Wild ω-Categories for the Homotopy Hypothesis in Type Theory

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Abstract

In classical homotopy theory, the *homotopy hypothesis* asserts that the fundamental ω-groupoid construction induces an equivalence between topological spaces and weak ω-groupoids. In the light of Voevodsky's *univalent foundations* program, which puts forward an interpretation of types as topological spaces, we consider the question of transposing the homotopy hypothesis to type theory. Indeed such a transposition could stand as a new approach to specifying higher inductive types. Since the formalisation of general weak ω-groupoids in type theory is a difficult task, we only take a first step towards this goal, which consists in exploring a shortcut through *strict* ω-categories.

The first outcome is a satisfactory type-theoretic notion of strict ω-category, which has hsets of cells in all dimensions. For this notion, defining the 'fundamental strict ω-category' of a type seems out of reach. The second outcome is an 'incoherently strict' notion of type-theoretic ω-category, which admits arbitrary types of cells in all dimensions. These are the 'wild' ω-categories of the title. They allow the definition of a 'fundamental wild ω-category' map, which leads to our (partial) homotopy hypothesis for type theory (stating an adjunction, not an equivalence).

All of our results have been formalised in the Coq proof assistant. Our formalisation makes systematic use of the machinery of coinductive types.

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1 Introduction

Martin-Löf type theory offers an alternative to the classical set-theoretic approach to mathematics. The univalent foundations program [23] advocates understanding sets as a very special kind of types (so-called 0-*types*, or *hsets*). Conversely, types should be understood in set-theoretic terms as some kind of topological spaces, hsets corresponding to those whose connected components are contractible. And, indeed, types have been interpreted as ω-groupoids of various flavours, most notably simplicial [13] and globular [3, 16, 24]. Now, on the set-theoretic side, the *homotopy hypothesis* states that topological spaces and ω-groupoids are equivalent [22]. This work addresses the question of coining a type-theoretic counterpart of this homotopy hypothesis.

We here propose a preliminary and partial version which barely expresses that the 'fundamental ω-groupoid' functor has a kind of left adjoint (instead of being an equivalence). For our proposal, it is sufficient to choose an appropriate target category T of 'ω-groupoids', together with an appropriate 'fundamental ω-groupoid' functor $\pi \colon \mathsf{Type} \to \mathsf{T}$. Indeed, we may then express our partial homotopy hypothesis as follows:

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▶ **Hypothesis 1.** *There exist* $L\colon \mathsf{T} \to \mathsf{Type}$ *and* $\eta\colon \forall C, \mathsf{T}(C, \pi(L(C)))$ *such that for each* $C\colon \mathsf{T}$ *and* $T\colon \mathsf{Type}$*, the map*

$$\eta(C)^\star\colon \mathsf{T}(\pi(L(C)), \pi(T)) \to \mathsf{T}(C, \pi(T))$$

given by precomposition with $\eta(C)$ *is an equivalence.*

The idea of a 'fundamental ω-category' map goes back to van den Berg/Garner and Lumsdaine [3, 16]. There, the 'fundamental' ω-categories extracted from types are definitely weak, because expressions like $f \circ (g \circ h)$ and $(f \circ g) \circ h$ (for suitable cells f, g, h and higher categorical composition $\circ$) may differ definitionally. However, they are always equal propositionally, so when reasoning inside type theory as we do, the distinction becomes invisible. Based on this observation, we here explore the shortcut of a 'fundamental *strict* ω-category' map.

Thus, our first task is to transpose to type theory the classical notion of strict ω-category. This is the subject of Section 3 (after a brief recap on set-theoretic strict ω-categories in Section 2). There we face a crucial choice concerning the type of cells in all dimensions. If we take these types to be hsets, we get an honest notion of *strict* ω-category. While if we allow these types to be arbitrary, we get an 'incoherently strict' notion, which we call *wild*. In wild ω-categories, some coherence diagrams usually showing up in definitions of *weak* ω-categories make sense, but are not required to commute – even weakly, hence the name 'wild'.

Now the crucial point is that we are able to define a 'fundamental wild ω-category' map, while a strict one seems out of reach – a difficulty previously observed by Altenkirch et al. [2] in a similar context.

Let us mention another crucial choice faced when defining both notions. Indeed, we have to express equations as commuting diagrams of morphisms between globular types. Because such morphisms form a coinductive type, there are two standard choices [8] for their equality (identity types and bisimilarity). However, we weren't able to define our 'fundamental ω-category' map using either of them, so we work with a third, coarser one (Definition 8). (Of course, using identity types or bisimilarity yields other, perfectly sensible definitions.)

Altogether this yields in Section 5 a first version of our hypothesis with T the type for wild ω-categories. Our hypothesis essentially asserts that new types may be constructed from the homotopical information carried by any wild ω-category. This is clearly akin to the assumption of existence for *higher inductive types* [23], as well as to the Rezk completion for precategories in [1] and we briefly discuss the relationship.

In Section 6, we get back to ω-groupoids, as opposed to ω-categories. We briefly discuss problems and solutions concerning the definition of wild ω-groupoids and a perhaps more primitive formulation of our homotopy hypothesis involving them. We finally conclude and sketch further directions in Section 7.

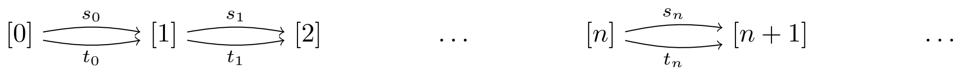
A note on the formalisation

This paper presents informally a mathematical development based not on set theory but on Martin-Löf type theory enriched with coinduction. All our definitions and statements have been formalised in the Coq proof assistant [21]. The formalisation is available as [9]. The code is composed of 2750 lines of definitions/theorems and less than 800 lines of proofs (as given by `coqwc`). Definitions and theorems constitute 75% of the formalisation, which is a lot. This is because `coqwc` only counts as proof what is coded using the tactic language, whereas most of our proofs are coded directly in Gallina. The reason for this is that the computational content of many of our proofs turned out to be crucial for their usability at later stages. This phenomenon has also been observed, e.g., of the Coq/HoTT library (`https://github.com/HoTT/HoTT`).

