteq s \in E$ we have $A(\sigma X) = A(\sigma s)$. Since E is in solved form, s is simple and cannot contain an eliminated simple variable. Thus $A(\sigma s) = As = s$. So it suffices to show that $A(\sigma X) = s$.

If $X = x$ for some simple variable x, then $\sigma x = s$ by the definition of σ. Since s is simple, it follows that $A(\sigma x) = A(s) = s$.

If $X = \alpha x$ for some instance variable αx, then $\sigma x = x$ because otherwise x would be eliminated and hence αx could not occur in E. So it follows that $A(\sigma(\alpha x)) = A(\alpha x) = s$.

It remains to show that σ is principal. Consider some other semi-unifier τ of E. Since τ unifies all equations of the form $x \doteq s \in E$, we have that $\tau x = \tau(\sigma x)$ for all simple variables x. Thus $\sigma \preceq \tau$. ◀

A system of equations E is *clashed* if one of the following conditions holds.

- $a \doteq s \in E$ or $s \doteq a \in E$ where $a \neq s$ and s is not an atom.
- $x \doteq s \in E$ where x or αx for some α occurs in s and s is not an atom.
- $\alpha x \doteq s \in E$ where αx occurs in $s \neq \alpha x$.

▶ **Proposition 26.** *A clashed system of equations has no semi-unifier.*

The rules in Figure 2 transform every anchored system of equations into an equivalent system of equations that is clashed or in solved form. To ensure termination, the rules must not be applied to a clashed system of equations. We write $s[t/x]$ for σs where σ is the unique substitution satisfying $\sigma x = t$ and $\sigma y = y$ for all y with $y \neq x$. We write $s[t/\alpha x]$ for the term that is obtained from s by replacing every occurrence of αx with t. We write $E[s/X]$ for $\{t[s/X] \doteq u[s/X] \mid t \doteq u \in E\}$.

$$
\begin{array}{lrcl}
R_{\text{bop}} & E, s_1 \cdot s_2 \doteq t_1 \cdot t_2 & \Longrightarrow & E, s_1 \doteq t_1, s_2 \doteq t_2 \\
R_{\text{refl}} & E, s \doteq s & \Longrightarrow & E \\
R_{\text{orient1}} & E, s \doteq \alpha x & \Longrightarrow & E, \alpha x \doteq s \text{ if } s \text{ is not an instance variable} \\
R_{\text{orient2}} & E, s \doteq x & \Longrightarrow & E, x \doteq s \text{ if } s \text{ is not an atom} \\
R_{\text{elim1}} & E, \alpha x \doteq s & \Longrightarrow & E[s/\alpha x], \alpha x \doteq s \text{ if } s \text{ is simple} \\
R_{\text{elim2}} & E, x \doteq s & \Longrightarrow & E[s/x], x \doteq s \text{ if } x \text{ does not occur in } s \text{ and } s \text{ is simple}
\end{array}
$$

Figure 2 Semi-Unification Rules.

▶ **Example 27.** We transform the following anchored system of equations into solved form.

$$
\begin{array}{rl}
 & \alpha x_1 \doteq a \cdot x_3,\ \alpha x_2 \doteq a \cdot (a \cdot x_4),\ \alpha x_3 \doteq a \cdot x_1,\ \alpha x_4 \doteq x_2,\ x_1 \doteq x_4 \\
\overset{R_{\text{elim2}}}{\Longrightarrow} & \alpha x_4 \doteq a \cdot x_3,\ \alpha x_2 \doteq a \cdot (a \cdot x_4),\ \alpha x_3 \doteq a \cdot x_4,\ \alpha x_4 \doteq x_2,\ x_1 \doteq x_4 \\
\overset{R_{\text{elim1}}}{\Longrightarrow} & x_2 \doteq a \cdot x_3,\ \alpha x_2 \doteq a \cdot (a \cdot x_4),\ \alpha x_3 \doteq a \cdot x_4,\ \alpha x_4 \doteq x_2,\ x_1 \doteq x_4 \\
\overset{R_{\text{elim2}}}{\Longrightarrow} & x_2 \doteq a \cdot x_3,\ a \cdot \alpha x_3 \doteq a \cdot (a \cdot x_4),\ \alpha x_3 \doteq a \cdot x_4,\ \alpha x_4 \doteq a \cdot x_3,\ x_1 \doteq x_4 \\
\overset{R_{\text{bop}}}{\Longrightarrow} & x_2 \doteq a \cdot x_3,\ a \doteq a,\ \alpha x_3 \doteq a \cdot x_4,\ \alpha x_4 \doteq a \cdot x_3,\ x_1 \doteq x_4 \\
\overset{R_{\text{refl}}}{\Longrightarrow} & x_2 \doteq a \cdot x_3,\ \alpha x_3 \doteq a \cdot x_4,\ \alpha x_4 \doteq a \cdot x_3,\ x_1 \doteq x_4
\end{array}
$$

