

Counting Euler Tours in Undirected Bounded Treewidth Graphs

Nikhil Balaji^{1,3}, Samir Datta¹, and Venkatesh Ganesan²

- 1 Chennai Mathematical Institute, Chennai, India
`{nikhil,sdatta}@cmi.ac.in`
- 2 Birla Institute of Technology and Science – Pilani, India
`venkatesh920@gmail.com`
- 3 Indian Institute of Technology, Bombay, India
`nbalaji@cse.iitb.ac.in`

Abstract

We show that counting Euler tours in undirected bounded tree-width graphs is tractable even in parallel - by proving a $\#\text{SAC}^1 \subseteq \text{NC}^2 \subseteq \text{P}$ upper bound. This is in stark contrast to $\#\text{P}$ -completeness of the same problem in general graphs.

Our main technical contribution is to show how (an instance of) dynamic programming on bounded *clique-width* graphs can be performed efficiently in parallel. Thus we show that the sequential result of Espelage, Gurski and Wanke [16] for efficiently computing Hamiltonian paths in bounded clique-width graphs can be adapted in the parallel setting to count the number of Hamiltonian paths which in turn is a tool for counting the number of Euler tours in bounded tree-width graphs. Our technique also yields parallel algorithms for counting longest paths and bipartite perfect matchings in bounded-clique width graphs.

While establishing that counting Euler tours in bounded tree-width graphs can be computed by non-uniform monotone arithmetic circuits of polynomial degree (which characterize $\#\text{SAC}^1$) is relatively easy, establishing a uniform $\#\text{SAC}^1$ bound needs a careful use of polynomial interpolation.

1998 ACM Subject Classification F.1.1 Models of Computation, F.1.3 Complexity Measures and Classes, G.2.2 Graph Theory

Keywords and phrases Euler Tours, Bounded Treewidth, Bounded clique-width, Hamiltonian cycles, Parallel algorithms

Digital Object Identifier 10.4230/LIPIcs.FSTTCS.2015.246

1 Introduction

An Euler tour of a graph is a closed walk on the graph that traverses every edge in the graph exactly once. Given a graph, deciding if there is an Euler tour of the graph is quite simple. Indeed, the famous Königsberg bridge problem that founded graph theory is a question about the existence of an Euler tour using each of these bridges exactly once. Euler settled this question in the negative and in the process gave a necessary and sufficient condition for a graph to be *Eulerian* (A connected graph is Eulerian if and only if all the vertices are of even degree). This gives a simple algorithm to check if a graph is Eulerian.

An equally natural question is to ask for the number of distinct Euler tours in a graph. For the case of directed graphs, the BEST theorem due to De Bruijn, Ehrenfest, Smith and Tutte gives an exact formula that gives the number of Euler tours in a directed graph [1, 25] which yields a polynomial time algorithm via a determinant computation. For undirected graphs,



© Nikhil Balaji, Samir Datta, and Venkatesh Ganesan;
licensed under Creative Commons License CC-BY

35th IARCS Annual Conf. Foundations of Software Technology and Theoretical Computer Science (FSTTCS 2015).
Editors: Prahladh Harsha and G. Ramalingam; pp. 246–260



Leibniz International Proceedings in Informatics

LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

no such closed form expression is known and the computational problem is $\#P$ -complete [8]. In fact, the problem is $\#P$ -complete even when restricted to 4-regular planar graphs [18]. So exactly computing the number of Euler tours is not in polynomial time unless $\#P = P$.

In this paper, we are concerned with the problem of counting Euler tours on graphs of bounded treewidth. Many problems which are NP-hard for general graphs, can be solved in polynomial time on bounded treewidth graphs. Indeed, a result of Courcelle [11] asserts that any graph property that is expressible in Monadic Second Order logic (with edge quantifiers) can be solved in linear time on bounded treewidth graphs. Elberfeld et al. [15] adapt the theorem of Courcelle in the parallel setting and prove a L bound. However, Eulerianity is provably not MSO -expressible [14] and hence the approaches mentioned above are not directly applicable in our context.

Our strategy to count Euler tours is as follows: Given a bounded treewidth graph G , we count the number of Euler tours of G by counting the number of Hamiltonian tours of the line graph of G , $L(G)$. In general, there is no bijection between these two quantities, but we show that G can be modified to obtain G' ($tw(G') \leq tw(G) + 3$) such that G, G' have the same number of Eulerian tours, which equals the number of Hamiltonian tours of $L(G')$. Henceforth, we will be primarily interested in line graphs of bounded treewidth graphs. It is known that such graphs are of bounded clique-width [24]

We base our proof on a proof that the decision version of Hamiltonicity is polynomial time computable in bounded clique-width graphs [16]. We prove that this algorithm can be parallelised and extended to the counting version. Next, we show that for line graphs of bounded tree-width graphs which form the family of interest, the clique-width expression can be inferred from the corresponding tree decomposition. The tree decomposition itself is obtainable by the L-version of Bodlaender's theorem [15].

Our main tool in establishing a uniform NC-bound for counting Hamiltonian cycles on bounded clique-width graphs hinges on *polynomial interpolation*. While polynomial interpolation has been used successfully to compute various graph polynomials [23], our use is somewhat indirect and subtle: it is used by the uniformity machine to populate a table whose entries do not depend on the input bounded clique-width expression but only the number of vertices in the corresponding graph and the clique-width. We then build a monotone arithmetic circuit that uses the clique-width expression of the graph and entries from this table to count the number of Hamiltonian cycles in the clique-width bounded graph. We then observe that since the number of distinct Hamiltonian tours of a graph is at most exponential in the number of vertices of the graph, and the circuit is monotone, the formal degree of the circuit must be a polynomial in the size of the input graph. This allows us to use a result from circuit complexity [2] to yield an upper bound of $\#SAC^1$ on the complexity of counting Euler tours on bounded treewidth graphs.

Our techniques also yield a parallel upper bound on the problems of counting longest paths/cycles and counting bipartite matchings in bounded clique-width graphs. These are well known problems (and $\#P$ -complete in general graphs) but their (counting) complexity has not been investigated in bounded clique-width graphs. While [13] studies the problem of counting longest paths and perfect matchings in bounded *tree-width DAGs*, we improve the results by resolving the problems for bounded *clique-width graphs* at the cost of replacing the L bound by a $\#SAC^1$ bound where we know that $L \subseteq \#SAC^1 \subseteq NC^2 \subseteq P$ ¹.