2 Set-theoretic strict ω-categories

Before presenting our definition of wild and strict ω-categories in type theory, we briefly sketch the traditional set-theoretic definition of strict ω-categories. We refer the reader to Lafont et al. [15] for further detail.

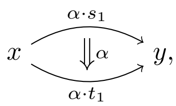
The base for strict ω-categories is the notion of globular sets, which are generally defined as presheaves over the so-called *globular category* Glob:



where for all n, $s_{n+1} \circ s_n = t_{n+1} \circ s_n$ and $s_{n+1} \circ t_n = t_{n+1} \circ t_n$.

▶ **Notation 1.** *We denote by s_n^p the composite $s_{p-1} \circ \ldots \circ s_n$, which returns the n-dimensional source of a p-cell. Similarly, we use t_n^p.*

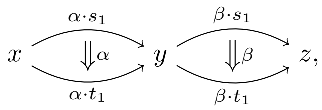
The idea is just that a globular set X has *objects*, the elements of $X[0]$, *1-cells* between them, and so on, each $(n+1)$-cell having *parallel* n-cells as source and target. E.g., a typical 2-cell looks like this:



where we use the standard shorthand notation, e.g., $\alpha \cdot s_1$ for $X(s_1)(\alpha)$. In the picture, $x = \alpha \cdot t_1 \cdot s_0 = \alpha \cdot s_1 \cdot s_0$ and $y = \alpha \cdot t_1 \cdot t_0 = \alpha \cdot s_1 \cdot t_0$.

We first concentrate on the data for composition and the so-called *interchange law*. A strict ω-category is a globular set X equipped (among other data), for all $n < p$, with a partial, binary *composition* operation on p-cells, defined on a pair (β, α) when the iterated source $\beta \cdot s_n^p$ of β matches the iterated target $\alpha \cdot t_n^p$ (recall Notation 1). The result is denoted by $\beta \circ_n \alpha$. Each such composition operation is required to be associative on the nose.

The source and target of such a composition are given by obvious globular intuition, generally not even spelled out. When $p = n+1$, composition is like categorical composition, i.e., we have $(\beta \circ_n \alpha) \cdot s_n = \alpha \cdot s_n$ and similarly $(\beta \circ_n \alpha) \cdot t_n = \beta \cdot t_n$. When $p > n+1$, composition is more like horizontal composition of 2-cells in a 2-category, as in



so we have

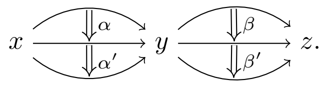
$$(\beta \circ_n \alpha) \cdot s_{p-1} = (\beta \cdot s_{p-1}) \circ_n (\alpha \cdot s_{p-1})$$

and similarly for t_{p-1}.

The crux of the definition of strict ω-category is the *interchange* law, which generalises the perhaps more well-known 2-categorical interchange law. It says that whenever $n < p < q$ and $\alpha, \alpha', \beta, \beta'$ are adequately composable q-cells, we have

$$(\beta' \circ_p \beta) \circ_n (\alpha' \circ_p \alpha) = (\beta' \circ_n \alpha') \circ_p (\beta \circ_n \alpha).$$

Graphically, for $n = 0$, $p = 1$, $q = 2$, both ways of composing the diagram below coincide:



Finally, we require identities $id_x : x \to x$ for all n-cells $x \in X[n]$. Let id_x^p denote the iteration $id_{id_{\cdots id_x}}$, p times (with $id_x^0 = x$). Identities should satisfy

$$id_x^p \circ_n \alpha = \alpha \qquad \beta \circ_n id_x^p = \beta \qquad id_{\beta \circ_n \alpha} = id_\beta \circ_n id_\alpha, \tag{1}$$

for all $x \in X[n]$, $p \geq 1$, α, and β such that $\alpha \cdot t_n^{n+p} = x$ and $\beta \cdot s_n^{n+p} = x$.

While this presentation is perfectly sensible when working in set theory, it is less convenient in type theory because many properties of the structure are imposed *by equations* rather than obtained *by construction*. For instance, compatibility between source and target maps in globular sets relies on equations. Using equations introduces a lot of non-definitional equalities that are very hard to deal with. On the contrary, we will see in the next section that our use of coinductive definitions allows us to define structures more computationally, thus avoiding the common pitfall of using equations.

3 Wild and strict ω-categories

In this section, we present our definition of wild and strict ω-categories. Our heavy use of coinduction is inspired by Cheng and Leinster's account [7] of Trimble's ω-categories, as well as by Lafont et al.'s presentation [15] of, e.g., weak equivalences. We start with globular types, a type-theoretic counterpart of globular sets. Essentially, a wild ω-category is a globular type equipped with identities and compositions in all dimensions, satisfying unitality and associativity axioms, and such that, at each dimension, composition preserves higher-dimensional identities and compositions. A strict ω-category is a wild ω-category whose underlying globular type is a globular hset, i.e., consists of hsets in all dimensions. The main goal of the present section is to give a precise meaning to the previous sketch of definition. Indeed, this is not completely straightforward, and we will explicitly discuss our design choices.

3.1 Globular types

The coinductive presentation of globular sets, which we'll here call globular types, is well-known and extremely simple:

▶ **Definition 1.** A *globular type* X consists of a type $|X|$, plus for all $x, y \in |X|$, a globular type $X[x, y]$.

Many of our definitions will, like the previous one, be coinductive or corecursive, and we will systematically omit to mention this feature. For instance, a morphism $X \twoheadrightarrow Y$ of globular types consists of

- a function $|f| : |X| \to |Y|$ between object types, and
- for all $x, x' \in |X|$, a morphism $f_{x,x'} : X[x, x'] \twoheadrightarrow Y[|f|(x), |f|(x')]$ of globular types.

