A Catalog of $\exists\mathbb{R}$-Complete Decision Problems About Nash Equilibria in Multi-Player Games

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Abstract

[Schaefer and Štefankovič, *Theory of Computing Systems, 2015*] provided an explicit formulation of $\exists\mathbb{R}$ as the class capturing the complexity of deciding the *Existential Theory of the Reals,* and established that deciding, given a 3-player *game,* whether or not it has a *Nash equilibrium* with no probability exceeding a given rational is $\exists\mathbb{R}$-complete. Four more decision problems about Nash equilibria for 3-player games were very recently shown $\exists\mathbb{R}$-complete via a chain of individual, problem-specific reductions in [Garg et al., *Proceedings of ICALP* 2015]; determining more such $\exists\mathbb{R}$-complete problems was posed there as an open problem. In this work, we deliver an extensive catalog of $\exists\mathbb{R}$-complete decision problems about Nash equilibria in 3-player games, thus resolving completely the open problem from [Garg et al., *Proceedings of ICALP* 2015]. Towards this end, we present a single and very simple, unifying reduction from the $\exists\mathbb{R}$-complete decision problem from [Schaefer and Štefankovič, *Theory of Computing Systems, 2015*] to (almost) *all* the decision problems about Nash equilibria that were before shown $\mathcal{NP}$-complete for 2-player games in [Bilò and Mavronicolas, *Proceedings of SAGT* 2012; Conitzer and Sandholm, *Games and Economic Behavior*, 2008; Gilboa and Zemel, *Games and Economic Behavior*, 1989]. Encompassed in the catalog are the four decision problems shown $\exists\mathbb{R}$-complete in [Garg et al., *Proceedings of ICALP* 2015].

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1 Introduction

1.1 Framework, Motivation and Contribution

The *Existential Theory of the Reals,* denoted as ETR, is the set of existential first-order sentences over the real numbers. In 1948, Alfred Tarski used his method of quantifier elimination [18] to show that the entire *First Order Theory of the Reals,* encompassing ETR, is decidable, albeit without an elementary bound on its complexity. To date the best known upper bound to decide ETR is $\mathcal{PSPACE}$, coming from the seminal work of Canny [4].Many geometric, graph-drawing and topological problems have been recognized to have the same complexity as ETR. Some of them concern recognizing *intersection graphs* of a certain type – see, e.g., [15, 16]; others concern deciding the stretchability of pseudolines [11]: given a family of plane curves, are they homeomorphic to a line arrangement? Based on these, the

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complexity class $\exists\mathbb{R}$ was defined by Schaefer and Štefankovič [17] as the set of problems with a polynomial-time, many-to-one reduction to ETR. Decision variants of fixed-point problems, including the *Brouwer fixed-point* problem and *Nash equilibria,* were shown $\exists\mathbb{R}$-complete in [17]; Nash equilibrium [12, 13] is undoubtedly the most influential solution concept in Game Theory, representing a state of a *game* where no *player* could unilaterally switch her *strategy* to improve her *payoff.* Both *search* and *decision problems* about Nash equilibria have been studied extensively in *Algorithmic Game Theory*; by the seminal results in [5, 7], their search problem is $\mathcal{PPAD}$-complete [14] even for 2-player games.

More specifically, Schaefer and Štefankovič [17, Corollary 3.5] identified the *first* $\exists\mathbb{R}$-complete decision problem about Nash equilibria in multi-player games: this is $\exists$ NASH IN A BALL, which asks, given an r*-player game* with $r \geq 3$ and a rational ϱ, whether or not it has a Nash equilibrium with no probability exceeding ϱ. The proof employed a reduction from BROUWER, another decision problem shown $\exists\mathbb{R}$-complete in [17], which asks whether or not a function, represented by a given straight-line program, has a fixed point in a specified ball. Very recently, Garg *et al.* [9] used a chain of *problem-specific* reductions, starting from $\exists$ NASH IN A BALL [17], to prove that four among the $\mathcal{NP}$-complete problems for 2-player games [1, 6, 10] are $\exists\mathbb{R}$-complete for r-player games with $r \geq 3$. Garg *et al.* [9, Appendix H] posed as an open problem the enlargement of the class of such $\exists\mathbb{R}$-complete problems.