¹ Note that $\#SAC^1$ is a function class and when we say $\#SAC^1 \subseteq NC^2$, what we actually mean is that any bit of the $\#SAC^1$ function family of interest is computable by a NC^2 circuit family

1.1 Previous Work

Chebolu, Cryan, Martin have given a polynomial time algorithm for counting Euler tours in undirected series-parallel graphs [9] and they have claimed to extend it to a polynomial time algorithm [10] for the counting Euler tours in bounded tree-width graphs. We would like to point out that the only incomplete, unrefereed manuscript available publicly [10] sketches an algorithm that does dynamic programming directly on the tree-decomposition. Since we show how to obtain the line graph of the bounded tree-width graph efficiently in parallel and then work on this bounded clique-width graph - our approach is fundamentally different from that of [9, 10]. Another difference is that their algorithm is not designed to be parallelisable.

Also notice that in a precursor to this paper [4], using totally different techniques (basically applications of the Logspace version of Courcelle's theorem [15]) it was claimed that the number of Euler tours in bounded tree-width directed and undirected graphs can be counted in Logspace but the approach had a serious flaw in the undirected version. Later versions [5, 6] of the paper claim the result only for directed graphs. This work proves a slightly weaker version of the result - the upper bound being $\#\text{SAC}^1$ rather than Logspace.

Given that counting Hamiltonian cycles on bounded clique-width graphs will suffice for our purposes, one result that is directly relevant is that of Flarup and Lyaudet [17]: They study the expressive power of Perfect Matching and Hamiltonian polynomials of graphs of bounded clique-width and show that they can simulate arithmetic polynomials, and are themselves contained in VP. This yields a GapSAC^1 bound (implicit) for counting Hamiltonian cycles in bounded clique-width graphs right away. There are two aspects in which the work of [17] differs from our work: Firstly, even though their techniques are also inspired from [16] like ours, they work with a slightly different notion of clique-width namely $W - m - \text{clique-width}^2$. Secondly, in the case of counting Euler tours, from a straight-forward application of [17] the best upper bound that can be obtained from the circuit families constructed in [17] is non-uniform GapSAC^1 , whereas we get an upper bound of Logspace-uniform³ $\#\text{SAC}^1$.

There is some similarity that this work bears with that of Makowsky et al. [23], in that both involve polynomial interpolation to count witnesses for a graph theory problem. The similarity is somewhat superficial because we use interpolation to obtain numbers *independent* of the input graph while they interpolate to compute a graph polynomial that crucially depends on the graph. The choice of graph theory problems is also quite different. In particular, [23] does not address the Hamiltonian cycle problem.

1.2 Our Results

This is the main theorem of this work:

► **Theorem 1.** *$\#\text{Hamiltonian Cycles (or Paths)}$ for bounded clique-width graphs is in $\#\text{SAC}^1$. Consequently, $\#\text{Euler Tours}$ for bounded tree-width graphs is also in $\#\text{SAC}^1$.*

As a bonus we also get the following:

► **Theorem 2.** *The following counts can be obtained in $\#\text{SAC}^1$ for bounded clique-width graphs (given a bounded clique-width expression for the graph):*

² These are weighted versions of clique-width and are used to produce weighted graphs. [17] motivate this variant of clique-width by observing that since K_n has clique-width 2, most graph polynomials are VNP-complete for bounded clique-width graphs.

³ In an earlier version of this paper, we had erroneously claimed a GapL upper bound for counting Euler tours. As pointed out to us by Ramprasad Satharishi, there is a rather serious gap with this approach.

1. #Hamiltonian Cycles
2. #Longest Paths/Cycles
3. #Cycle Covers
4. #Perfect Matchings (for bipartite graphs)

1.3 Overview of Algorithm

Every Euler tour in a graph yields a Hamiltonian cycle in its line graph. Though this map is not bijective we show that we can make it so by altering the input graph slightly. It is well known [20] that the line graphs of bounded tree-width graphs have bounded clique-width. We show how to obtain a bounded clique-width decomposition for the line graph of a bounded tree-width graph in Logspace using the Logspace version of Courcelle's Theorem [15] by first obtaining a bounded tree-width decomposition via a Logspace version of Bodlaender's theorem [15].

Our main algorithm replaces the sequential procedure from [16] to decide if a bounded clique-width graph has a Hamiltonian path. Instead, it computes the number of Hamiltonian cycles. The procedure uses elementary counting coupled with polynomial interpolation to compute some matrices which are independent of the input graph depending only on its size. The matrices are then combined with vectors maintaining counts, along the structure tree of the clique-decomposition. A degree bound for the monotone arithmetic circuit then suffices to prove the #SAC¹ bound.

1.4 Organization

The rest of the paper is organized as follows: In Section 2, we introduce some definitions and results that will be helpful in understanding the rest of the paper. Section 3 shows how to obtain a clique-width expression for the line graph of a bounded treewidth graph in Logspace. Section 4 presents a #SAC¹ implementation of our algorithm to count the number of Hamiltonian tours in graphs of bounded clique-width. We conclude with some unresolved questions related to this work in Section 5.

2 Preliminaries

► **Definition 3** (Line Graph). For an undirected graph $G = (V, E)$, the line graph of G denoted $L(G) = (L_V, L_E)$ is the graph where $L_V = E$ and $(e_i, e_j) \in L_E$ if and only if there exists a vertex $v \in V$ such that both e_i and e_j are incident on v .

► **Definition 4** (Treewidth). Given an undirected graph $G = (V_G, E_G)$ a tree decomposition of G is a tree $T = (V_T, E_T)$ (the vertices in $V_T \subseteq 2^{V_G}$ are called *bags*), such that

1. Every vertex $v \in V_G$ is present in at least one bag, i.e., $\cup_{X \in V_T} X = V_G$.
2. If $v \in V_G$ is present in bags $X_i, X_j \in V_T$, then v is present in every bag X_k in the unique path between X_i and X_j in the tree T .
3. For every edge $(u, v) \in E_G$, there is a bag $X_r \in V_T$ such that $u, v \in X_r$.

The width of a tree decomposition is $\max_{X \in V_T} (|X| - 1)$. The treewidth of a graph is the minimum width over all possible tree decomposition of the graph.