Of course, morphisms of globular types compose.

We also need a definition of the cartesian product of globular types:

▶ **Definition 2.** The *product* $X \times Y$ of two globular types X and Y is defined by

- $|X \times Y| = |X| \times |Y|$ and
- for all $x, x' \in |X|$ and $y, y' \in |Y|$, $(X \times Y)[(x, y), (x', y')] = X[x, x'] \times Y[y, y']$.

3.2 Data for composition and identities

Let us now start our definition of wild ω-categories.

In the following definition of composition for globular types, we choose to tie together infinitely many elementary compositions into a morphism of globular types:

▶ **Definition 3.** To equip a globular type X with *ω-categorical composition* is to give:
- for all $x, y, z \in |X|$, a morphism $\mathsf{comp}_X(x, y, z) \colon X[x, y] \times X[y, z] \twoheadrightarrow X[x, z]$ of globular types,
- and, for all $x, y \in |X|$, ω-categorical composition on $X[x, y]$.

In the terminology of Section 2, the function $|\mathsf{comp}_X|$ between object types represents composition for $p = n + 1$, whereas compositions for $p > n + 1$ are packed in higher-dimensional components $(\mathsf{comp}_X)_{(f,g),(f',g')}$. Similarly:

▶ **Definition 4.** To equip a globular type X with *ω-categorical identities* is to give:
- for all $x \in |X|$, an element $id^X_x \in |X[x, x]|$, and
- for all $x, y \in |X|$, ω-categorical identities on $X[x, y]$.

▶ **Definition 5.** An *ω-precategory* is a globular type equipped with ω-categorical composition and identities.

▶ Remark. Our use of 'precategory' conflicts with [1], where it is used for 'not-necessarily-univalent' categories. However, it coincides with Cheng's [6].

Most notions defined so far on globular types lift to ω-precategories, namely the object type $|X|$, the hom-ω-precategory $X[x, y]$, and cartesian product $X \times Y$. We lift notations accordingly. However, lifting the notion of morphism of globular types requires some work, which we now do, yielding the notion of ω-functor.

3.3 Omega-functors

It is not straightforward to define what it means for a morphism $F \colon X \twoheadrightarrow Y$ of globular types between ω-precategories to preserve composition.

Let us treat the object level first, and consider any $x, x', x'' \in |X|$. The idea is that F preserves composition at x, x', x'' iff the diagram

$$\begin{array}{ccc} X[x,x'] \times X[x',x''] & \xrightarrow{\mathsf{comp}_X(x,x',x'')} & X[x,x''] \\ {\scriptstyle F_{x,x'} \times F_{x',x''}} \big\downarrow & & \big\downarrow {\scriptstyle F_{x,x''}} \\ Y[|F|x,|F|x'] \times Y[|F|x',|F|x''] & \xrightarrow[\mathsf{comp}_Y(|F|x,|F|x',|F|x'')]{} & Y[|F|x,|F|x''], \end{array} \tag{2}$$

commutes.

To give a meaning to this commutation, we need to choose a notion of equality of globular morphisms. Because globular morphisms form a coinductive type, identity types are expected to be too fine as a notion of equality between them [8]. The standard choice for this is bisimilarity:

▶ **Definition 6.** Two globular morphisms $F, G \colon X \twoheadrightarrow Y$ are *bisimilar* iff
- for all $x \in |X|$, $|f|(x) = |g|(x)$, and
- for all $x, x' \in |X|$, $f_{x,x'}$ and $(p_x)_*((p_{x'})_*(g_{x,x'}))$ are bisimilar,

where $p_*(-)$ denotes transport [23] along p and p_x is the given equality $|f|(x) = |g|(x)$.

Transport is necessary here in order to compare $f_{x,x'}: X[x,x'] \twoheadrightarrow Y[|f|(x),|f|(x')]$ and $g_{x,x'}: X[x,x'] \twoheadrightarrow Y[|g|(x),|g|(x')]$.

Bisimilarity would be a very reasonable notion of equality between globular morphisms. However, we were not able to construct the 'fundamental wild ω-category' of a type for the resulting notion of wild ω-category. So we here decide to use the following extensional and inductive definition (based on globular cells).

▶ **Definition 7.** We define the type of globular n-cells of an ω-precategory A, noted $cell_n\, A$, by induction on the natural number n as

- $cell_0\, A := |A|$, and
- $cell_{n+1}\, A := \sum_{a,a':|A|} cell_n(A[a,a'])$.

Given a globular morphism $F: A \twoheadrightarrow B$ and $c : cell_n\, A$, the globular cell Fc of B obtained by applying F to c can by defined inductively by

- $|F|c$ when $n = 0$, and
- $(|F|a, |F|a', F_{a,a'}c')$ when $c = (a, a', c')$.

▶ **Definition 8.** Two globular morphisms $F, G: A \twoheadrightarrow B$ are *extensionally equal* iff

$$\text{for all } n \in \mathbb{N} \text{ and } c \in cell_n\, A,\ Fc = Gc.$$

We may now settle the following (obviously proof-relevant) definition:

▶ **Definition 9.** A morphism $F: X \twoheadrightarrow Y$ of globular types between ω-precategories *preserves composition* iff

- for all $x, x', x'' \in |X|$, the square (2) commutes extensionally, and
- for all $x, x' \in |X|$, $F_{x,x'}$ preserves composition.

Preservation of composition will be used below in Definition 12 to express the interchange law. E.g., consider any ω-precategory X, and $x, y, z \in |X|$. Viewing $\mathsf{comp}_X(x, y, z)$ as a morphism of globular types between ω-precategories, saying that it preserves composition amounts to stating the interchange law of Section 2, specialised to $n = 0$. For instance, on objects, it means that, taking X in (2) to be $X[x,y] \times X[y,z]$, for all $f, f', f'' \in X[x,y]$, $g, g', g'' \in X[y,z]$, $a \in X[x,y][f,f']$, $a' \in X[x,y][f',f'']$, $b \in X[y,z][g,g']$, and $b' \in X[y,z][g',g'']$, we have $(b' \bullet b) \circ (a' \bullet a) = (b' \circ a') \bullet (b \circ a)$ (using some hopefully clear notation).

