A new approach to modelling nonlinear time series: Introducing the ExpAR-ARCH and ExpAR-GARCH models and applications

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Abstract

The analysis of time series has long been the subject of interest in different fields. For decades time series were analysed with linear models. Nevertheless, an issue that has been raised is whether there exist other models that can explain and fit real data better than linear ones. In this paper, new nonlinear time series models are proposed (namely the ExpAR-ARCH and the ExpAR-GARCH), which are combinations of a nonlinear model in the conditional mean and a nonlinear model in the conditional variance and have the potential of explaining observed data in various fields. Simulated data of these models are presented, while different algorithms (the Nelder-Mead simplex direct search method, the Quasi-Newton line search algorithm, the Active-Set algorithm, the Sequential Quadratic Programming algorithm, the Interior Point algorithm and a Genetic Algorithm) are used and compared in order to check their estimation performance when it comes to these suggested nonlinear models. Moreover, an application to the Dow Jones data is considered, showing that the new models can explain real data better than the AR-ARCH and AR-GARCH models.

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1 Introduction

During the last century considerable achievements have been made in both theoretical and empirical linear time series analysis. The Autoregressive (AR) model of Yule (1927) and the Autoregressive Moving Average (ARMA) model of Box and Jenkins (1970) are the two most noticeable examples of linear models which have found many applications in real life data. Linear models have many advantages, such as good fitting and predictive ability, which is the main reason why they have been used so much. However, there are time series that exhibit nonlinear characteristics, in which case, linear time series models can be too restrictive and if our aim is a more profound analysis of how series are generated, we need to allow for more general models.

Hence, the limitations of linear models have raised the issue of whether there exist other models that can explain and predict better time series with such characteristics. This issue resulted in the expansion of the linear models in the literature and in the development of various nonlinear models, e.g. nonlinear models in conditional mean and nonlinear models in conditional variance, all of which have attempted to explain and forecast more accurately specific time series.

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The classic nonlinear models in conditional mean are the Threshold Autoregressive (TAR) model of Tong (1977) (along with its special cases, the Self-Exciting Threshold Autoregressive (SETAR) model (Tong and Lim (1980)) and the Smooth Transition Autoregression (STAR) model (Chan and Tong (1986))), the Exponential Autoregressive (ExpAR) model of Ozaki (1980) and the Bilinear model of Granger and Andersen (1978). On the other hand, the most characteristic examples of nonlinear models in conditional variance are the Autoregressive Conditional Heteroskedasticity (ARCH) and the Generalised Autoregressive Conditional Heteroskedasticity (GARCH) models, introduced by Engle (1982) and Bollerslev (1986) respectively.

Since the introduction of these models, there have been introduced many nonlinear models, some trying to explain nonlinearities in the conditional mean and others attempting to interpret nonlinearities in the conditional variance. Nevertheless, there are some time series exhibiting asymmetries which could be better explained by models that have both a nonlinear conditional mean and a changing conditional variance, but there has not been much work on combining these two forms of nonlinearity. Tong (1990), p. 116, was the first to suggest combining the first-generation models in order to produce second-generation models, as he called them, giving as examples the specification of a SETAR-ARCH model and a Bilinear-ARCH model, which combine a SETAR or a Bilinear model, respectively, for the conditional mean with a conditional variance following an ARCH model. Since then, such models have gradually become popular and are being used more widely, mainly in applications to financial data.

The class of these second-generation models that has been applied most though is the TAR-GARCH family, and especially the STAR-GARCH and STAR-STGARCH models. Applications of TAR-GARCH-type and SETAR-GARCH-type models can be found in Li and Lam (1995), in Li and Li (1996), in Amendola and Niglio (2000), in Osinska and Witkowski (2004), in Chiang and Doong (2001), and in Munoz, Marquez and Acosta (2007), while applications of STAR-GARCH-type models can be found in Lee and Li (1998), in Lundbergh and Terasvirta (1999), and in Busetti and Manera (2003).

A different class of models that combines a nonlinear conditional mean and conditional variance is the Exponential Autoregressive model with GARCH errors, which, however, has not been much used or developed. The first model of this class, introduced by LeBaron (1992) with the purpose of exploring the relationship between volatility and serial correlation for different stock return series at daily and weekly frequencies, was a combination of Bollerslev's (1986) GARCH model, Ozaki's (1980) ExpAR model and Stock's (1988) time deformation model. Later, Koutmos (1997) used an Exponential Autoregressive model for the conditional mean with a Threshold GARCH model for the conditional standard deviation (EAR-TGARCH) along with a Generalised Error Distribution, which was a generalised version of LeBaron's (1992) model, in order to study the daily stock returns in some equity markets of the Pacific Basin area and to examine if the behaviour of these markets are similar to the behaviour of developed ones.

In this paper, we suggest the ExpAR-ARCH and ExpAR-GARCH models, which are combinations of the pure ExpAR model of Ozaki (1980) for the conditional mean and the ARCH or GARCH model respectively for the conditional variance, and which have the potential of explaining and forecasting nonlinear time series of various fields. It should be highlighted that these models are different from the ones proposed by LeBaron (1992) and by Koutmos (1997) in the variable contained in the exponential term of the conditional mean model. Our models are in accordance with the ExpAR model suggested by Ozaki (1980) containing the lag of the variable in the exponential term, while LeBaron's (1992) model and

Koutmos' (1997) model contain the conditional variance instead.

The paper is organised as follows: In section 2, the models are introduced. In section 3, the estimation method is presented. Some simulation results are shown in section 4, while in section 5 an application to real data is considered. Finally, some concluding remarks are made in section 6.

2 Models

In this section, the suggested models are presented. These consist of a nonlinear model for the conditional mean (ExpAR) and a nonlinear model for the conditional variance (ARCH or GARCH).