► **Definition 5** (NLC-width). Let k be a positive integer. The class NLC_k of labeled graphs $G = (V, E, lab_G)$ where $lab_G : V \rightarrow [k]$, is recursively defined as follows:

1. The single vertex graph labeled by a label a , \bullet_a for $a \in [k]$ is in NLC_k .

2. Let $G = (V_G, E_G, lab_G) \in NLC_k$ and $H = (V_H, E_H, lab_H) \in NLC_k$ be two vertex-disjoint labeled graphs and $S \subseteq [k]^2$, then $G \times_S H = (V', E', lab') \in NLC_k$, where $V' = V_G \cup V_H$ and

$$E' = E_G \cup E_H \cup \{(u, v) | u \in V_G, v \in V_H, (lab_G(u), lab_H(v)) \in S\}$$

and for all $u \in V'$,

$$lab'(u) = \begin{cases} lab_G(u), & \text{if } u \in V_G \\ lab_H(u), & \text{if } u \in V_H \end{cases}$$

3. Let $G = (V_G, E_G, lab) \in NLC_k$ and $R : [k] \rightarrow [k]$ be a function, then $\circ_R(G) := (V_G, E_G, lab')$ defined by $lab'(u) = R(lab(u))$ for all $u \in V_G$ is in NLC_k .

The NLC-width⁴ of a labeled graph G is the least integer k such that $G \in NLC_k$. An expression Y built with $\bullet_a, \times_S, \circ_R$, for integers $a \in [k]$, $S \subseteq [k]^2$ and $R : [k] \rightarrow [k]$ is called a NLC-width k expression. The graph defined by expression Y is denoted by $val(Y)$.

► **Definition 6** (Clique Width). Let k be a positive integer. The class CW_k of labeled graphs $G = (V, E, lab_G)$ where $lab_G : V \rightarrow [k]$ is recursively defined as follows:

1. The single vertex graph labeled by a label a , \bullet_a for $a \in [k]$ is in CW_k .
2. Let $G = (V_G, E_G, lab_G) \in CW_k$ and $H = (V_H, E_H, lab_H) \in CW_k$ be two vertex-disjoint labeled graphs. Then $G \oplus H = (V', E', lab') \in CW_k$, where $V' = V_G \cup V_H$ and $E' = E_G \cup E_H$ and for all $u \in V'$

$$lab'(u) = \begin{cases} lab_G(u), & \text{if } u \in V_G \\ lab_H(u), & \text{if } u \in V_H \end{cases}$$

3. Let a, b be distinct positive integers and $G = (V_G, E_G, lab) \in CW_k$ be a labeled graph. Then,

(a) $\rho_{a \rightarrow b}(G) := (V_G, E_G, lab') \in CW_k$ where for all $u \in V_G$

$$lab'(u) = \begin{cases} lab_G(u), & \text{if } lab_G(u) \neq a \\ b, & \text{if } lab_G(u) = a \end{cases}$$

(b) $\eta_{a,b}(G) := (V_G, E', lab_G) \in CW_k$ where,

$$E' = E_G \cup \{(u, v) | u, v \in V_G, lab(u) = a, lab(v) = b\}$$

The clique-width of a labeled graph G is the least integer k such that $G \in CW_k$. An expression X built with $\bullet_a, \oplus, \rho_{a \rightarrow b}, \eta_{a,b}$ for integers $a, b \in [k]$ is called a clique-width k expression. By $val(X)$, we denote the graph defined by expression X .

► **Definition 7** (Chordal graph, Chordal completion). A graph is said to be chordal if every cycle with at least 4 vertices always contains a chord. A chordal completion of a graph G is a chordal graph with the same vertex set as G which contains all edges of G .

► **Definition 8** (Perfect Elimination Ordering, Elimination Tree [19]). Let $G = (V, E)$ be a graph and $o = (v_1, v_2, \dots, v_n)$ be an ordering of the vertices of G . Let $N^-(G, o, i)$ and

⁴ NLC stands for *Node Label Controlled*, has its origins in graph grammars, was defined by Wanke [28].

$N^+(G, o, i)$ for $i = 1, \dots, n$ be the set of neighbors v_j of vertex v_i with $j < i$ and $j > i$ respectively.

$$\begin{aligned} N^-(G, o, i) &= \{v_j | (v_i, v_j) \in E \text{ and } j < i\} \\ N^+(G, o, i) &= \{v_j | (v_i, v_j) \in E \text{ and } j > i\} \end{aligned}$$

The vertex order o is said to be a Perfect Elimination Ordering (PEO) if for all $i \in [n]$, $N^+(G, o, i)$ induces a complete subgraph of G . The structure of G can then be characterized by a tree $T(G, o) = (V_T, E_T)$ defined as follows:

$$\begin{aligned} V_T &= V \\ E_T &= \{(v_i, v_j) \in E | i < j \text{ and } \forall j', i < j' < j, (v_i, v_{j'}) \notin E\} \end{aligned}$$

Such a $T(G, o)$ is called the Elimination Tree associated with the graph G .

For more information on Chordal graphs and PEO, we refer the reader to Golumbic's book [19].

► **Definition 9** (Cycle Cover). A *cycle cover* \mathcal{C} of $G = (V, E)$ is a set of vertex-disjoint cycles that cover the vertices of G . I.e., $\mathcal{C} = \{C_1, C_2, \dots, C_k\}$, where $V(C_i) = \{c_{i_1}, \dots, c_{i_{r(i)}}\} \subseteq V$ such that $(c_{i_1}, c_{i_2}), (c_{i_2}, c_{i_3}), \dots, (c_{i_{r(i)-1}}, c_{i_{r(i)}}), (c_{i_{r(i)}}, c_{i_1}) \in E(C_i) \subseteq E$ and $\bigsqcup_{i=1}^k V(C_i) = V$. The least numbered vertex $h_i \in V(C_i)$, is called the head of the cycle.

► **Definition 10** ($\#\text{SAC}^1$). $\#\text{SAC}^1$ is the class of functions from $\{0, 1\}^n$ to nonnegative integers computed by polynomial-size logarithmic-depth, semi unbounded arithmetic circuits⁵, using $+$ (unbounded fan-in) and \times gates (fan-in 2) and the constants 0 and 1.

For further background on circuit complexity, we refer the reader to [27].