We may treat preservation of identities in a similar way:

▶ **Definition 10.** A morphism $F: X \twoheadrightarrow Y$ of globular types between ω-precategories *preserves identities* iff

- for all $x \in |X|$, $|F_{x,x}|(id^X_x) = id^Y_{|F|(x)}$, and
- for all $x, y \in |X|$, $F_{x,y}$ preserves identities.

This will again be used in Definition 12 to enforce the third law of (1). Indeed, for any ω-precategory X, and $x, y, z \in |X|$, saying that $\mathsf{comp}_X(x, y, z)$ preserves identities entails, e.g., that the identity on any pair $(f, g) \in |X[x,y] \times X[y,z]|$ should be mapped by

$$X[x,y][f,f] \times X[y,z][g,g] \xrightarrow{(\mathsf{comp}_X(x,y,z))_{(f,g),(f,g)}} X[x,z][g \circ f, g \circ f]$$

to the identity on $g \circ f$ (abbreviating $|\mathsf{comp}_X(x,y,z)|(f,g)$ to $g \circ f$).

▶ **Definition 11.** A morphism of globular types between ω-precategories is an *ω-functor* iff it preserves composition and identities.

3.4 Wild ω-categories

It is now routine to extend the previous techniques to define *associativity* as extensional commutation in all dimensions of all diagrams

$$\begin{array}{ccc}
X[x,y] \times X[y,z] \times X[z,t] & \xrightarrow{\mathsf{comp}_X(x,y,z) \times X[z,t]} & X[x,z] \times X[z,t] \\
{\scriptstyle X[x,y] \times \mathsf{comp}_X(y,z,t)} \downarrow & & \downarrow {\scriptstyle \mathsf{comp}_X(x,z,t)} \\
X[x,y] \times X[y,t] & \xrightarrow[\mathsf{comp}_X(x,y,t)]{} & X[x,t]
\end{array}$$

and *unitality* as extensional commutation in all dimensions of all diagrams

$$\begin{array}{ccc}
X[x,y] & \xrightarrow{\langle \ulcorner id_x \urcorner \circ !, X[x,y] \rangle} & X[x,x] \times X[x,y] \\
{\scriptstyle \langle X[x,y], \ulcorner id_y \urcorner \circ ! \rangle} \downarrow & \searrow\!\!= & \downarrow {\scriptstyle \mathsf{comp}_X(x,x,y)} \\
X[x,y] \times X[y,y] & \xrightarrow[\mathsf{comp}_X(x,y,y)]{} & X[x,y],
\end{array}$$

where $! \colon A \twoheadrightarrow 1$ denotes the unique morphism to the terminal globular type with 1 at all stages (for all A), and $\ulcorner id_x \urcorner \colon 1 \twoheadrightarrow X[x,x]$ maps the unique element of $|1|$ to id_x, the unique endo 1-cell over it to id_{id_x}, and so on.

We may at last define the type ω-wCat of wild ω-categories:

▶ **Definition 12.** A *wild ω-category* is an ω-precategory, satisfying associativity and unitality, whose compositions are ω-functors in all dimensions. Morphisms ω-$\mathsf{wCat}(C, D)$ between wild ω-categories C and D are simply ω-functors between the underlying ω-precategories.

The complete formal definition of wild ω-categories is given in the file `omega_categories.v` [9].

3.5 Strict ω-categories

The definition of wild ω-categories is not satisfactory as a type-theoretic account of strict ω-categories. As a matter of fact, wild ω-categories are not even weak ω-categories. Indeed, they appear to lack some coherence conditions. For instance, for any wild ω-category X, $f \in |X[x,y]|$ and $g \in |X[y,z]|$, there are two proofs of $g \circ (id_y \circ f) = g \circ f$ (the less trivial one goes to $(g \circ id_y) \circ f$ and then simplifies). These proofs induce by transport two 2-cells, say l and r. In weak higher categories, one imposes that l and r are related by a 'coherence' 3-cell. This is not the case in our wild ω-categories, which may thus be viewed as 'incoherently' weak ω-categories.

One perhaps reassuring perspective is that wild ω-categories can be restricted to ω-categories where all higher coherences are trivially satisfied. This is the case when the involved types are all hsets, which leads to:

▶ **Definition 13.** A *strict* ω-category is a wild ω-category X such that $|X|$ is an hset and for all $x, y \in |X|$, $X[x,y]$ is a strict ω-category. We call ω-sCat the type of strict ω-categories.

Strict ω-categories are intuitively close to set-theoretic strict ω-categories, and, as suggested by a referee, we expect their interpretation in standard models such as the simplicial model [13] to coincide with set-theoretic ω-categories. However, as we have seen, our definition relies on extensional equality (Definition 8), where identity types or bisimilarity could have been used. We thus in fact have three notions of strict ω-categories, and it is not entirely clear that all three are interpreted in the simplicial model as set-theoretic strict ω-categories.

As previously observed in a similar context [2], it seems impossible to define any satisfactory 'fundamental strict ω-category'. That is why we devote the rest of the paper to the definition of the 'fundamental wild ω-category' and use it to propose (partial) homotopy hypotheses.

4 Fundamental wild ω-category

We have defined wild ω-categories, and now turn to the definition of the 'fundamental wild ω-category' map $\pi\colon \mathsf{Type} \to \omega\text{-}\mathsf{wCat}$. This is essentially an internal variant of van den Berg and Garner's [3] and Lumsdaine's [16] constructions, with wild ω-categories instead of weak ω-categories. For any type T, we first easily define the globular type underlying $\pi(T)$. We then explain the definition of composition, which is a good example of how we had to generalise some of our statements in order to be able to tie the coinduction loop. We refer the reader to the formalisation for complete definitions and proofs. The part on composition is technical and may safely be skipped: the sequel only makes use of Theorem 16.