Let y_t be a time series generated by a stationary process. The Exponential Autoregressive model of order s with heteroscedastic errors is defined as:

$$y_t = c + \sum_{i=1}^{s} \{\phi_i + \pi_i \cdot exp(-\gamma \cdot y_{t-1}^2)\} \cdot y_{t-i} + u_t, \quad (1)$$

where

$$u_t = \epsilon_t \cdot \sqrt{h_t}, \quad (2)$$

$$\epsilon_t \sim n.i.d.(0,1), \text{ and} \quad (3)$$

$$h_t = h(n) = n' \cdot z_t. \quad (4)$$

In the case of the ExpAR(s)-ARCH(q) model, we have

$$n = (\alpha_0, \alpha_1, \ldots, \alpha_q)', \quad (5)$$

$$z_t = (1, u_{t-1}^2, \ldots, u_{t-q}^2), \quad (6)$$

$$\alpha_0 > 0 \text{ and } \alpha_i \geq 0, i > 0, i = 1, \ldots, q, \quad (7)$$

while in the case of the ExpAR(s)-GARCH(p, q) model, we have

$$n = (\alpha_0, \alpha_1, \ldots, \alpha_q, \beta_1, \ldots, \beta_p)', \quad (8)$$

$$z_t = (1, u_{t-1}^2, \ldots, u_{t-q}^2, h_{t-1}, \ldots, h_{t-p}), \quad (9)$$

$$\alpha_0 > 0, \alpha_i \geq 0, i > 0, i = 1, \ldots, q, \text{and } \beta_j \geq 0, j > 0, j = 1, \ldots, p. \quad (10)$$

3 Estimation

The estimation procedure that is used in this research is maximum likelihood. Assuming that the sequence u_t is identically normal distributed and conditioning on the observation at time $t = 0$, y_0, the conditional log-likelihood function is

$$L_T(\theta) = \sum_{i=1}^{T} l_t(\theta), \tag{11}$$

where

$$l_t = -\frac{1}{2} \cdot \log 2\pi - \frac{1}{2} \cdot \log h_t - \frac{1}{2} \cdot \frac{u_t^2}{h_t}, \tag{12}$$

is the log-likelihood at time t, which means that the overall conditional log-likelihood function is

$$L_t(\theta) = -\frac{T}{2} \cdot \log 2\pi - \frac{1}{2} \cdot \sum_{i=1}^{T} \log h_t - \frac{1}{2} \cdot \sum_{i=1}^{T} \frac{u_t^2}{h_t}. \tag{13}$$

The gradient of the overall conditional log-likelihood is defined as

$$G_T = [\partial L_T / \partial b', \partial L_T / \partial \omega'], \tag{14}$$

while the gradient of the log-likelihood function at time t is given by

$$g_T = [\partial l_t / \partial b', \partial l_t / \partial \omega'], \tag{15}$$

where b is the vector of conditional mean parameters and ω is the vector of the conditional variance parameters.

In the previous formulae of the estimation part, u_t should be replaced by

$$y_t - c - \sum_{i=1}^{s} \{\phi_i + \pi_i \cdot exp(-\gamma \cdot y_{t-1}^2)\} \cdot y_{t-i}. \tag{16}$$

However, maintaining u_t makes the notation less complicated and keeps it close to Bollerslev (1986).

4 Simulations

4.1 Methodology

In this section simulated series of the proposed models are presented. Since new models are introduced, it is important to simulate them. In this way we can see what characteristics real data, which could be described by them, would have. It should be emphasised that here only the first order models are considered, as it is well-known that low orders of nonlinear models can capture the biggest part of nonlinearity.

The simulated series are compared with the well-known AR-ARCH and AR-GARCH models, which can be considered as benchmarks. The comparison is made by using the same

Table 1 Moments of the simulated series.

	ExpAR(1)-ARCH(1)	ExpAR(1)-GARCH(1,1)	AR(1)-ARCH(1)	AR(1)-GARCH(1,1)
Mean	0.5381	0.4838	0.4213	0.4179
Standard deviation	0.3045	0.7523	0.3113	0.7636
Skewness	-0.2438	-0.2637	0.0222	-0.0511
Kurtosis	42.916	83.326	43.487	81.298

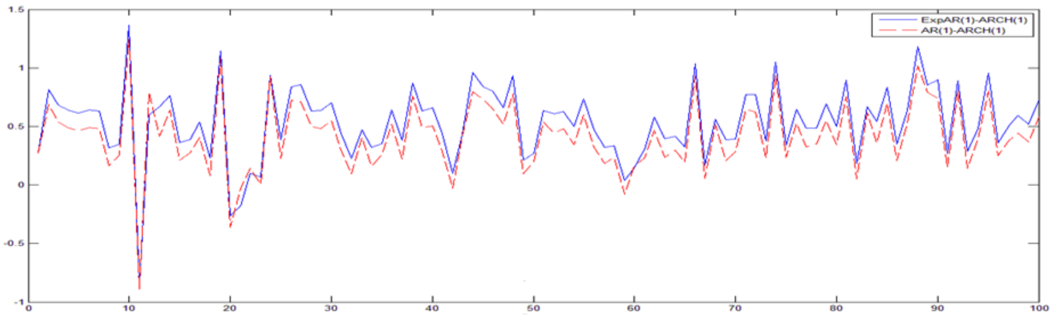

Figure 1 Simulated series: ExpAR(1)-ARCH(1) and AR(1)-ARCH(1).

values for the common parameters (same c, ϕ_1, α_0 and α_1 for all the models, same β_1 for the models with GARCH errors, and same π_1 and γ for the models that are described by an ExpAR model for the conditional mean) and the same number of simulated data ($T = 3400$).

The parameter values were chosen as follows: In order to be consistent with the conditional variance model restrictions, we chose non-negative conditional variance parameters values. We also wanted a stationary conditional mean model and a stationary conditional variance model. Hence, we set the ϕ_1 parameter to a value which is smaller than one in absolute value, and we set the α_1 parameter and the sum of the α_1 and β_1 parameters to be smaller than one in the case of ARCH and GARCH errors respectively.

Moreover, since estimation plays a very important role when fitting data to models, it is essential to estimate the coefficients of the models in order to see how close the assumed and estimated parameters are and to suggest effective estimating methods. Furthermore, various algorithms have been used and compared in order to check their estimation performance when it comes to the suggested nonlinear models and to the classic AR-ARCH and AR-GARCH models. More specifically, the algorithms used are the Nelder-Mead simplex direct search method (NM), the Quasi-Newton line search algorithm (QN), the Active-Set algorithm (AS), the Sequential Quadratic Programming algorithm (SQP), the Interior Point algorithm (IP) and a Genetic Algorithm (GA).

The NM and the QN methods solve unconstrained optimisation problems, while the GA, the AS, the SQP and the IP algorithms solve the constrained optimisation problem. The GA was used to solve the unconstrained optimisation problem as well, but it never gave good results, as it gave some negative estimates for the ARCH or GARCH parameters, resulting in complex values of the log-likelihood function and of the standard errors, and therefore its results are not displayed.