► **Proposition 11** ([2, 26]). Any function $f : \{0, 1\}^n \rightarrow R$, where R is a semi-ring, computed by arithmetic circuits of size s and degree d can be computed by semi-unbounded arithmetic circuits of size $\text{poly}(s, d)$ and depth $O(\log d)$. In particular, all functions computed by polynomial sized circuits of polynomial degree are exactly those in $\#\text{SAC}^1$.

► **Fact 12** (Kronecker substitution [12]). Let $P(x_1, x_2, \dots, x_n)$ be a multivariate polynomial of degree d . We replace every occurrence of variable x_i by x^{d^i} . This yields a unique univariate polynomial $Q(x)$ of degree at most $d^{O(n)}$ such that P can be efficiently recovered from the knowledge of coefficients of Q . When the number of variables is a constant, the degree of the multivariate polynomial and the univariate polynomial are polynomially related.

3 From Euler Tours to Hamiltonian cycles

It is possible to construct a graph G such that G has no Eulerian tours, but $L(G)$ has a Hamiltonian cycle⁶. Proposition 13 gives necessary and sufficient conditions for when a line graph of a given graph is Hamiltonian.

► **Proposition 13** ([21]). $L(G)$ is Hamiltonian if and only if G has a closed trail that contains at least one end point of every edge.

⁵ Note that such circuits have degree that is at most a polynomial in the number of input variables.

⁶ Indeed, there is a 2-connected graph – K_4 with one of the edges removed – which is non-Eulerian but its line graph is Hamiltonian.

Given a graph G , we want to construct a graph G' such that every closed trail in G' that contains at least one end point of every edge is exactly an Eulerian tour of G' . The following Lemma guarantees exactly this:

► **Lemma 14.** *Given an undirected graph G , construct a graph $G' = (V', E')$ from G as follows: Replace every edge $e = (u, v)$ of G by path of length three. Then G and G' have the same number of Eulerian tours and the Eulerian tours of G' are in bijection with the Hamiltonian tours of $L(G')$.*

Notice that G is a minor of G' , and the tree decomposition of G' can be obtained from that of G by locally adding to each bag containing an edge e of G , the extra vertices and edges of the path of length three. Hence, the following is immediate:

► **Proposition 15.** *G has bounded treewidth iff G' has bounded treewidth.*

► **Proposition 16** ([20]). *If G is of treewidth k , then $L(G)$ has clique-width $f(k) = 2k + 2$.*

► **Proposition 17** ([15]). *Given a bounded treewidth graph G , a balanced tree decomposition⁷ of G is obtainable in \mathbb{L} .*

We first need the Perfect Elimination Ordering(PEO) of the vertices of the graph. It is known that a graph has a PEO if and only if it is chordal. Since we can do a chordal completion of a bounded treewidth graph (while preserving treewidth), such an ordering of the vertices always exists. Recently Arvind et al. gave a Logspace procedure for obtaining a PEO in k -trees (which are maximal treewidth- k graphs). We adapt this for graphs that are chordal completions of bounded treewidth graphs:

► **Lemma 18** (Adapted from [3]). *Given a balanced tree decomposition of a bounded treewidth graph G , a Perfect Elimination Ordering and the corresponding elimination tree of a chordal completion of G , which is a balanced binary tree of depth $O(\log n)$, can be computed in \mathbb{L} .*

► **Lemma 19** (Adapted from [20]). *Given the tree decomposition of a graph G along with a elimination tree, the clique-width expression X of $L(G)$ is obtainable in \mathbb{L} . The parse tree of this clique-width expression has height at most $O(\log n)$*

We show in the subsequent Lemma that the method in [20] is amenable to a Logspace implementation when provided with a PEO of the vertices of the graph.

► **Lemma 20** (Adapted from [20]). *The NLC-width of the line graph $L(G)$ of a graph G of treewidth k is at most $k + 2$ and such a NLC-width expression is obtainable in \mathbb{L} .*

Gurski and Wanke [20] observe that it is sufficient to look at G that are k -trees here because the line graph of every subgraph of G then is an induced subgraph of the line graph of G and the class NLC_k is closed under taking induced subgraphs for every $k \geq 1$ (See Theorem 4 in [20]). Our method involves dealing with bounded treewidth graphs that are chordal, which are a strict superclass of k -trees and we observe that the property mentioned above still holds in this case.

► **Proposition 21.** *Given a graph G of NLC-width at most k by an NLC-width expression Y , we can obtain the clique-width expression X of G , where $|X| \leq 2k + 2$ in \mathbb{L} .*

⁷ A tree decomposition of a graph is said to be balanced if the tree underlying the decomposition is balanced

To sum up, these are the main preprocessing steps:

1. Obtain a balanced binary tree decomposition of the input treewidth k graph G in Logspace via Proposition 17 [15].
2. Obtain the tree decomposition of G' (as required by Proposition 13 and specified by Lemma 14) from the tree decomposition of G .
3. Perform a chordal completion of G' by adding edges to every bag.
4. Obtain a PEO tree of G' of height $O(\log n)$, where every vertex has at most k children via Lemma 18.
5. Construct a NLC width $(k + 2)$ expression for $L(G')$ via Lemma 20
6. From the NLC width $(k + 2)$ expression, construct a clique-width $(2k + 2)$ expression for $L(G')$ via Proposition 21 (The surplus edges added during the chordal completion are removed at this step).

The proofs of Lemma 14, 18, 19, 20 and Proposition 21 can be found in the full version of the paper [7].

4 The #SAC¹ upper bound

Let X be the clique-width k expression for a labeled graph $G = (V, E, lab)$ such that G is $\text{val}(X)$ and let $|V| = n$. Let G be of clique-width k . Hence by Definition 6, G can be constructed from the graph with n isolated labeled vertices, using at most k labels. Notice that X can be viewed as a tree (we will refer to this as the parse tree of the clique-width expression) with the n isolated labeled vertices at the leaves and every internal node is labeled with one of the operations $o = \{\bullet_i, \oplus, \eta_{i,j}, \rho_{i \rightarrow j} : i, j \in [k] \wedge i \neq j\}$. To each internal vertex of the tree, we can associate a graph (possibly disconnected) which is a subgraph of G , and at the root of the tree, we get G itself. The size of the tree is polynomial in n and k . Our objective in this section will be to count the number of Hamiltonian cycles in G , when provided with the clique-width expression X . We will count along the parse tree of the clique-width expression.