We start by defining the globular type underlying the fundamental wild ω-category.

▶ **Definition 14.** For any type T, the underlying globular type of $\pi(T)$ (still denoted by $\pi(T)$ by abuse of notation) has T itself as its type of objects, and $(\pi(T))[x, y] = \pi(x = y)$ for all $x, y \in T$.

We now turn to defining composition on $\pi(T)$. The definition of composition seems to introduce a lot of choices. Indeed, let us start by fixing $x, y, z \in T$ and look at what $\pi(x = y) \times \pi(y = z) \twoheadrightarrow \pi(x = z)$ is on objects (i.e., on 1-cells in $\pi(T)$). The obvious choice is concatenation of equality proofs, which we denote by $(a\colon x = y), (b\colon y = z) \mapsto (a \centerdot b)$ [23]. But actually, here we need to choose between two different definitions of concatenation, depending on whether a or b is eliminated first. Fortunately, it is well known that both definitions are equal, so the choice does not really matter.

Actually, this situation occurs at every dimension: the definition of composition is not unique, but all potential candidates are equal. This claim is justified by the work of Lumsdaine [16], where he constructs the operad $P_{ML^{Id}}$ of all definable composition laws over a (generic) type and shows that this operad is contractible. Contractibility means that all possible choices of composition are equal. The difficulty is to show that our particular choice of composition gives rise to a wild ω-category. For lack of space, we will only sketch the definition of our compositions, referring to the formalisation [9] for details.

Let us start with the definition of $\pi(x = y) \times \pi(y = z) \twoheadrightarrow \pi(x = z)$ in low dimensions, for $x, y, z \in T$. On objects, we have seen that the obvious choice is concatenation. On 1-cells, consider $a, a'\colon x = y$, $b, b'\colon y = z$, together with $e\colon a = a'$ and $f\colon b = b'$. How to compose e and f into a proof of $a \centerdot b = a' \centerdot b'$? We consider, for all types A, B, C and map $\varphi\colon A \to B \to C$, the obvious function $\mathsf{ap2}_{A,B,C,\varphi}$ of type

$$\forall a, a' \in A, b, b' \in B, e \in (a = a'), f \in (b = b'), \varphi\ a\ b = \varphi\ a'\ b'.$$

Applying this with $A = (x = y)$, $B = (y = z)$, $C = (x = z)$, and $\varphi = \centerdot$, we indeed get $\mathsf{ap2}\ a\ a'\ b\ b'\ e\ f$ of type $a \centerdot b = a' \centerdot b'$ (omitting the subscript of $\mathsf{ap2}$ for readability). To now deal with 2-cells, considering $e'\colon a = a'$, $f'\colon b = b'$, $u\colon e = e'$, and $v\colon f = f'$, we again apply $\mathsf{ap2}$, with $A = (e = e')$, $B = (f = f')$, $C = (\varphi'\ e\ f = \varphi'\ e'\ f')$, with $\varphi'\ e\ f = \mathsf{ap2}\ a\ a'\ b\ b'\ e\ f$ for all e, f. In the next dimension, we'll need a different φ'' with one more layer of $\mathsf{ap2}$.

It would be obvious how to formalise this process coinductively if it weren't for the first level, where $\centerdot$ is used. The trick is thus to abstract over this. Here, things become slightly more verbose, and we apologise to the reader: for all types A, B, C, functions $\varphi\colon A \to B \to C$, and elements $a, a' \in A$ and $b, b' \in B$, we coinductively define a morphism of globular types

$$\mathsf{comp2}_{A,B,C,\varphi}(a, a', b, b')\colon \pi(a = a') \times \pi(b = b') \twoheadrightarrow \pi(\varphi\ a\ b = \varphi\ a'\ b')$$

(the '2' refers to $\mathsf{ap2}$) by

- mapping $e\colon a = a'$ and $f\colon b = b'$ to $\mathsf{ap2}\ \varphi\ e\ f\colon \varphi\ a\ b = \varphi\ a'\ b'$,
- and then defining $(\mathsf{comp2}_{A,B,C,\varphi}(a, a', b, b'))_{(e,f),(e',f')}$ to be

$$\mathsf{comp2}_{A[a,a'],B[b,b'],C[\varphi\ a\ b,\varphi\ a'\ b'],\mathsf{ap2}\ \varphi}(e, e', f, f'). \tag{3}$$

This is well-defined, since $(\mathsf{comp2}_{A,B,C,\varphi}(a, a', b, b'))_{(e,f),(e',f')}$ should have type

$$(\pi(a = a') \times \pi(b = b'))[(e, f), (e', f')] \twoheadrightarrow \pi(\varphi\ a\ b = \varphi\ a'\ b')[\mathsf{ap2}\ \varphi\ e\ f, \mathsf{ap2}\ \varphi\ e'\ f'],$$

i.e.,

$$\pi(a = a')[e, e'] \times \pi(b = b')[f, f'] \twoheadrightarrow \pi(\varphi\ a\ b = \varphi\ a'\ b')[\mathsf{ap2}\ \varphi\ e\ f, \mathsf{ap2}\ \varphi\ e'\ f'],$$

or equivalently

$$\pi(e = e') \times \pi(f = f') \twoheadrightarrow \pi(\mathsf{ap2}\ \varphi\ e\ f = \mathsf{ap2}\ \varphi\ e'\ f'),$$

which is indeed the type of (3). It is now routine to define, for all types A and elements $a, a', a'' \in A$,

$$\mathsf{hcomp}_A(a, a', a'')\colon \pi(a = a') \times \pi(a' = a'') \twoheadrightarrow \pi(a = a'')$$

by

- mapping $e\colon a = a'$ and $f\colon a' = a''$ to $e \cdot f$, with
- $\mathsf{hcomp}_A(a, a', a'')_{(e,f),(e',f')} = \mathsf{comp2}_{(a=a'),(a'=a''),(a=a''),(\lambda e.\lambda f.e \cdot f)}(e, e', f, f')$.