▶ **Proposition 28.** *If $E \Longrightarrow E'$, then E and E' have the same semi-unifiers.*

▶ **Proposition 29.** *If $E, X \doteq s$ is anchored and s is simple, then $E[s/X], X \doteq s$ is anchored.*

▶ **Lemma 30.** *If $E \Longrightarrow E'$ and E is anchored, then E' is anchored.*

Proof. For R_{elim1} and R_{elim2}, this follows immediately from Proposition 29. For the other rules, the claim is trivial. ◀

▶ **Lemma 31.** *There is no infinite reduction sequence $E_1 \Longrightarrow E_2 \Longrightarrow \cdots$ such that E_i is not clashed for all $i \in \{1, 2, \ldots\}$.*

Proof. We assign a triple of natural numbers $(n_1, n_2, n_3 + n_4 + n_5)$ to every system of equations $E = \{s_1 \doteq t_1, \ldots, s_n \doteq t_n\}$ where

- n_1 is the number of simple variables in E that are not eliminated,
- n_2 is the number of instance variables in E that are not eliminated,
- n_3 is the sum of the sizes of every term s_i that is not an instance variable for $i \in \{1, 2, \ldots\}$,
- n_4 is the sum of the sizes of every term s_i that is not an atom for $i \in \{1, 2, \ldots\}$ and
- n_5 is the number of equations in E.

With the usual lexicographical ordering, this yields a well-founded ordering on systems of equations. It is easy to show that every rule application respects the ordering and hence an infinite reduction sequence is impossible. ◀

We call a system of equations E *terminal* if there is no system of equations E' with $E \Longrightarrow E'$.

▶ **Proposition 32.** *If E is anchored and terminal, then E is either clashed or in solved form.*

Proof. Let E be anchored, terminal and not clashed. We show that E is in solved form, that is, all equations have the form $X \doteq s$ where X is eliminated and s is simple.

Let $s \doteq t \in E$. Then s is an atom because otherwise R_{bop}, R_{refl}, R_{orient1} or R_{orient2} would be applicable contradicting the fact that E is terminal. Also, t is simple because if it contained an instance variable αx, then E would also contain an equation $\alpha x \doteq u$ because E is anchored. This is a contradiction because R_{elim1} would be applicable. Since s is an atom and t is simple, s must be eliminated because otherwise R_{elim1} or R_{elim2} would be applicable. ◀

Thus we can transform every anchored system of equations E either into solved form or into a clashed system of equations. In the first case, we have computed a principal semi-unifier of E. In the second case, E has no semi-unifier.

11 Conclusion

The paper presents the first unification algorithm for nonnested recursion schemes. Based on a novel coinductive definition of $\mathcal{S}$-equivalence, we establish the existence of principal $\mathcal{S}$-unifiers. Our method for solving $\mathcal{S}$-unification problems works by a reduction to a new decidable semi-unification problem we call AnSUP (anchored semi-unification problem). AnSUP is quite different from other decidable semi-unification problems we know of.

- The uniform fragment [7, 16] only allows for a single substitution variable (or, in the usual representation, a single inequality). In contrast to this, our reduction requires many substitution variables.
- The acyclic fragment [9] and its extension to the R-acyclic fragment [14] disallow cyclic inequalities in the following sense. There must not be a sequence of inequalities $s_1 \dot{\preceq} t_1, \ldots, s_n \dot{\preceq} t_n$ such that t_n and s_1 share a variable and for all $i \in \{1, \ldots, n-1\}$, t_i and s_{i+1} share a variable. In contrast to this, we allow arbitrary cycles and need them in the reduction.
- The left-linear fragment [6] does not allow that a variable occurs twice in the left-hand side s of an inequality $s \dot{\preceq} t$. Adding inequalities of the form $(s, s) \dot{\preceq} (s, t)$ is impossible since this would allow for the same expressive power as unrestricted semi-unification, which is undecidable. So it is impossible to express ordinary unification with the left-linear fragment.
- The quasi-monadic fragment [13] does not allow terms containing two different variables.
- There is a decidable fragment that only allows two variables [11].

To the best of our knowledge, AnSUP is the only fragment that allows for cyclic inequalities, an unrestricted number of variables and subsumes ordinary unification.

Our algorithm for anchored semi-unification needs $\mathcal{O}(n^3)$ steps, since all three components of the triple used in the termination proof (Lemma 31) are linearly bounded by the problem size. However, since our semi-unification rules build on the naive rules for ordinary unification, the term size can grow exponentially in the number of steps. For example, our algorithm performs poorly on the ordinary unification problem $x_1 \doteq x_2 \cdot x_2,\ x_2 \doteq x_3 \cdot x_3, \ldots,\ x_{n-1} \doteq x_n \cdot x_n$. We expect that using the same techniques as for ordinary unification [1], it is possible to design an algorithm for anchored semi-unification with polynomial complexity.

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