A full story is known for decision problems about Nash equilibria for 2-player games; they are $\mathcal{NP}$-complete; see [1, 6, 10] for an extensive catalog. Their membership in $\mathcal{NP}$ is due to the fact that the Nash equilibria for a 2-player game involve rational probabilities; this allows, given the supports, polynomial time verification of the Nash equilibrium property. This is no longer the case for r-player games with $r \geq 3$, which may have Nash equilibria with irrational probabilities. Hence, these decision problems are only known to be $\mathcal{NP}$-hard over r-player games with $r \geq 3$, and their precise complexity characterization has remained elusive. (Two notable exceptions are the problems of deciding the existence of a *rational* Nash equilibrium [2] and a *uniform* Nash equilibrium [3], which belong to $\mathcal{NP}$ for r-player games with $r \geq 3$, and this finalizes their complexity classification.) *In this work, we show that they are (almost) all $\exists\mathbb{R}$-complete, delivering an extended catalog of $\exists\mathbb{R}$-complete decision problems about Nash equilibria for r-player games with* $r \geq 3$ (Theorem 10).

1.2 Techniques and Significance

We employ a *game reduction* (Section 3) that maps, given an arbitrary number $\delta > 0$, a pair of 3-player games $\widetilde{\mathsf{G}}$ and $\widehat{\mathsf{G}}$, called the *subgames,* to a 3-player game G with a larger set of strategies for each player; both games $\widetilde{\mathsf{G}}$ (with δ added to each utility) are "embedded" in G as subgames. The reduction guarantees certain correspondences between the Nash equilibria for $\widetilde{\mathsf{G}}$ and $\widehat{\mathsf{G}}$, respectively, and those for G. Specifically, a Nash equilibrium for G subsumes either a Nash equilibrium for $\widetilde{\mathsf{G}}$ or one for $\widehat{\mathsf{G}}$ (Lemma 7); in the other direction, a Nash equilibrium for $\widehat{\mathsf{G}}$ always induces one for G (Lemma 8); but a Nash equilibrium for $\widetilde{\mathsf{G}}$ induces one for G if and only if none of its probabilities exceeds $\frac{1}{2}$ (Lemma 9).

We proceed to embed the game reduction into a polynomial time many-to-one reduction from $\exists$ NASH IN A BALL to a catalog of decision problems about Nash equilibria for r-player games with $r \geq 3$, thus establishing their $\exists\mathbb{R}$-hardness (Section 4). We are given an instance $\widetilde{\mathsf{G}}$ of $\exists$ NASH IN A BALL, called the *inbox game.* We construct a game $\widehat{\mathsf{G}}$, called the *gadget game,* which may depend on $\widetilde{\mathsf{G}}$. Finally, we apply the game reduction on $\widetilde{\mathsf{G}}$ and $\widehat{\mathsf{G}}$ to get the game G. The correspondences between the Nash equilibria for $\widetilde{\mathsf{G}}$ and $\widehat{\mathsf{G}}$, respectively, and those for G are used to deduce the properties of the Nash equilibria for G, which are found to depend on whether or not the inbox game $\widetilde{\mathsf{G}}$ is a positive instance for $\exists$ NASH IN A

BALL. The established equivalence between $\widetilde{\mathsf{G}}$ being a positive instance for ∃ NASH IN A BALL and the induced properties of G imply the ∃ℝ-hardness of the properties. The *single, unifying reduction* we employ to establish the ∃ℝ-hardness of all decision problems in the catalog (Sections 3 and 4) is extremely simple, as well as its corresponding proof; thus, it simplifies tremendously the corresponding chain of (problem-specific) reductions in [9], which had involved proofs but only yielded *four* ∃ℝ-hard problems, which are encompassed in the catalog we present. The catalog includes (almost) *all* the decision problems about Nash equilibria for 2-player games shown $\mathcal{NP}$-complete in [1, 6, 10].

2 Background and Preliminaries

2.1 The Class ∃ℝ

The *Existential Theory of the Reals,* denoted as ETR, is the set of true sentences of the form $(\exists x_1, \ldots, x_n)(\varphi(x_1, \ldots, x_n)$, where φ is a quantifier-free $(\vee, \wedge, \neg)$-boolean formula over the signature $(0, 1, +, *, <, \leq, =)$ interpreted over the real numbers. ∃ℝ is the complexity class associated with ETR: A decision problem *belongs to* ∃ℝ if there is a polynomial-time, many-to-one reduction from it to ETR, and it is ∃ℝ-*hard* if there is a polynomial-time many-to-one reduction from each problem in ∃ℝ to it; it is ∃ℝ-*complete* if it belongs to ∃ℝ and it is ∃ℝ-hard. Since satisfiability of a propositional boolean formula (SAT) can be expressed in ETR, $\mathcal{NP} \subseteq \exists\mathbb{R}$; so, ETR is for ∃ℝ what SAT is for $\mathcal{NP}$. Thus, an ∃ℝ-complete problem is decided in $\mathcal{NP}$ if and only if ETR in $\mathcal{NP}$. By Canny's result [4], $\exists\mathbb{R} \subseteq \mathcal{PSPACE}$. We refer the reader to [17] for more background on the class ∃ℝ.