This is again well-defined, because the latter has type

$$\pi(e = e') \times \pi(f = f') \twoheadrightarrow \pi(e \cdot f = e' \cdot f'),$$

i.e.,

$$\pi(a = a')[e, e'] \times \pi(a' = a'')[f, f'] \twoheadrightarrow \pi(a = a'')[e \cdot f, e' \cdot f'],$$

as expected.

▶ **Definition 15.** This yields composition structure on $\pi(T)$, for any T:

- for all $x, y, z \in T$, we let $\mathsf{comp}_{\pi(T)}(x, y, z) = \mathsf{hcomp}_T(x, y, z)$;
- for all $x, y \in T$, we get composition structure on $\pi(x = y)$ by coinduction hypothesis.

One may similarly define identity structure and show:

▶ **Theorem 16.** *This ω-precategory structure makes $\pi(T)$ into a wild ω-category.*

Proof. Let us say a few words about the proof. It resorts to a high level of generalisation to tie the coinduction loop for each law of ω-categories. In particular, a more explicit coinductive definition of the interchange law is developed and proved equivalent to the compact version that composition preserves composition. This explicit interchange law, plus a proof that composition in all dimensions sends pairs of proofs by reflexivity to some proof by reflexivity, enables us to prove the interchange law by coinduction, using elimination of identity types, a.k.a. path induction. The other laws are dealt with similarly. The complete proof is given in the file `type_to_omega_cat.v` [9]. ◀

5 Partial Homotopy Hypothesis

The classical homotopy hypothesis states an equivalence between spaces and ω-groupoids. It can be formulated either at the level of so-called homotopy categories [12], or at the level of model categories [11] or even at the level of $(\infty, 1)$-categories [22]. On the type-theoretic side, we propose our first partial homotopy hypothesis:

▶ **Hypothesis 1.** *There exist* $L\colon \omega\text{-}\mathsf{wCat} \to \mathsf{Type}$ *and* $\eta\colon \forall C, \omega\text{-}\mathsf{wCat}(C, \pi(L(C)))$ *such that for each wild* ω*-category* C *and type* T*, the map*

$$\eta(C)^{\star}\colon \omega\text{-}\mathsf{wCat}(\pi(L(C)), \pi(T)) \to \omega\text{-}\mathsf{wCat}(C, \pi(T))$$

given by precomposition with $\eta(C)$ *is an equivalence.*

We conjecture that Hypothesis 1 is consistent, and more precisely that it holds in the groupoid model [10]. Indeed, in this model, small types are small discrete groupoids and (small) wild ω-categories are the set-theoretic, strict ω-categories of Section 2. The fundamental wild ω-category $\pi(T)$ of any small discrete groupoid T is a discrete globular set, equipped with the only possible additionnal structure. Thus, morphisms from any strict ω-category C into $\pi(T)$ are just maps from $C[0]$ to T compatible with (the equivalence relation induced by) $C[1]$. Hence we can define $L(C)$ to be the quotient $C[0]/C[1]$, $\eta(C)$ being induced by the quotient map.

This indicates in particular that Hypothesis 1 is consistent with the Univalence Axiom. But it also shows that the asserted adjunction may be far from an equivalence. An easy way of strengthening our hypothesis is to require $\eta(C)$ to be a weak equivalence, in the sense of [15]:

▶ **Definition 17.** An ω-functor $F\colon C \twoheadrightarrow D$ is a *weak equivalence* iff

- for all $d \in |D|$, there exists $c \in |C|$ such that $|D[|F|(c), d]|$ is inhabited, and
- for all $c, c' \in |C|$, $F_{c,c'}$ is a weak equivalence.

Another possibility is to state a proper adjunction between L and π and ask both its unit and counit to be weak equivalences in the appropriate sense. We haven't yet investigated such strengthened hypotheses.

For now, let us modestly check how our hypothesis implies the existence of a type corresponding to the standard type-theoretic circle. For this purpose we introduce the wild ω-category S^1 as follows: it has a single object $\star$, and $S^1[\star, \star]$ is the discrete wild ω-category on $\mathbb{N}$-many objects with composition given by addition. Of course, we could have worked with $\mathbb{Z}$ instead. Assuming our hypothesis, we prove the expected, non-dependent induction principle for $L(S^1)$, together with (propositional versions of) computational rules.

▶ **Theorem 18.** *There exists a term inhabiting the non-dependent recursion principle of the circle*

$$L(S^1)_{\mathsf{rec}}\colon \forall T \in \mathsf{Type}, b \in T, l \in (b = b), L(S^1) \to T,$$

satisfying (propositionally) the expected computational laws:

$$L(S^1)_{\beta,\star}\colon \forall T, b, l, L(S^1)_{\mathsf{rec}}\ T\ b\ l\ (\eta\ \star) = b,$$
$$L(S^1)_{\beta,1}\colon \forall T, b, l, p_*((\mathsf{ap}\ (L(S^1)_{\mathsf{rec}}\ T\ b\ l)\ (\eta_{\star,\star}\ 1))) = l,$$

where $p_*(-)$ *denotes transport [23] along* $p = L(S^1)_{\beta,\star}\ T\ b\ l$*.*

Proof. Any ω-functor from S^1 is determined by the images of $\star$ and $1 \in S^1[\star,\star]$. Using the inverse of the equivalence given by Hypothesis 1, this induces an essentially unique ω-functor $\pi(L(S^1)) \twoheadrightarrow \pi(T)$, from which we extract the underlying map $L(S^1) \to T$. Using the retraction part of the equivalence given by Hypothesis 1, we deduce the computational laws. The complete proof[1] is given in the file `homotopy_hypothesis.v` [9]. ◀

This suggests that Hypothesis 1 may imply existence of certain *higher inductive types* [23]. The basic idea of higher inductive types is to generate not only the elements of an inductive type, but also equality proofs between them, and so on. Lumsdaine and Shulman [17] propose a semantics for them (in particular) in $(\infty,1)$-toposes, using so-called strictly Reedy-functorial path objects. Closer to implementation, Sojakova [19] proposes an operational definition of higher inductive types as so-called homotopy-initial algebras. Both accounts fix a particular syntax for higher inductive types.