In Table 1 there can be found the moments of each simulated series, while Figures 1-6 show the simulated series and their autocorrelation and partial autocorrelation plots. Table 2 shows the arbitrary initial values used when running the algorithms. The values of the real parameters for the simulations are displayed in the first column of Tables 3-6, while in the remaining columns the results obtained from the algorithms are reported, including the

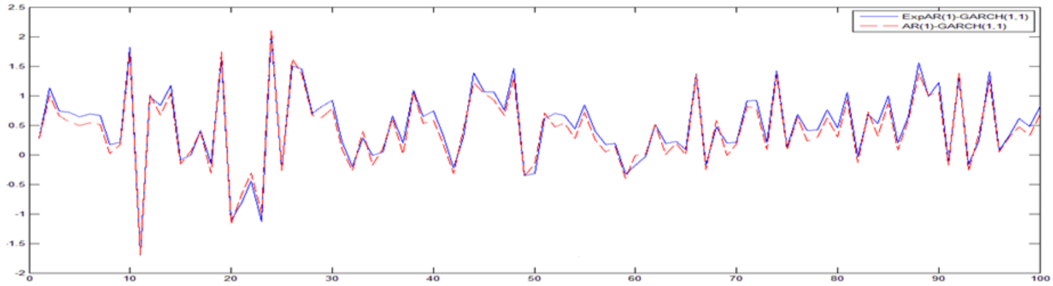

Figure 2 Simulated series: ExpAR(1)-GARCH(1,1) and AR(1)-GARCH(1,1).

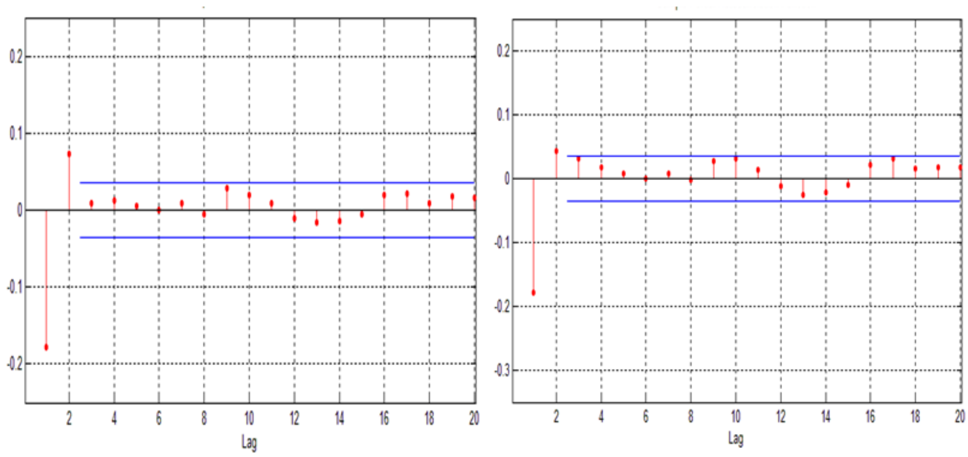

Figure 3 Autocorrelation (a) and Partial Autocorrelation (b) plot of ExpAR(1)-ARCH(1).

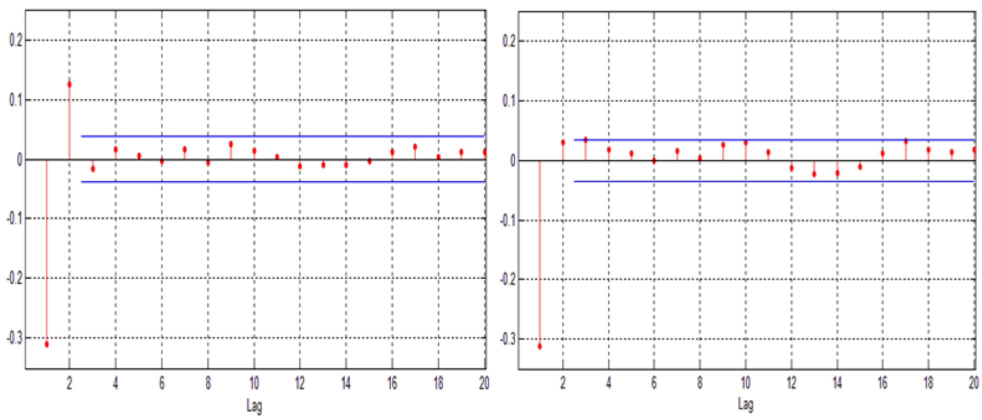

Figure 4 Autocorrelation (a) and Partial Autocorrelation (b) plot of AR(1)-ARCH(1).

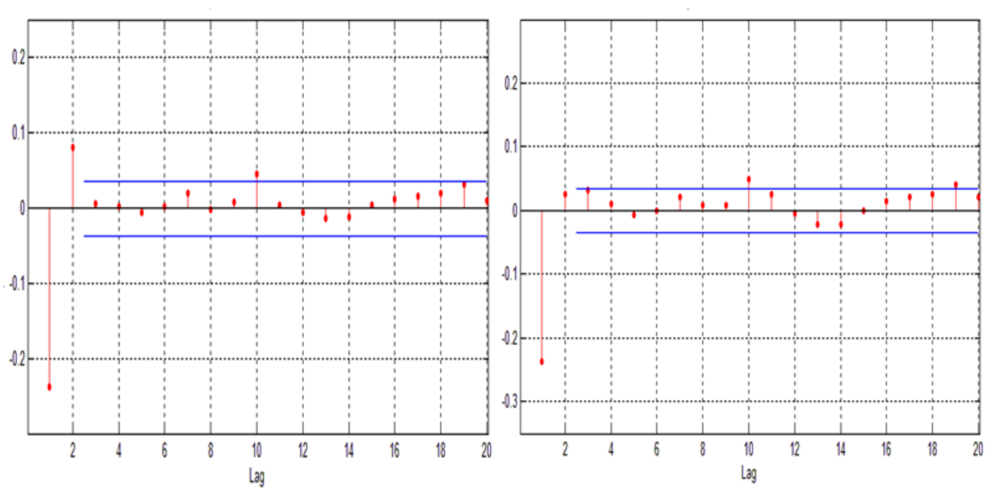

Figure 5 Autocorrelation (a) and Partial Autocorrelation (b) plot of ExpAR(1)-GARCH(1,1).