2.2 Games and Nash Equilibria

A *game* is a triple $\mathsf{G} = \left\langle [r], \{\Sigma_i\}_{i\in[r]}, \{\mathsf{U}_i\}_{i\in[r]} \right\rangle$, where *(i)* $[r] = \{1, \ldots, r\}$ is a finite set of *players* with $r \geq 2$, and *(ii)* for each player $i \in [r]$, Σ_i is the set of *strategies* for player i, and U_i is the *payoff function* $\mathsf{U}_i : \times_{k\in[r]}\Sigma_k \to \mathbb{R}$ for player i. Denote as $\underline{u}(\mathsf{G}) = \min_{\mathbf{s}\in\mathbf{\Sigma}, i\in[r]} \{\mathsf{U}_i(\mathbf{s})\}$ the *minimum payoff* for G. The game G is *win-lose* if for each player $i \in [r]$, U_i is a function $\mathsf{U}_i : \times_{k\in[r]}\Sigma_k \to \{0, 1\}$. For each player $i \in [r]$, denote $\mathbf{\Sigma}_{-i} = \times_{k\in[r]\setminus\{i\}}\Sigma_k$; denote $\mathbf{\Sigma} = \times_{k\in[r]}\Sigma_k$. A *profile* is a tuple $\mathbf{s} \in \mathbf{\Sigma}$ of r strategies, one per player. The vector $\mathsf{U}(\mathbf{s}) = \langle \mathsf{U}_1(\mathbf{s}), \ldots, \mathsf{U}_r(\mathbf{s})\rangle$ is the *payoff vector* for $\mathbf{s}$. A *partial profile* $\mathbf{s}_{-i}$ is a tuple of $r-1$ strategies, one for each player other than i; so $\mathbf{s}_{-i} \in \mathbf{\Sigma}_{-i}$. For a profile $\mathbf{s}$ and a strategy $t \in \Sigma_i$, denote as $\mathbf{s}_{-i} \diamond t$ the profile obtained by substituting strategy t for s_i in $\mathbf{s}$. The game G has the *positive payoff property* [1] if for each player $i \in [r]$ and each partial profile $\mathbf{s}_{-i} \in \mathbf{\Sigma}_{-i}$, there is a strategy $t = t(\mathbf{s}_{-i})$ such that $\mathsf{U}_i(\mathbf{s}_{-i} \diamond t) > 0$.

A *mixed strategy* for player $i \in [r]$ is a probability distribution σ_i on her strategy set Σ_i: a function $\sigma_i : \Sigma_i \to [0, 1]$ such that $\sum_{s\in\Sigma_i} \sigma_i(s) = 1$. Denote as $\mathsf{Support}(\sigma_i)$ the set of strategies $s \in \Sigma_i$ with $\sigma_i(s) > 0$. A *mixed profile* $\boldsymbol{\sigma} = \{\sigma_i\}_{i\in[r]}$ is a tuple of mixed strategies, one per player. So, a profile is the degenerate case of a mixed profile where all probabilities are either 0 or 1. A *partial mixed profile* $\boldsymbol{\sigma}_{-i}$ is a tuple of $r-1$ mixed strategies, one per player other than i. For a mixed profile $\boldsymbol{\sigma}$ and a mixed strategy τ_i of player $i \in [r]$, denote as $\boldsymbol{\sigma}_{-i} \diamond \tau_i$ the mixed profile obtained by substituting τ_i for σ_i in the mixed profile $\boldsymbol{\sigma}$.

A mixed profile $\boldsymbol{\sigma}$ induces a probability measure $\mathbb{P}_{\boldsymbol{\sigma}}$ on $\mathbf{\Sigma}$ in the natural way; so, for a profile $\mathbf{s}$, $\mathbb{P}_{\boldsymbol{\sigma}}(\mathbf{s}) = \prod_{k\in[r]} \sigma_k(s_k)$. Say that the profile $\mathbf{s} \in \mathbf{\Sigma}$ is *supported* in the mixed profile $\boldsymbol{\sigma}$ if $\mathbb{P}_{\boldsymbol{\sigma}}(\mathbf{s}) > 0$. Under the mixed profile $\boldsymbol{\sigma}$, the payoff of each player becomes a random variable. So, associated with $\boldsymbol{\sigma}$ is the *expected payoff* for each player $i \in [r]$, denoted as $\mathsf{U}_i(\boldsymbol{\sigma})$, which is the expectation according to $\mathbb{P}_{\boldsymbol{\sigma}}$ of her payoff; so, clearly, $\mathsf{U}_i(\boldsymbol{\sigma}) = \sum_{\mathbf{s}\in\Sigma} \left(\prod_{k\in[r]} \sigma_k(s_k)\right) \cdot \mathsf{U}_i(\mathbf{s})$.