If, along the lines of Theorem 18, we could show that the partial homotopy hypothesis entails adequate induction principles, it could be understood as *specifying* higher inductive types, in a syntax-independent way. Of course, this would only be a definition from the internal point of view, i.e., the corresponding computational behaviour would not be accounted for.

So, given any candidate syntax for (or combinatorial description of) higher inductive types, this opens the option of describing 'the corresponding' wild ω-category, from which Hypothesis 1 would yield the desired type and reasoning principles.

▶ Remark. It is well-known that set-theoretic strict ω-categories cannot represent all homotopy types. E.g., they do not model the homotopy type of the 2-sphere [18]. Depending on the ambient type theory, wild ω-categories may be much more expressive than strict ω-categories. Nevertheless, we suspect that even in such cases, they may not adequately represent all types.

Let us consider a few example syntaxes.

To start with, Ahrens et al.'s categories [1] straightforwardly embed into wild ω-categories, and are enough to specify (a groupoidal version of) S^1 as above (but not S^2). Their categories may express 'non-freeness' properties of composition. E.g., we may consider the category obtained by quotienting S^1 under $1+1=1$, i.e., replace $\mathbb{N}$ by booleans and addition with sup. Or similarly quotient under $1+1=0$, i.e., work with $\mathbb{Z}/2\mathbb{Z}$. Also, it seems plausible to extend their Yoneda-based *Rezk completion* procedure—which (in their terms) constructs a (univalent) category from a precategory—to a proof of our hypothesis for categories. Please note, however, that they work in homotopy type theory, and their construction uses univalence (because the category of sets needs to be univalent) and higher inductive types (through propositional truncation).

A different 'syntax' is offered by globular types themselves, and we may hope for a type-theoretic analogue of the standard adjunction computing the free strict ω-category associated to any globular set. In contrast this syntax does not allow to express non-freeness properties. E.g., we may express S^1, but none of the quotients of S^1 evoked above. Also, we cannot express the higher inductive type for S^2 with one base point b and an equality proof on refl_b, because we cannot talk about identities. However, we can perfectly consider the globular set with two base points 0 and 1, two 1-cells $s, t \colon 0 \to 1$, and two 2-cells $s \to t$.

[1] The proof that the constructed ω-functor from S^1 actually preserves composition remains incomplete at the time of writing.

Finally, the most expressive such syntax would probably be offered by *computads* [20], a.k.a. *polygraphs* [5, 15]. A computad is essentially a graph, together with a set of 2-cells between parallel paths, together with a set of 3-cells between parallel paths of 2-cells, and so on. In particular, one may talk about identities and composition in all dimensions. And indeed, any strict ω-category is weakly equivalent to some strict ω-category which is free on a computad [15].

6 Towards ω-groupoids

In the fundamental wild ω-category $\pi(T)$ associated to any type T, all cells are invertible. We will see below that defining general ω-groupoids is not straightforward. Nevertheless, following Ahrens et al. [1], we have an economical, yet a bit restrictive definition:

▶ **Definition 19.** A wild ω-category X is a *univalent ω-groupoid* when for all $x, y \in |X|$,
- the map $(x = y) \to |X[x, y]|$ (induced by transporting id_x) is an equivalence (of types) [23],
- and $X[x, y]$ is a univalent ω-groupoid.

This univalence automatically entails existence of inverses in a very strong sense. In particular, for any wild ω-category X and $x, y, z \in |X|$, the diagram

$$\begin{array}{ccc} (x = y) \times (y = z) & \longrightarrow & (x = z) \\ \downarrow & & \downarrow \\ |X[x, y]| \times |X[y, z]| & \longrightarrow & |X[x, z]| \end{array}$$

(extensionally) commutes in Type, so that, if X is univalent, inverses in the ω-groupoidal sense have to be inverses in the identity type sense. The above definition has to be about *ω-groupoids*, as it implies that all cells are invertible. We prove:

▶ **Proposition 1.** *The ω-category $\pi(T)$ is a univalent ω-groupoid, for all types T.*

▶ **Proposition 2.** *For all univalent ω-groupoids G, there is a type T and an extensional equivalence $\pi(T) \simeq G$.*

By extensional equivalence we here mean an equivalence $e\colon |\pi(T)| \simeq |G|$ of types, such that for all $x, y \in T$ the map $\pi(x = y) \to |G[e\ x, e\ y]|$ induced by e is an extensional equivalence.

Proof. Take $T = |G|$. ◀

We conclude this section by considering perhaps more primitive versions of our hypothesis using some notion of *ω-groupoid* rather than ω-categories. Of course, univalent ω-groupoids are not interesting for this purpose, so we seek a definition of what it means for some (possibly higher) cell in a wild ω-category to be invertible. We first review possible notions of ω-groupoids and transpose a result of Cheng [6] showing that two of them are equivalent. We then get back to ω-groupoidal statements of our hypothesis.