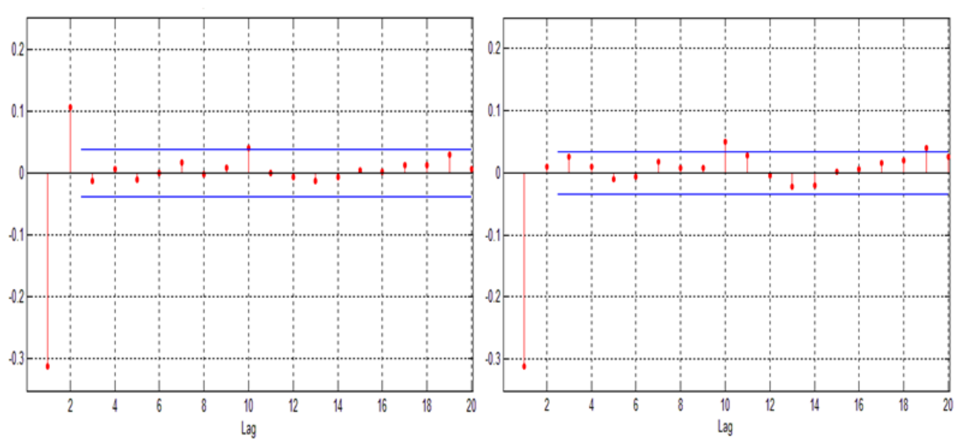

Figure 6 Autocorrelation (a) and Partial Autocorrelation (b) plot of AR(1)-GARCH(1,1).

estimates for the parameters, the respective standard errors, the value of the log-likelihood function, and the number of iterations/generations required for the convergence of the algorithm to the optimum. In addition, once we obtain the estimates from the algorithms, the algorithms are run again, but then the initial values are not arbitrary. Instead, they are chosen accordingly to which algorithm gave better results in terms of closeness to the real values, so that the accuracy and speed of the algorithms are tested when the initial guess for the values of the parameters is indeed close to the real parameters.

4.2 Results

4.2.1 Using arbitrary initial values for the algorithms

According to the results obtained, we notice that for the ExpAR(1)-ARCH(1) model only the QN algorithm gave estimates which are close to all the true parameter values. Moreover, the QN algorithm required the lowest number of iterations in order to converge to a solution and gave the highest log-likelihood value. On the other hand, the AS algorithm failed and didn't give any results, while the remaining algorithms gave relatively good estimates only for the constant c and the parameters of the ARCH(1) model, α_0 and α_1. Moreover, the SQP and IP

Table 2 Initial values of the algorithms.

	ExpAR(1)-ARCH(1)	ExpAR(1)-GARCH(1,1)	AR(1)-ARCH(1)	AR(1)-GARCH(1,1)
c	0.0010	0.0010	0.0010	0.0010
ϕ_1	0.6000	0.6000	0.6000	0.6000
π_1	0.3000	0.3000	-	-
γ	1	1	-	-
α_0	0.0010	0.0010	0.0010	0.0010
α_1	0.1000	0.1000	0.1000	0.1000
β_1	-	0.2000	-	0.2000

Table 3 Estimation results for ExpAR(1)-ARCH(1).

	ExpAR(1)-ARCH(1)					
	NM	**QN**	**AS**	**SQP**	**IP**	**GA**
c= 0.55	0.5958 (0.0094)	0.5689 (0.0112)	NaN (NaN)	0.5957 (0.0094)	0.5957 (0.0094)	0.6463 (0.0102)
$\phi_1 = -0.30$	18.3029 (0+9.7672i)	-0.3366 (0.0732)	NaN (NaN)	18.8319 -20.0583	19.3354 -9.3605	-0.7759 (0.1661)
$\pi_1 = 0.50$	-18.2896 (0+9.7662i)	0.4922 (0.0558)	NaN (NaN)	-18.8185 -20.0555	-19.3219 -9.3596	0.6571 (0.1729)
$\gamma = 1.20$	-0.0109 (0+0.0057i)	1.1357 (0.3388)	NaN (NaN)	-0.0106 (0.0111)	-0.0103 (0.0050)	0.1857 (0.0543)
$\alpha_0 = 0.05$	0.0502 (0.0013)	0.0505 (0.0013)	NaN (NaN)	0.0502 (0.0013)	0.0502 (0.0013)	0.0546 (0.0015)
$\alpha_1 = 0.40$	0.4463 (0.0234)	0.4398 (0.0235)	NaN (NaN)	0.4463 (0.0234)	0.4463 (0.0234)	0.3675 (0.0202)
Log-likelihood	2.1720e+03	2.1799e+03	NaN	2.1720e+03	2.1720e+03	2.1461e+03
Iterations/ Generations	1018	69	400	170	115	112

methods gave the same estimates and standard errors for the parameters c, α_0 and α_1, while the estimates obtained by these two algorithms for the remaining parameters are close to each other, although very far away from the true values. Furthermore, the estimates obtained by the NM method are close to the estimates obtained by the SQP and IP algorithms for all the parameters. The GA gave good estimates only for the ARCH parameters.

In the case of the ExpAR(1)-GARCH$(1,1)$ model, the QN algorithm again performed quite well. In this model, however, we observe that the IP algorithm gave estimates that are close to the true values of the parameters as well. In fact, here the highest log-likelihood value was obtained by the IP algorithm and the second highest log-likelihood value was given by the QN algorithm, although the latter required slightly fewer iterations. Yet the difference between these two log-likelihood values is rather unimportant. In addition, the Genetic Algorithm under constrained optimisation gave good estimates for the parameters of the conditional mean model, although it overestimated the parameter γ. However, it didn't give very good estimates for the GARCH parameters, in contrast to the QN, IP and SQP methods. The AS algorithm failed here as well.

Table 4 Estimation results for ExpAR(1)-GARCH(1,1).

	ExpAR(1)-GARCH(1,1)					
	NM	**QN**	**AS**	**SQP**	**IP**	**GA**
c= 0.55	0.2307 (0.0091)	0.5733 (0.0101)	NaN (NaN)	0.6248 (0.0093)	0.5740 (0.0101)	0.5844 (0.0094)
$\boldsymbol{\phi_1 = -0.30}$	6.2888 (-1.9683)	-0.3454 (0.0258)	NaN (NaN)	26.7355 (-10.5873)	-0.3432 (0.0255)	-0.3011 (0+0.0228i)
$\boldsymbol{\pi_1 = 0.50}$	-6.0639 (-1.9649)	0.4714 (0.0384)	NaN (NaN)	-26.9161 (-10.5871)	0.4659 (0.0384)	0.5108 (0.0444)
$\boldsymbol{\gamma = 1.20}$	-0.0072 (0.0021)	1.0593 (0.1730)	NaN (NaN)	-0.0007 (0.0013)	1.0652 (0.1738)	2.0012 (0+0.5525i)
$\boldsymbol{\alpha_0 = 0.05}$	0.5076 (0+0.0279i)	0.0535 (0.0039)	NaN (NaN)	0.0561 (0.0041)	0.0536 (0.0039)	0.1210 (0.0098)
$\boldsymbol{\alpha_1 = 0.40}$	0.3328 (0.0090)	0.4497 (0.0219)	NaN (NaN)	0.4612 (0.0225)	0.4493 (0.0219)	0.5254 (0.0239)
$\boldsymbol{\beta_1 = 0.50}$	-0.0243 (0+0.0105i)	0.4713 (0.0194)	NaN (NaN)	0.4593 (0.0197)	0.4713 (0.0194)	0.2755 (0.0240)
Log-likelihood	-4.3460e+03	-2.94089e+03	NaN	-3.0024e+03	-2.94088e+03	-3.0296e+03
Iterations/ Generations	1417	74	400	109	80	116

Table 5 Estimation results for AR(1)-ARCH(1).