To this end, we call a subset of edges $E' \subseteq E$ *path-cycle covers*, if in the subgraph $G' = (V, E', lab)$ every vertex in G' has degree at most 2. To every such G' , we associate a multiset M consisting of multisets $\langle lab(v_1), lab(v_r) \rangle$ one each for every path/cycle $p = v_1, \dots, v_r$, $r \geq 1$, in G' , where v_1, v_r have degree at most 1 in G' if they exist (p being a cycle otherwise). Let $F(X)$ be the set of all multisets M for all such subsets $E' \subseteq E$.

Let K be the set of all possible labels of the end points, in the labeled graph produced at the output of each node in the parse tree. We refer to elements of K as *types*. Note that every M consists of at most $|K|$ distinct *types* and $F(X)$ has at most $(n + 1)^{|K|}$ distinct multisets each with at most n multisets of size 2. Here $K = K_0 \uplus K_1 \uplus K_2$ is the set of distinct *types* where K_2 accounts for types of the form $\langle i, j \rangle$ (for $i \neq j$) corresponds to paths whose end points are i and j ; K_0 for the empty type $\langle \rangle = \emptyset$ corresponds to a *cycle*; K_1 for types of the form $\langle i, i \rangle$ which could be either paths whose end points are both labeled i , or isolated vertices with the label i . Observe that, $|K_2| = \binom{k}{2}$, $K_1 = 2k$ and $K_0 = 1$, where we distinguish between the cases of single isolated vertex of label i and multiple vertex paths with end points labeled i for technical reasons, leading to the extra factor of 2. Our notation is consistent with [16] in all cases except for the empty type, since in [16] cycles are not permitted.

Our objective is to count the number of path-cycle covers, $\#X[M]$, corresponding to a multiset M in the graph $\text{val}(X)$. In particular,

$$\sum_{i,j \in [K]} \#X[M_{i,j}]$$

where $M_{i,j} = \langle\langle i, j \rangle\rangle$ is a multiset containing a single type $\langle i, j \rangle$, yields the number of Hamiltonian paths with end points coloured i, j in $\text{val}(X)$. We denote by $\#X$ the vector indexed by M and hence has $(n+1)^K$ entries where $\#X[M]$ (where $M \in [0, n]^K$) stores the count of the number of path/cycle covers of type specified by M in the graph $\text{val}(X)$. Let C_o be a $(n+1)^K \times (n+1)^K$ matrix which for each pair of multisets M, M' denotes the number of ways to form M' from M under an operation $o \in \{\eta_{i,j}, \rho_{i \rightarrow j} : i, j \in [k] \wedge i \neq j\}$. C_o is defined uniquely for the two kinds of operations η, ρ and is independent of the input graph $\text{val}(X)$.

Then the following is an easy consequence of the definitions:

► **Proposition 22.** *The value of $\#X$ is given by:*

1. if $X = \bullet_i$ then if $M = \langle\langle i, i \rangle\rangle$ then $\#X[M] = 1$; else $\#X[M] = 0$.
2. else if $X = X_1 \oplus X_2$ then

$$\#X[M] = \sum_{M' \in [0, n]^K : M' \subseteq M} \#X_1[M'] \#X_2[M \setminus M']$$

3. else if $X = \rho_{i \rightarrow j}(X_1)$ then $(C_{\rho_{i \rightarrow j}})^T \#X_1$
4. else $X = \eta_{i,j}(X_1)$ then $(C_{\eta_{i,j}})^T \#X_1$

Proof. The first item is immediate. For the second, notice that each multiset of types M in the disjoint union of two graphs is formed by picking multisets M', M'' from the two graphs respectively and taking their multi-union. Thus the number of distinct ways to form M is obtained by considering all possible decompositions of M into sets M', M'' one from each graph. Since, this is a decomposition $M'' = M \setminus M'$, the correctness of the second item follows.

For the third and the fourth items, notice that we have a matrix C such that $C[M, M']$ is the number of ways to convert a multiset M to a multiset M' . Thus the number of ways to form M' is to take the product of $\#X[M]C[M, M']$ and add up the products over all M . This is the stated form in matrix notation. ◀

Proposition 22 enables us to prove the $\#\text{SAC}^1$ upper bound:

► **Lemma 23.** *For a bounded clique-width expression X , for every multiset of types, M , the value $\#X[M]$ of the number of path-cycle covers at any node along the parse tree of the clique-width expression can be computed in $\#\text{SAC}^0$ where the inputs to the $\#\text{SAC}^0$ circuit are entries of the matrix C_o for $o \in \{\bullet_i, \oplus, \eta_{i,j}, \rho_{i \rightarrow j} : i, j \in [k] \wedge i \neq j\}$. The number of path-cycle covers in the input graph can hence be counted in $\#\text{SAC}^1$.*

The proof of Lemma 23 can be found in the full version of the paper [7]. We now turn to the proof of our main Theorem 1

Proof. (of Theorem 1) To count Euler tours on bounded treewidth graphs, we can count Hamiltonian cycles in the line graph (via Lemma 14). Here we need to compute the quantity $\#X[\langle\emptyset\rangle]$ (since the empty multiset represents a cycle, the path-cycle cover consisting of a single cycle must be a Hamiltonian cycle itself). This follows from Lemma 23. ◀

Proof. (of Theorem 2) Hamiltonian cycles can be counted in #SAC¹ by Lemma 23. Longest Cycles (Paths) can be counted by considering multisets which consist of a single cycle (respectively, path) and the minimum number of isolated vertices respectively. To see this observe that for every cycle (respectively, path) C in the graph there is a multiset consisting of a single empty type (respectively, non-empty type) and $|V(G)| - |V(C)|$ isolated vertices respectively.

Counting cycle covers is equally simple. We just need to add up the counts for multisets consisting only of empty types. This, is of course because an empty type represents a cycle.

Perfect Matchings in bipartite graphs can therefore be counted by counting the cycle covers in a biadjacency matrix. ◀

4.1 Computing $C_{\rho_{i \rightarrow j}}$ and $C_{\eta_{i,j}}$

It is easy to compute $C_{\rho_{i \rightarrow j}}$ by the following,

► **Proposition 24.** $C(\rho_{i \rightarrow j})$ is a $\{0, 1\}$ -matrix such that the entry corresponding to M_1, M_2 is equal to 1 iff $\rho_{i \rightarrow j}(M_1) = M_2$ (it is 0 otherwise).