A *pure Nash equilibrium* is a profile $\mathbf{s} \in \Sigma$ such that for each player $i \in [r]$ and for each strategy $t \in \Sigma_i$, $\mathsf{U}_i(\mathbf{s}) \geq \mathsf{U}_i(\mathbf{s}_{-i} \diamond t)$. A *mixed Nash equilibrium,* or *Nash equilibrium* for short, is a mixed profile $\boldsymbol{\sigma}$ such that for each player $i \in [r]$ and for each mixed strategy τ_i, $\mathsf{U}_i(\boldsymbol{\sigma}) \geq \mathsf{U}_i(\boldsymbol{\sigma}_{-i} \diamond \tau_i)$. A Nash equilibrium $\boldsymbol{\sigma}$ is *fully mixed* if $\mathsf{Support}(\sigma_i) = \Sigma_i$ for each player $i \in [r]$. Denote as $\mathcal{NE}(\mathsf{G})$ the set of Nash equilibria for G. We shall make extensive use of the following characterization of Nash equilibria.

▶ **Lemma 1.** *A mixed profile $\boldsymbol{\sigma}$ is a Nash equilibrium if and only if for each player $i \in [r]$,* (1) *for each strategy $t \in \mathsf{Support}(\sigma_i)$, $\mathsf{U}_i(\boldsymbol{\sigma}) = \mathsf{U}_i(\boldsymbol{\sigma}_{-i} \diamond t)$, and* (2) *for each strategy $t \notin \mathsf{Support}(\sigma_i)$, $\mathsf{U}_i(\boldsymbol{\sigma}) \geq \mathsf{U}_i(\boldsymbol{\sigma}_{-i} \diamond t)$.*

We also recall a simple technical fact from [1, Lemma 2.2].

▶ **Lemma 2.** *Fix a win-lose game G with the positive payoff property. Then, in a Nash equilibrium $\boldsymbol{\sigma}$, for each player $i \in [r]$, $\mathsf{U}_i(\boldsymbol{\sigma}) > 0$.*

For an arbitrary number δ, we denote as $\mathsf{G} + \delta$ the game obtained from G by adding δ to each possible value of the payoff function. We recall a very simple, well-known fact:

▶ **Lemma 3.** *Consider the r-player games G and $\widehat{\mathsf{G}} = \mathsf{G} + \delta$, for some number δ. Then, $\mathcal{NE}(\mathsf{G}) = \mathcal{NE}(\widehat{\mathsf{G}})$. Moreover, for every Nash equilibrium $\boldsymbol{\sigma} \in \mathcal{NE}(\widehat{\mathsf{G}})$ and player $i \in [r]$, $\widehat{\mathsf{U}}_i(\boldsymbol{\sigma}) = \mathsf{U}_i(\boldsymbol{\sigma}) + \delta$.*

2.3 Decision Problems about Nash Equilibria

Here are the formal statements of the decision problems we shall consider, in the style of Garey and Johnson [8], where I. and Q. stand for Instance and Question, respectively.

∃ NASH IN A BALL

I.:	A game G and a rational number $k \in (0, 1)$.
Q.:	Is there a Nash equilibrium $\boldsymbol{\sigma}$ such that for each player $i \in [r]$, $\max_{s\in\Sigma_i} \sigma_i(s) \leq k$?

∃ SECOND NASH

I.:	A game G.
Q.:	Is there a second Nash equilibrium?

∃ NASH WITH LARGE PAYOFFS

I.:	A game G and a number u.
Q.:	Is there a Nash equilibrium $\boldsymbol{\sigma}$ such that for each player $i \in [r]$, $\mathsf{U}_i(\boldsymbol{\sigma}) \geq u$?

∃ NASH WITH SMALL PAYOFFS

I.:	A game G and a number u.
Q.:	Is there a Nash equilibrium $\boldsymbol{\sigma}$ such that for each player $i \in [r]$, $\mathsf{U}_i(\boldsymbol{\sigma}) \leq u$?

∃ NASH WITH LARGE TOTAL PAYOFF

I.:	A game G and a number u.
Q.:	Is there a Nash equilibrium $\boldsymbol{\sigma}$ such that $\sum_{i\in[r]} \mathsf{U}_i(\boldsymbol{\sigma}) \geq u$?