	AR(1)-ARCH(1)					
	NM	**QN**	**AS**	**SQP**	**IP**	**GA**
c= 0.55	0.5628 (0.0059)	0.5631 (0.0059)	0.5627 (0.0059)	0.5627 (0.0059)	0.5627 (0.0059)	0.5660 (0.0060)
$\boldsymbol{\phi_1 = -0.30}$	-0.3286 (0.0121)	-0.3298 (0.0121)	-0.3285 (0.0121)	-0.3285 (0.0121)	-0.3285 (0.0121)	-0.3321 (0.0122)
$\boldsymbol{\alpha_0 = 0.05}$	0.0505 (0.0013)	0.0506 (0.0013)	0.0505 (0.0013)	0.0505 (0.0013)	0.0505 (0.0013)	0.0498 (0.0014)
$\boldsymbol{\alpha_1 = 0.40}$	0.4390 (0.0234)	0.4398 (0.0235)	0.4390 (0.0234)	0.4390 (0.0234)	0.4390 (0.0234)	0.4912 (0.0264)
Log-likelihood	2.1800e+03	2.1800e+03	2.1800e+03	2.1800e+03	2.1800e+03	2.1773e+03
Iterations/ Generations	250	51	71	49	58	97

Table 6 Estimation results for AR(1)-GARCH(1,1).

	AR(1)-GARCH(1,1)					
	NM	**QN**	**AS**	**SQP**	**IP**	**GA**
$c = 0.55$	0.3062 (1.1215e-05-1.2259e-12i)	0.5661 (0.0077)	0.5660 (0.0077)	0.5660 (0.0077)	0.5660 (0.0077)	0.5951 (0.0078)
$\phi_1 = -0.30$	0.2855 (1.3589e-05+2.7361e-12i)	-0.3360 (0.0121)	-0.3359 (0.0121)	-0.3359 (0.0121)	-0.3359 (0.0121)	-0.3771 (0.0120)
$\alpha_0 = 0.05$	-0.0730 (1.0032e-09+1.3493e-15i)	0.0533 (0.0039)	0.0534 (0.0039)	0.0534 (0.0039)	0.0534 (0.0039)	0.0626 (0.0048)
$\alpha_1 = 0.40$	11.451 (1.2296e-07 + 5.7040e-12i)	0.4486 (0.0218)	0.4483 (0.0218)	0.4483 (0.0218)	0.4483 (0.0218)	0.4706 (0.0237)
$\beta_1 = 0.50$	0.5182 (2.1610e-12+4.8252e-06i)	0.4727 (0.0193)	0.4726 (0.0193)	0.4727 (0.0193)	0.4727 (0.0193)	0.4428 (0.0211)
Log-likelihood	-2.8144e-06-9.9903e+02i	-2.9413e+03	-2.9413e+03	-2.9413e+03	-2.9413e+03	-2.9520e+03
Iterations/ Generations	558	57	87	58	37	73

When running the algorithms for estimating the AR(1)-ARCH(1) model, we notice that all the methods performed similarly and well. In fact, the IP, SQP and AS algorithms gave exactly the same estimates and standard errors, while the NM gave slightly different estimates for the constant c and the autoregressive parameter ϕ_1. The QN method and the GA performed similarly, but it could be said that the differences are rather unimportant, so that any of these algorithms could be used to estimate an AR(1)-ARCH(1) model. However, the algorithm that required the lowest number of iterations is the SQP (49) and then the QN (51).

Similar to the AR(1)-ARCH(1) model, when estimating the AR(1)-GARCH$(1,1)$ the IP, SQP and AS algorithms gave exactly the same estimates, while the QN method gave slightly different estimates, but again the differences are rather unimportant. All these four algorithms performed well. However, here the GA gave estimates that deviate more from the true values and the NM method did not perform well at all, even giving a negative value for the α_0 parameter. Here, the IP algorithm required the minimum number of iterations (37), while the QN and the SQP methods followed, requiring, 57 and 58 iterations respectively.

It is easily noticed that when estimating the above models, the QN algorithm, although solving the unconstrained problem, not only gave estimates close to the real values of the parameters in every single case, compared to the IP, SQP, AS, NM and GA methods, but also overall required a small number of iterations in order to converge to the solution. Hence, since the QN method performed well in every case, it seems logical to use the estimates obtained from it as initial guesses and run the algorithms again in order to check if there is any improvement in their performance, when we know that the initial values we give to the algorithms are indeed close to the true values. The initial values used for the second round of optimisations are shown in Table 7, while the respective estimation results can be found in Tables 8-11.

4.2.2 Using non-arbitrary initial values for the algorithms

When the initial guesses were closer to the true values of the parameters, we notice that for both the ExpAR(1)-ARCH(1) and the ExpAR(1)-GARCH$(1,1)$ models all the algorithms, apart from the GA which does not require initial values, performed quite well and in fact they gave the same estimates, while requiring a smaller number of iterations to reach a

Table 7 Initial values for the algorithms on the second round.

	ExpAR(1)-ARCH(1)	ExpAR(1)-GARCH(1,1)	AR(1)-ARCH(1)	AR(1)-GARCH(1,1)
c	0.5689	0.5733	0.5631	0.5661
ϕ_1	-0.3366	-0.3454	-0.3298	-0.3360
π_1	0.4922	0.4714	-	-
γ	11.357	10.593	-	-
α_0	0.0505	0.0535	0.0506	0.0533
α_1	0.4398	0.4497	0.4398	0.4486
β_1	-	0.4713	-	0.4727

Table 8 Estimation results for ExpAR(1)-ARCH(1) on the second round.