Brown et al. [4] use *strict inverses*, i.e., for them an ω-groupoid is an ω-category in which for every n-cell $f\colon x \to y$ there is an n-cell $g\colon y \to x$ such that $g \circ_{n-1} f = id_x$ and $f \circ_{n-1} g = id_y$ (for $n > 0$). Others [20, 12, 18] consider *weak inverses*, in several apparently different ways. A recent preprint [14] shows that two such definitions coincide, namely those of Street [20], and Kapranov and Voevodsky [12]. The latter had previously been shown by Simpson [18] to be equivalent to an apparently stronger definition. Finally, Cheng [6] shows that these definitions are further equivalent to a seemingly weaker definition *when required of the whole ω-category.* A bit more precisely, Cheng defines the notion of a *dual* to an n-cell in an ω-category, which is in general weaker than that of a *weak inverse* in the sense of [20, 12, 14]. But she shows that for a given ω-category, having all duals is equivalent to having all weak inverses. We now show how to recover this equivalence in our setting.

It is easy to define what it means for a wild ω-category to have all duals.

▶ **Definition 20.** In a wild ω-category X, for all $x, y \in |X|$ and $f \in |X[x,y]|$, a *dual* for f is a 1-cell $g \in |X[y,x]|$ such that there exist 2-cells inhabiting

$$|X[y,y][f \circ g, id_y]| \qquad \text{and} \qquad |X[x,x][g \circ f, id_x]|.$$

We then say that X *has all duals* iff for all $x, y \in |X|$ and $f \in |X[x,y]|$, f has a dual, and for all $x, y \in |X|$, $X[x,y]$ has all duals.

The notion of weak inverse is a bit harder. We follow the coinductive presentation of [15] (a definition considered 'unsound' by Cheng, but which Coq readily accepts!).

▶ **Definition 21.** We pose the following mutually coinductive definitions:
- two objects x and y of a wild ω-category X are *equivalent*, notation $x \sim y$, iff there exists a reversible 1-cell $f \in |X[x,y]|$;
- a 1-cell $f \in |X[x,y]|$ is *reversible* when it has a weak inverse;
- a *weak inverse* for a 1-cell $f \in |X[x,y]|$ is a 1-cell $g \in |X[y,x]|$ such that $g \circ f \sim id_x$ and $f \circ g \sim id_y$.

▶ **Definition 22.** A wild ω-category *has all weak inverses* when for all $x, y \in |X|$, any $f \in |X[x,y]|$ is reversible, and $X[x,y]$ has all weak inverses.

▶ **Proposition 3.** *Any wild ω-category has all duals iff it has all weak inverses.*

This allows us to comfortably state a first definition of ω-groupoid:

▶ **Definition 23.** A *wild ω-groupoid* is a wild ω-category with all duals.

One possible problem with this definition is that, ideally, being an ω-groupoid should be a mere property of ω-categories, i.e., for all X, the type 'X has all weak inverses' should be a mere proposition [23]. The very definition of univalence [23] suggests that having all duals might not suffice and that one may have to resort to some sensible notion of ω-adjunction in this context, but there does not seem to be any commonly accepted such notion in the literature, and we leave the question open.

There is one possible solution if we wish to delve into HoTT — except for this paragraph, our whole development remains within Martin-Löf type theory with coinduction. Namely, we could *truncate* the naive definition above. Indeed, according to the naive definition above, a wild ω-groupoid is a wild ω-category, equipped with a choice of duals for all cells in all dimensions. So we may define *brutal ω-groupoids* to denote wild ω-categories for which there exists, in the mere propositional sense, duals for all cells in all dimensions. It would however be preferable to have a mere property without resorting to truncation, as is done in the standard treatment of the univalence axiom.

We conclude this section by stating a groupoidal variant of our hypothesis. Since we have several candidate definitions for general ω-groupoids, we state our hypothesis taking this as a parameter. So let $\omega\text{-}\mathsf{Gpd}$ denote some type of ω-groupoids.

▶ **Hypothesis 2.** *There exist $L\colon \omega\text{-}\mathsf{Gpd} \to \mathsf{Type}$ and $\eta : \forall G \in \omega\text{-}\mathsf{Gpd}, \omega\text{-}\mathsf{wCat}(G, \pi(L(G)))$ such that for each ω-groupoid G and type T, the following map is an equivalence:*

$$\eta(G)^\star \colon \omega\text{-}\mathsf{wCat}(\pi(L(G)), \pi(T)) \to \omega\text{-}\mathsf{wCat}(G, \pi(T)).$$

▶ Remark. We here use ω-functors as morphisms of ω-groupoids. An alternative would be to require morphisms of ω-groupoids to preserve weak inverses. If we could refine the type of inverses to a given morphism into a mere proposition, then both possibilities would coincide.

7 Conclusion and Future Work

We have defined wild and strict ω-categories, as well as wild and univalent ω-groupoids. We have constructed a 'fundamental ω-groupoid' map from types into all of these notions but strict ω-categories. We have stated a few variants of our partial homotopy hypothesis which postulates some correspondence between types and the given ω-dimensional structure. We have loosely related such hypotheses to the existence of higher inductive types and to the Rezk completion for one-dimensional categories.

The main remaining issue in our development is that we have used a rather coarse notion of equality between globular morphisms (Definition 8) in order to be able to define our 'fundamental wild ω-category' map. We wonder whether this could be done using a more standard notion like bisimilarity.

As explained in the introduction, this paper grew out of an attempt to use strict, as opposed to weak, ω-groupoids in the statement of a type-theoretic homotopy hypothesis. Beyond the issues raised by the definition of type-theoretic ω-groupoids (Section 6), it now seems likely that the lack of coherence of wild ω-categories disqualifies them for the full homotopy hypothesis.

Our main future challenge is thus clearly to propose a type-theoretic notion of weak ω-category allowing the definition of a 'fundamental weak ω-category' map. Assuming that we succeed in defining weak ω-categories, stating a full version of the homotopy hypothesis would first require us to define weak ω-groupoids properly. Thus, we expect the discussion of Section 6 about weak inverses to also be relevant in the weak case. Namely, we will need to investigate whether 'having all duals' is a mere property of weak ω-categories, and, if not, how to refine it into one. Finally, a full version of the homotopy hypothesis would essentially assert the existence of an infinite-dimensional generalisation of the Rezk completion. We wonder whether the construction of [1] could be adapted to this setting.

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