Let $W_{\vec{\alpha}}(t')$ denote the number of ways to form one path/cycle of type $t' \in K$, given a multiset of paths/cycles consisting of $\vec{\alpha}(t)$ paths/cycles for every type $t \in K$.

Next, we show how to compute $C_{\eta_{i,j}}$:

► **Lemma 25.** There is a Logspace Turing machine that takes input $W_{\vec{\alpha}}(t')$ for every $\vec{\alpha} \in [0, n]^{|K|}, t' \in K$ and outputs the entries of the matrix $C_{\eta_{i,j}}$.

Proof. We show that each entry can be computed in DLOGTIME-uniform TC⁰ which is contained in L (see e.g. Vollmer [27]). Our main tool in this lemma is an application of polynomial interpolation.

Notice that the rows/columns of the matrix C are indexed by multisets of *types*. Here a *type* is an element from K . Therefore any such multiset can be described by a vector $\vec{\alpha}$ of length $|K|$. Here each entry of the vector represents the number of paths/cycles with that type inside the multiset.

In the following we will consistently make use of the notation, $\vec{z}^{\vec{a}}$ to denote: $\prod_{i \in I} z_i^{a_i}$, where I is the index set for both \vec{z}, \vec{a} .

We have the following:

► **Lemma 26.** $C[M, M']$ is the coefficient of $\vec{x}^{\vec{c}'} \vec{y}^{\vec{c}}$ in the following polynomial $p_{\vec{c}, \vec{c}'}(\vec{x}, \vec{y})$:

$$\prod_{t, t' \in K} \prod_{\vec{\alpha} \in [0, n]^{|K|}} \sum_{\vec{d}_{\vec{\alpha}}} \left(W_{\vec{\alpha}}(t') x_{t'} y_{t'}^{\alpha(t)} \right)^{d_{\vec{\alpha}}(t')}$$

To fix the notation we reiterate (items 1, 2, 3, 4 were defined previously and we introduce some new notation in items 5, 6, 7, 8, 9):

1. K is the set of types, where $|K| = \binom{k}{2} + 2k + 1$.
2. $t, t' \in K$ are types in the input, output multiset (respectively M, M').
3. A *allocation*, $\alpha(t) \in [0, n]$ is the number of path-cycle covers of type $t \in K$.
4. $\vec{\alpha} \in [0, n]^{|K|}$ is a possible allocation vector indexed by K in which each entry is $\alpha(t)$.
5. $d_{\vec{\alpha}}(t') \in [0, n]$ is the number of paths of type t' formed from each allocation of type $\vec{\alpha}$.
6. $\vec{d}_{\vec{\alpha}} \in [0, n]^{|K|}$ is a vector indexed by K in which each entry is one of $d_{\vec{\alpha}}(t')$.
7. $W_{\vec{\alpha}}(t')$ is the number of ways to form a single path/cycle of type t' from an allocation vector $\vec{\alpha}$.

8. $\vec{W}_{\vec{\alpha}}$ is the vector indexed by K in which each entry is one of $W_{\vec{\alpha}}(t')$.
 9. $\vec{c}, \vec{c}' \in [0, n]^{|K|}$ are vectors indicating number of paths/cycles in M, M' respectively.

To see that Lemma 25 follows from Lemma 26 we use Kronecker substitution (see Fact 12) to convert the multivariate polynomial $p_{\vec{c}', \vec{c}}(\vec{x}, \vec{y})$ with $2|K|$ variables to a univariate polynomial. Then we use Lagrange interpolation to find the coefficient of an arbitrary term - in particular, the term corresponding to $\vec{x}^{\vec{c}'} \vec{y}^{\vec{c}}$ in TC^0 (see e.g. Corollary 6.5 in [22]). ◀

Proof. (of Lemma 26) Consider the following expression:

$$\sum_{\vec{d} \in \mathbf{D}} \prod_{t' \in K} \prod_{\vec{\alpha} \in [0, n]^{|K|}} W_{\vec{\alpha}}(t')^{d_{\vec{\alpha}}(t')} \quad (1)$$

where the sum is taken over $\mathbf{D} \subseteq [0, n]^{|K|}$ consisting of all \vec{d} 's satisfying:

$$\forall t', \quad \sum_{\vec{\alpha} \in [0, n]^{|K|}} d_{\vec{\alpha}}(t') = c'(t') \quad (2)$$

$$\forall t, \quad \sum_{\vec{\alpha} \in [0, n]^{|K|}} \sum_{t' \in K} \alpha(t) d_{\vec{\alpha}}(t') = c(t) \quad (3)$$

► **Claim 27.** $C[M, M']$ equals Expression (1).

Proof. (of Claim) The Condition 2 above asserts that the number of paths/cycles of type t' present in M' equals the sum over all $\vec{\alpha}$ of the number of paths/cycles of type t' using resources described by $\vec{\alpha}$; the Condition 3 is essentially a conservation of resource equation for every type t saying that all the resources present in M are used one way or the other in M' .

Let P, P' be path-cycle covers represented by M, M' respectively such that we can obtain P' from P , i.e., P' is one possible path-cycle cover that can be obtained by an $\eta_{i,j}$ operation on P . This transformation is described by a unique \vec{d} . Then the pair contributes precisely one to $C[M, M']$. On the other hand $\{P, P'\}$ satisfies (2),(3) so contributes exactly one to the summand corresponding to the unique \vec{d} in Expression 1. Since the pair P, P' corresponds to a unique \vec{d} and contributes exactly one, the remaining summands would evaluate to zero. This can be explained by observing that for all $\vec{d}' \neq \vec{d}$, the number of paths of type t in \vec{d}' is not equal to the corresponding number in \vec{d} for atleast one t . Hence, they would contribute nothing to pair P, P' . ◀

To complete the proof notice that the coefficient of $\vec{x}^{\vec{c}'} \vec{y}^{\vec{c}}$ is precisely expression 1 under the conditions 2, 3. Now, we explain the reasoning behind expression 1. We have $d_{\vec{\alpha}}(t')$ paths of type t' , each of which can be formed in $W_{\vec{\alpha}}(t')$ ways. Note that each of these $d_{\vec{\alpha}}(t')$ paths are formed from different $\vec{\alpha}$ (though the values of each of these $d_{\vec{\alpha}}(t')$ vectors $\vec{\alpha}$ is the same, they are inherently different as they are composed of mutually exclusive vertex sets) and we consider each valid set of $d_{\vec{\alpha}}(t')$ vectors $\vec{\alpha}$, exactly once. Hence, we multiply with $W_{\vec{\alpha}}(t')^{d_{\vec{\alpha}}(t')}$ to get the final count. ◀