	ExpAR(1)-ARCH(1)					
	NM	QN	AS	SQP	IP	GA
c= 0.55	0.5721 (0.0112)	0.5721 (0.0112)	0.5721 (0.0112)	0.5721 (0.0112)	0.5721 (0.0112)	0.8084 (0.0041)
$\boldsymbol{\phi_1 = -0.30}$	-0.3749 (0.0812)	-0.3749 (0.0812)	-0.3748 (0.0811)	-0.3748 (0.0812)	-0.3749 (0.0812)	-0.3334 (0+0.0060i)
$\boldsymbol{\pi_1 = 0.50}$	0.5113 (0.0642)	0.5113 (0.0642)	0.5112 (0.0641)	0.5113 (0.0642)	0.5113 (0.0642)	-0.3054 (0+0.0377i)
$\boldsymbol{\gamma = 1.20}$	0.9700 (0.2915)	0.9700 (0.2915)	0.9701 (0.2912)	0.9701 (0.2917)	0.9700 (0.2914)	26.610 (0+0.3613i)
$\boldsymbol{\alpha_0 = 0.05}$	0.0505 (0.0013)	0.0505 (0.0013)	0.0505 (0.0013)	0.0505 (0.0013)	0.0505 (0.0013)	0.0553 (0.0015)
$\boldsymbol{\alpha_1 = 0.40}$	0.4399 (0.0234)	0.4399 (0.0234)	0.4399 (0.0234)	0.4399 (0.0234)	0.4399 (0.0234)	0.4605 (0.0243)
Log-likelihood	2.1801e+03	2.1801e+03	2.1801e+03	2.1801e+03	2.1801e+03	1.9529e+03
Iterations/ Generations	212	33	12	13	31	113

solution, as would be expected. Moreover, now the AS method not only did not fail, but required the minimum number of iterations for both models as well.

Nevertheless, the solutions obtained now are slightly worse than the ones obtained before from the QN algorithm, as now the estimates for the parameters of the conditional mean model, and especially the estimate for the γ parameter in the case of the ExpAR(1)-ARCH(1) model, deviate slightly more from the true values, although the values of the log-likelihood are somewhat higher than before for both models. In addition, in contrast to the ExpAR(1)-ARCH(1) model, the estimate for the γ parameter for the ExpAR(1)-GARCH$(1,1)$ model was slightly improved.

Furthermore, for the ExpAR(1)-ARCH(1) model the GA here gave good estimates only for the ϕ_1 parameter and for the parameters of the conditional variance model, while for the ExpAR(1)-GARCH$(1,1)$ model it gave better estimates for the c, γ, α_0, α_1 and β_1 parameters, but worse estimates for ϕ_1 and π_1.

Table 9 Estimation results for ExpAR(1)-GARCH(1,1) on the second round.

	ExpAR(1)-GARCH(1,1)					
	NM	**QN**	**AS**	**SQP**	**IP**	**GA**
c= 0.55	0.5740 (0.0101)	0.5740 (0.0101)	0.5740 (0.0101)	0.5740 (0.0101)	0.5740 (0.0101)	0.5656 (0.0108)
$\phi_1 = -0.30$	-0.3432 (0.0255)	-0.3432 (0.0255)	-0.3432 (0.0255)	-0.3432 (0.0255)	-0.3432 (0.0255)	-0.2130 (0.0167)
$\pi_1 = 0.50$	0.4659 (0.0384)	0.4659 (0.0384)	0.4659 (0.0384)	0.4659 (0.0384)	0.4659 (0.0384)	0.2653 (0.0478)
$\gamma = 1.20$	10.652 (0.1738)	10.652 (0.1738)	10.652 (0.1738)	10.653 (0.1738)	10.652 (0.1738)	11.389 (0.1594)
$\alpha_0 = 0.05$	0.0536 (0.0039)	0.0536 (0.0039)	0.0536 (0.0039)	0.0536 (0.0039)	0.0536 (0.0039)	0.0602 (0.0046)
$\alpha_1 = 0.40$	0.4493 (0.0219)	0.4493 (0.0219)	0.4493 (0.0219)	0.4493 (0.0219)	0.4493 (0.0219)	0.4322 (0.0210)
$\beta_1 = 0.50$	0.4713 (0.0194)	0.4713 (0.0194)	0.4713 (0.0194)	0.4713 (0.0194)	0.4713 (0.0194)	0.4680 (0.0200)
Log-likelihood	-2.9409e+03	-2.9409e+03	-2.9409e+03	-2.9409e+03	-2.9409e+03	-2.9735e+03
Iterations/ Generations	150	39	10	11	34	107

Table 10 Estimation results for AR(1)-ARCH(1) on the second round.

	AR(1)-ARCH(1)					
	NM	**QN**	**AS**	**SQP**	**IP**	**GA**
c= 0.55	0.5627 (0.0059)	0.5627 (0.0059)	0.5627 (0.0059)	0.5627 (0.0059)	0.5627 (0.0059)	0.5546 (0.0060)
$\phi_1 = -0.30$	-0.3285 (0.0121)	-0.3285 (0.0121)	-0.3285 (0.0121)	-0.3285 (0.0121)	-0.3285 (0.0121)	-0.3121 (0.0124)
$\alpha_0 = 0.05$	0.0505 (0.0013)	0.0505 (0.0013)	0.0505 (0.0013)	0.0505 (0.0013)	0.0505 (0.0013)	0.0530 (0.0014)
$\alpha_1 = 0.40$	0.4390 (0.0234)	0.4390 (0.0234)	0.4390 (0.0234)	0.4390 (0.0234)	0.4390 (0.0234)	0.4165 (0.0230)
Log-likelihood	2.1800e+03	2.1800e+03	2.1800e+03	2.1800e+03	2.1800e+03	2.1775e+03
Iterations/ Generations	48	19	7	8	13	164

Table 11 Estimation results for AR(1)-GARCH(1,1) on the second round.

	AR(1)-GARCH(1,1)					
	NM	**QN**	**AS**	**SQP**	**IP**	**GA**
c= 0.55	0.5660 (0.0077)	0.5660 (0.0077)	0.5660 (0.0077)	0.5660 (0.0077)	0.5660 (0.0077)	0.5606 (0.0077)
$\boldsymbol{\phi_1 = -0.30}$	-0.3359 (0.0121)	-0.3359 (0.0121)	-0.3359 (0.0121)	-0.3359 (0.0121)	-0.3359 (0.0121)	-0.3266 (0.0122)
$\boldsymbol{\alpha_0 = 0.05}$	0.0534 (0.0039)	0.0534 (0.0015)	0.0534 (0.0039)	0.0534 (0.0039)	0.0534 (0.0039)	0.0603 (0.0044)
$\boldsymbol{\alpha_1 = 0.40}$	0.4483 (0.0218)	0.4483 (0.0248)	0.4483 (0.0218)	0.4483 (0.0218)	0.4483 (0.0218)	0.4607 (0.0217)
$\boldsymbol{\beta_1 = 0.50}$	0.4726 (0.0193)	0.4727 (0.0193)	0.4727 (0.0193)	0.4727 (0.0193)	0.4727 (0.0193)	0.4399 (0.0200)
Log-likelihood	-2.9413e+03	-2.9413e+03	-2.9413e+03	-2.9413e+03	-2.9413e+03	-2.9435e+03
Iterations/ Generations	67	22	7	8	16	85

In the case of the AR(1)-ARCH(1) and AR(1)-GARCH$(1,1)$ models all the algorithms, apart from the GA, agreed again, this time giving exactly the same estimates. More specifically, for the AR(1)-ARCH(1) model there was a slight improvement of the estimates, but for the AR(1)-GARCH$(1,1)$ model the estimates were almost the same, compared to the ones obtained before. Moreover, the GA performed well in both cases, especially in the case of the AR(1)-ARCH(1) model. Overall, in both models the estimates were good, with AS and SQP methods requiring the minimum number of iterations.

Hence, when estimating the parameters of the AR(1)-ARCH(1) or AR(1)-GARCH$(1,1)$ model, the choice of initial values and the choice of the algorithm is rather not important. However, when estimating the parameters of the two new nonlinear models, when using arbitrary initial values, the algorithms that perform overall best are the QN and IP, while when using initial values that are indeed close to the true parameter values, most algorithms seem to agree.

5 Applications to real data

As an illustration of the practical potential of the two new models, in this section we consider an application using financial data, in order to examine whether these nonlinear models can explain real time series data. Moreover, we use once again the AR-ARCH and AR-GARCH models as benchmarks with which to compare the results and to see if the new models can give a better fit to the series than these well-known and commonly used models. What is more, we use the Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC) for model selection, in order to help us decide which model describes the data better. According to these, the preferred model is the one with the minimum criteria values.

Our data set consists of daily figures for the Dow Jones stock price index and, more specifically, the data used are the first differences of the daily high values for the period between 3 January 2000 and 13 May 2013. The moments of the first differences of this time series can be found in Table 12 (prices in USD), while the histogram can be seen in Figure 7. The autocorrelation and partial autocorrelation plots, which indicate the use of first order

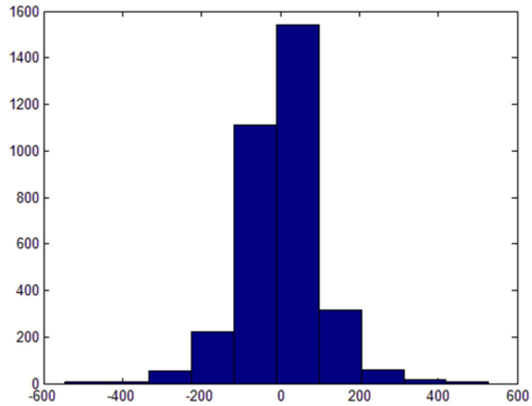

Figure 7 Histogram of the first differences of Dow Jones data.

Table 12 Moments of the first differences of Dow Jones data.

Sample size	3338
Mean	10.402
Standard deviation	966.342
Skewness	63.924
Kurtosis	-0.0545

models, can be found in Figures 8a and 8b respectively. In Table 13 there can be found the estimation results for the time series, including the maximum likelihood estimates, the standard errors, the log-likelihood value, the number of iterations required for the algorithm to converge to a solution and the AIC and BIC values for every model.

It can be seen from Table 13 that the estimates obtained for all the common parameters between the two models with ARCH errors (ExpAR-ARCH and AR-ARCH) are similar, and the estimates obtained for all the common parameters between the two models with GARCH errors (ExpAR-GARCH and AR-GARCH) are close to each other as well.

The lowest number of iterations was obtained for the AR(1)-GARCH$(1,1)$ model (55) and then for the ExpAR(1)-GARCH$(1,1)$ model (96), while the highest log-likelihood value was obtained for the ExpAR(1)-GARCH$(1,1)$ model (-36024) and then for the AR(1)-GARCH$(1,1)$ model (-36028). The lowest AIC value was given for the ExpAR(1)-GARCH$(1,1)$ model (72061.074) and then for the AR(1)-GARCH$(1,1)$ model (72065.032), while the lowest BIC value was obtained for the AR(1)-GARCH$(1,1)$ model (72095.69) and then for the ExpAR(1)-GARCH$(1,1)$ model (72103.866). Hence, according to the Akaike Information Criterion, the preferred model is the ExpAR(1)-GARCH$(1,1)$, while according to the Bayesian Information Criterion the preferred model is the AR(1)-GARCH$(1,1)$. However, we should bear in mind that the latter result could be due to the fact that the BIC penalises a higher number of parameters more than the AIC.

Table 14 shows the t-statistics for the estimates of the ExpAR(1)-GARCH(1,1) model. It is worth noting that the estimates of all the parameters for the preferred model, according to the AIC, are statistically significant at a 5% level.

Table 15 shows the values of the LR tests, according to which under the null hypothesis the true model is either the AR(1)-ARCH(1), or the ExpAR(1)-ARCH(1) or the AR(1)-GARCH(1,1) against the alternative that the true model is the ExpAR(1)-GARCH(1,1). According to the results, we can reject the null hypothesis that the true model is the AR(1)-

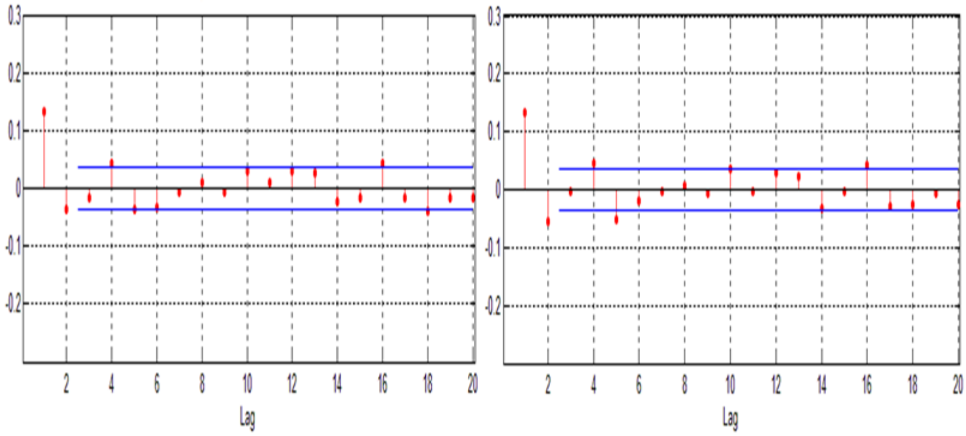

Figure 8 Autocorrelation (a) and Partial Autocorrelation (b) plot of the first differences of Dow Jones data.