4.2 Calculation of $W_{\vec{\alpha}}(t')$

$W_{\vec{\alpha}}(t')$ denotes the number of ways to form exactly one type $t' \in K$ in M' given a multiset of types consisting of $\vec{\alpha}(t)$ types for every type $t \in K$ in M . For simplicity of notation, let $t = \langle i, j \rangle \in K$ be a type and let $\beta(i) = \alpha(\langle i, i \rangle) + \alpha^{(=0)}(\langle i, i \rangle)$ be the total number of multisets of type $\langle i, i \rangle$, where $\alpha(\langle i, i \rangle)$ (respectively $\alpha^{(=0)}(\langle i, i \rangle)$) denote paths (respectively single nodes) labeled $\langle i, i \rangle$ in $\vec{\alpha}$. Note that this distinction is not necessary for types where the end points have different labels.

► **Lemma 28.** For an operation η_{i_0, j_0} in the clique-width expression and for any type $t' = \langle i, j \rangle$, $W_{\vec{\alpha}}(t')$ is given by

$$W_{\vec{\alpha}}(\langle i, j \rangle) = \llbracket \langle i, j \rangle \rrbracket_{\vec{\alpha}} W_{\vec{\alpha}}$$

where,

$$W_{\vec{\alpha}} = \binom{\alpha(\langle i_0, j_0 \rangle) + \beta(i_0) + \beta(j_0)}{\alpha(\langle i_0, j_0 \rangle)} \alpha(\langle i_0, j_0 \rangle)! \beta(i_0)! \beta(j_0)! 2^{\alpha(\langle i_0, i_0 \rangle) + \alpha(\langle j_0, j_0 \rangle)}$$

and, $\llbracket \langle i, j \rangle \rrbracket_{\vec{\alpha}}$ is given by^{8, 9}

- $\llbracket \langle a, b \rangle \rrbracket_{\vec{\alpha}} = \llbracket \beta(a) = \beta(b) \rrbracket$
- $\llbracket \langle a, a \rangle^{(=0)} \rrbracket_{\vec{\alpha}} = \llbracket \alpha_{a,a}^{(=0)} = 1 \wedge \alpha(\langle a, b \rangle) = 0 \rrbracket$
- $\llbracket \langle a, a \rangle \rrbracket_{\vec{\alpha}} = \llbracket \beta(a) = \beta(b) + 1 \rrbracket$
- $\llbracket \langle i, a \rangle \rrbracket_{\vec{\alpha}} = \llbracket (\alpha(\langle i, a \rangle) = 1 \wedge \beta(a) = \beta(b)) \vee (\alpha(\langle i, b \rangle) = 1 \wedge \beta(a) = \beta(b) + 1) \rrbracket$
- $\llbracket \langle i, j \rangle \rrbracket_{\vec{\alpha}} = \llbracket (\alpha(\langle i, a \rangle) = 1 \wedge \alpha(\langle a, j \rangle) = 1 \wedge \beta(a) = \beta(b) + 1) \vee (\alpha(\langle i, a \rangle) = 1 \wedge \alpha(\langle b, j \rangle) = 1 \wedge \beta(a) = \beta(b)) \rrbracket$
- $\llbracket \langle i, i \rangle \rrbracket_{\vec{\alpha}} = \llbracket (\alpha(\langle i, a \rangle) = 2 \wedge \beta(a) = \beta(b) + 1) \vee (\alpha(\langle i, a \rangle) = 1 \wedge \alpha(\langle i, b \rangle) = 1 \wedge \beta(a) = \beta(b)) \rrbracket$
- $\llbracket \langle \emptyset \rangle \rrbracket_{\vec{\alpha}} = \llbracket \beta(a) = \beta(b) \rrbracket$

where, $\{a, b\} = \{i_0, j_0\}$ in some order.

Proof. (of Lemma 28) Let's look at $W_{\vec{\alpha}}(\langle a, a \rangle)$ in detail. The $W_{\vec{\alpha}}(t)$ for all the other types t are computed similarly. Type $\langle a, a \rangle$ can be formed from the alternating sequence of types $\langle a, a \rangle, \langle b, b \rangle, \langle a, a \rangle, \dots, \langle a, a \rangle$ interleaved with some (possibly zero) $\langle a, b \rangle$ types. Thus, the equality $\beta(a) = \beta(b) + 1$ should hold while $\alpha(\langle a, b \rangle)$ can be any arbitrary non-negative integer. When $\alpha(\langle a, b \rangle) = 0$, the condition $\alpha(\langle a, a \rangle) \geq 1 \vee \alpha(\langle a, a \rangle)^{(=0)} > 1$ should hold to ensure that we are not considering the type $(\langle a, a \rangle)^{(=0)}$.

The number of ways of interspersing $\alpha(\langle a, b \rangle)$ types among $\beta(a) + \beta(b)$ types is

$$\binom{\alpha(\langle a, b \rangle) + \beta(a) + \beta(b)}{\alpha(\langle a, b \rangle)}$$

We can do this for all permutations of the $\langle a, b \rangle, \langle a, a \rangle$ and $\langle b, b \rangle$ types hence we multiply by: $\alpha(\langle a, b \rangle)! \beta(a)! \beta(b)!$. Finally, we can flip the orientation of paths of types $\langle a, a \rangle$ and $\langle b, b \rangle$ as they are equivalent respectively to their flipped orientations. Note that single nodes cannot be flipped. The proof is therefore completed by multiplying with: $2^{\alpha(\langle a, a \rangle) + \alpha(\langle b, b \rangle)}$. Lastly, a boundary case occurs when $\alpha(\langle a, b \rangle) = 0$ where every path can be flipped. Here, it is easy to see that in considering every permutation of types while accounting for flips, we end up counting each path twice (including its reverse). Hence, in this case we divide by 2. ◀

5 Conclusion and Open Ends

- Can the $\#SAC^1$ bound be improved, to say, GapL or Logspace?
- How far can the Euler tour result be extended? To bounded clique-width graphs? Chordal graphs?
- Can the determinant of bounded clique-width adjacency matrices be computed in better than $\#SAC^1$? (it is known to be L-hard even for bounded tree-width graphs from [6]).