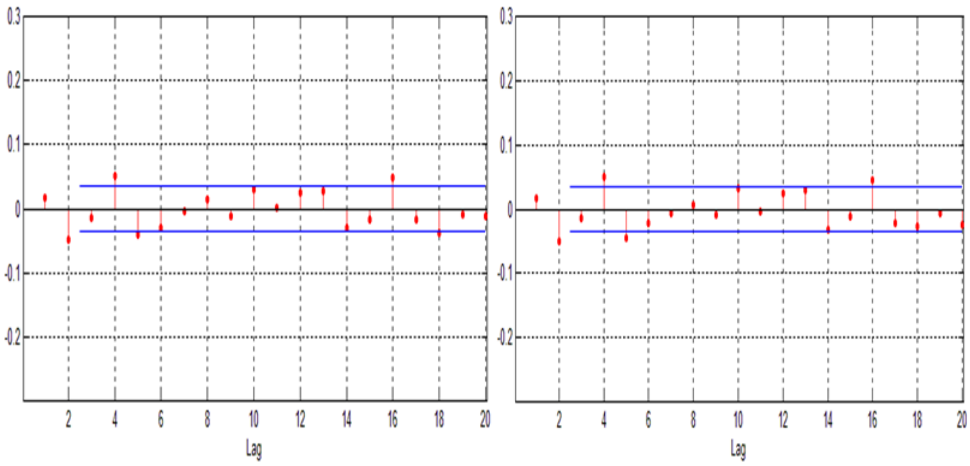

Figure 9 Autocorrelation (a) and Partial Autocorrelation (b) plot of the residuals of the ExpAR(1)-GARCH(1,1) model.

ARCH(1) or the ExpAR(1)-ARCH(1) at a 1% level. We can also reject the null hypothesis that the true model is the AR(1)-GARCH(1,1) model at a 5% level. Consequently, at a 5% level we can accept the alternative hypothesis that the true model is the ExpAR(1)-GARCH(1,1), which verifies the model selection according to the AIC.

In addition, Figure 9 shows the autocorrelation and partial autocorrelation plots of the residuals of the estimated ExpAR(1)-GARCH(1,1) model, which are useful tools to assess the presence of autocorrelation at individual lags. According to these plots, most sample autocorrelations and partial autocorrelations fall inside the 95% confidence bounds and change sign indicating the residuals to be random. Hence, the choice of the ExpAR(1)-GARCH(1,1) model for this time series seems to be appropriate.

6 Conclusions

In this paper two new nonlinear time series models, namely the ExpAR-ARCH and ExpAR-GARCH, have been suggested. Simulated series have been shown and several methods have

Table 13 Estimation results for the first differences of Dow Jones data.

	ExpAR(1)-ARCH(1)	**ExpAR(1)-GARCH(1,1)**	**AR(1)-ARCH(1)**	**AR(1)-GARCH(1,1)**
c	4.0217	4.8205	4.0059	4.6790
	(1.1359)	(0.9304)	(1.1487)	(0.9163)
ϕ_1	0.1072	0.1220	0.1072	0.1233
	(0.0142)	(0.0127)	(0.0142)	(0.0127)
π_1	210.2257	3.0392	-	-
	(37.4497)	(1.3967)	(-)	(-)
γ	9.3089	0.0231	-	-
	(4.0451)	(0.0095)	(-)	(-)
α_0	7.6998e+03	69.1401	7.7063e+03	69.5382
	(6.1816)	(9.0288)	(7.3042)	(5.4370)
α_1	0.1638	0.0580	0.1632	0.0580
	(0.0162)	(0.0046)	(0.0162)	(0.0045)
β_1	-	0.9344	-	0.9344
	(-)	(0.0046)	(-)	(0.0041)
Log-likelihood	-3.6747e+04	-3.6024e+04	-3.6748e+04	-3.6028e+04
Iterations	241	96	133	55
AIC	73505.116	72061.074	73503.804	72065.032
BIC	73541.795	72103.866	73528.33	72095.69

Table 14 t-statistics for the estimates of the ExpAR(1)-GARCH(1,1) model.

	c	ϕ_1	π_1	γ	α_0	α_1	β_1
Estimates	4.8205	0.1220	3.0392	0.0231	69.1401	0.0580	0.9344
	(0.9304)	(0.0127)	(1.3967)	(0.0095)	(9.0288)	(0.0046)	(0.0046)
t-statistic	5.1811	9.6063	2.1760	2.4316	7.6577	12.6087	203.1304

Table 15 Likelihood Ratio tests, (*): reject null hypothesis at $\alpha = 0.01$, (**): reject null hypothesis at $\alpha = 0.05$, but not at $\alpha = 0.01$.

	AR(1)-ARCH(1)	**ExpAR(1)-ARCH(1)**	**AR(1)-GARCH(1,1)**
ExpAR(1)-GARCH(1,1)	1448	1446	8
	(*)	(*)	(**)

been used and compared in the estimation, showing that the algorithms that performed better when using arbitrary initial values are the Quasi-Newton and the Interior Point, while most algorithms gave similar results when using initial guesses that are indeed close to the true parameter values. The results have been compared to the AR-ARCH and AR-GARCH models, in the case of which the choice of initial values or of algorithm is surprisingly not so important.

It has also been shown that the new models, and in fact low orders of them, can describe specific financial time series data. In addition, according to the Akaike Information Criterion, the ExpAR-GARCH model can even fit better than the well-known and widely used AR-ARCH and AR-GARCH models. Furthermore, some diagnostic tests have been used in order to verify our model selection.

All in all, the ExpAR-ARCH and ExpAR-GARCH models can be a useful tools in describing nonlinear behaviour in financial time series and have the potential of describing and fitting various real time series data. It should be noted that our suggested models can be extended by allowing for other forms of conditional variance. What is more, it is of interest to apply the ExpAR model with conditional heteroscedastic errors to other important financial and economic time series and to check the new models' forecasting performance as well. This will be the purpose of future investigation.

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