⁸ In this section, the notation $\llbracket S \rrbracket$ represents the Boolean value of the statement S . $\llbracket t \rrbracket_{\vec{\alpha}}$ represents a Boolean valued normalizing factor associated with the type t under the allocation vector $\vec{\alpha}$.

⁹ We adopt a convention in which types t' (other than the type $\langle i_0, j_0 \rangle$) not explicitly included in the expressions have an allocation $\alpha_{t'}$ equalling zero.

Acknowledgements. We would like to thank K. Narayan Kumar, Aniket Mane, M. Praveen and Prakash Saivasan for illuminating discussions regarding this paper. We would like to thank Eric Allender, Vikraman Arvind, Nutan Limaye, Meena Mahajan, Pierre McKenzie, Partha Mukhopadhyay, Ramprasad Saptharishi, Srikanth Srinivasan and V.Vinay for reading a follow-up work that resulted from this paper and for their comments, from which we discovered an error in a previous version of this paper. Thanks are also due to anonymous referees for several comments that helped improve the content and presentation of the paper. This work is partially funded by a grant from the Infosys Foundation.

References

- 1 van T. Aardenne-Ehrenfest and N. G. de Bruijn. Circuits and trees in oriented linear graphs. *Simon Stevin: Wis-en Natuurkundig Tijdschrift*, 28:203, 1951.
- 2 Eric Allender, Jia Jiao, Meena Mahajan, and V. Vinay. Non-commutative arithmetic circuits: Depth reduction and size lower bounds. *Theor. Comput. Sci.*, 209(1-2):47–86, 1998.
- 3 Vikraman Arvind, Bireswar Das, Johannes Köbler, and Sebastian Kuhnert. The isomorphism problem for k-trees is complete for logspace. *Information and Computation*, 217:1–11, 2012.
- 4 Nikhil Balaji and Samir Datta. Tree-width and logspace: Determinants and counting Euler tours. *CoRR*, abs/1312.7468, 2013.
- 5 Nikhil Balaji and Samir Datta. Bounded treewidth and space-efficient linear algebra. *CoRR*, abs/1412.2470, 2014.
- 6 Nikhil Balaji and Samir Datta. Bounded treewidth and space-efficient linear algebra. In *Theory and Applications of Models of Computation*, pages 297–308. Springer, 2015.
- 7 Nikhil Balaji, Samir Datta, and Venkatesh Ganesan. Counting Euler tours in undirected bounded treewidth graphs. *arXiv:1510.04035v1 [cs.CC]*, 2015. <http://arxiv.org/abs/1510.04035v1>.
- 8 Graham Brightwell and Peter Winkler. Counting Eulerian circuits is #p-complete. In *ALLENEX/ANALCO*, pages 259–262, 2005.
- 9 Prasad Chebolu, Mary Cryan, and Russell Martin. Exact counting of Euler tours for generalized series-parallel graphs. *J. Discrete Algorithms*, 10:110–122, 2012.
- 10 Prasad Chebolu, Mary Cryan, and Russell Martin. Exact counting of Euler tours for graphs of bounded treewidth. *CoRR*, abs/1310.0185, 2013.
- 11 Bruno Courcelle. The monadic second-order logic of graphs. i. recognizable sets of finite graphs. *Information and computation*, 85(1):12–75, 1990.
- 12 David Cox, John Little, and Donal O’shea. *Ideals, varieties, and algorithms*, volume 3. Springer, 1992.
- 13 B. Das, S. Datta, and P. Nimbhorkar. Log-space algorithms for paths and matchings in k-trees. *Theory Comput. Syst.*, 53(4):669–689, 2013.
- 14 Heinz-Dieter Ebbinghaus and Jörg Flum. *Finite model theory*. Perspectives in Mathematical Logic. Springer, 1995.
- 15 Michael Elberfeld, Andreas Jakoby, and Till Tantau. Logspace versions of the theorems of Bodlaender and Courcelle. In *FOCS*, pages 143–152, 2010.
- 16 Wolfgang Espelage, Frank Gurski, and Egon Wanke. How to solve NP-hard graph problems on clique-width bounded graphs in polynomial time. In *WG*, pages 117–128. Springer, 2001.
- 17 Uffe Flarup and Laurent Lyaudet. On the expressive power of permanents and perfect matchings of matrices of bounded pathwidth/cliqewidth. *Theory of Computing Systems*, 46(4):761–791, 2010.
- 18 Qi Ge and Daniel Štefankovič. The complexity of counting Eulerian tours in 4-regular graphs. *Algorithmica*, 63(3):588–601, 2012.
- 19 Martin Charles Golumbic. *Algorithmic graph theory and perfect graphs*, volume 57. Elsevier, 2004.
- 20 Frank Gurski and Egon Wanke. Line graphs of bounded clique-width. *Discrete Mathematics*, 307(22):2734–2754, 2007.
- 21 Frank Harary and C St JA Nash-Williams. On Eulerian and Hamiltonian graphs and line graphs. *Canadian Mathematical Bulletin*, 8:701–709, 1965.
- 22 W. Hesse, E. Allender, and D.A.M. Barrington. Uniform constant-depth threshold circuits for division and iterated multiplication. *Journal of Computer and System Sciences*, 65:695–716, 2002.

- 23 J. A. Makowsky, U. Rotics, I. Averbouch, and B. Godlin. Computing graph polynomials on graphs of bounded clique-width. In *WG*, pages 191–204, 2006.
- 24 Sang-il Oum and Paul D. Seymour. Approximating clique-width and branch-width. *J. Comb. Theory, Ser. B*, 96(4):514–528, 2006.
- 25 WT Tutte and CAB Smith. On unicursal paths in a network of degree 4. *The American Mathematical Monthly*, 48(4):233–237, 1941.
- 26 V. Vinay. Counting auxiliary pushdown automata. In *Structure in Complexity Theory*, pages 270–284, 1991.
- 27 Heribert Vollmer. *Introduction to circuit complexity – a uniform approach*. Texts in theoretical computer science. Springer, 1999.
- 28 Egon Wanke. k -nlc graphs and polynomial algorithms. *Discrete Applied Mathematics*, 54(2):251–266, 1994.