∃ NASH WITH SMALL TOTAL PAYOFF

I.:	A game G and a number u.
Q.:	Is there a Nash equilibrium $\boldsymbol{\sigma}$ such that $\sum_{i\in[r]} \mathsf{U}_i(\boldsymbol{\sigma}) \leq u$?

∃ NASH WITH LARGE SUPPORTS

I.:	A game G and an integer $k \geq 1$.
Q.:	Is there a Nash equilibrium $\boldsymbol{\sigma}$ such that for each player $i \in [r]$, $\lvert\mathsf{Support}(\sigma_i)\rvert \geq k$?

∃ NASH WITH SMALL SUPPORTS

I.:	A game G and an integer $k \geq 1$.
Q.:	Is there a Nash equilibrium $\boldsymbol{\sigma}$ such that for each player $i \in [r]$, $\lvert\mathsf{Support}(\sigma_i)\rvert \leq k$?

∃ NASH WITH RESTRICTING SUPPORTS

I.:	A game G and a subset of strategies $\mathsf{T}_i \subseteq \Sigma_i$ for each player $i \in [r]$.
Q.:	Is there a Nash equilibrium $\boldsymbol{\sigma}$ such that for each player $i \in [r]$, $\mathsf{T}_i \subseteq \mathsf{Support}(\sigma_i)$?

∃ NASH WITH RESTRICTED SUPPORTS

I.:	A game G and a subset of strategies $\mathsf{T}_i \subseteq \Sigma_i$ for each player $i \in [r]$.
Q.:	Is there a Nash equilibrium $\boldsymbol{\sigma}$ such that for each player $i \in [r]$, $\mathsf{Support}(\sigma_i) \subseteq \mathsf{T}_i$?

Given two mixed profiles $\boldsymbol{\sigma}$ and $\widehat{\boldsymbol{\sigma}}$, denote as $\mathsf{Diff}\,(\boldsymbol{\sigma}, \widehat{\boldsymbol{\sigma}}) := \{i \in [r] : \sigma_i \neq \widehat{\sigma}_i\}$ the set of players with different mixed strategies in $\boldsymbol{\sigma}$ and $\widehat{\boldsymbol{\sigma}}$. A Nash equilibrium $\boldsymbol{\sigma}$ is *Strongly Pareto-Optimal* if for each mixed profile $\widehat{\boldsymbol{\sigma}}$ where there is player $i \in [r]$ with $\mathsf{U}_i(\widehat{\boldsymbol{\sigma}}) > \mathsf{U}_i(\boldsymbol{\sigma})$ for some player $i \in [r]$, there is a player $j \in \mathsf{Diff}(\boldsymbol{\sigma}, \widehat{\boldsymbol{\sigma}})$ such that $\mathsf{U}_j(\widehat{\boldsymbol{\sigma}}) \leq \mathsf{U}_j(\boldsymbol{\sigma})$; so, there is no other profile where at least one player is strictly better off and every player using a different strategy is strictly better off. We have two additional decision problems.

∃ NON-PARETO-OPTIMAL NASH

I.:	A game G.
Q.:	Is there a Nash equilibrium which is not Pareto-Optimal?

∃ NON-STRONGLY PARETO-OPTIMAL NASH

I.:	A game G.
Q.:	Is there a Nash equilibrium which is not Strongly Pareto-Optimal?

Restricted to 2-player games with rational utilities, all these problems are $\mathcal{NP}$-complete. ∃ NASH IN A BALL was the first problem shown $\exists\mathbb{R}$-complete. The problems ∃ SECOND NASH, ∃ NASH WITH LARGE PAYOFFS, ∃ NASH WITH RESTRICTING SUPPORTS and ∃ NASH WITH RESTRICTED SUPPORTS are $\exists\mathbb{R}$-complete for r-player games with $r \geq 3$ [9].

2.4 The Game $\widehat{\mathsf{G}}[m]$

For an integer $m \geq 2$, define the 3-player win-lose game $\widehat{\mathsf{G}}[m]$ as follows: For each player $i \in [3]$, $\widehat{\Sigma}_i = [m]$, and the strategy sets are *cyclic* so that strategy 0 coincides with strategy m; the payoff functions are: (1) $\widehat{\mathsf{U}}_1(\mathbf{s}) = 1$ if and only if $s_1 = s_2$; (2) $\widehat{\mathsf{U}}_2(\mathbf{s}) = 1$ if and only if $s_2 = s_3 - 1$; (3) $\widehat{\mathsf{U}}_3(\mathbf{s}) = 1$ if and only if $s_3 = s_1 - 1$. Note that $\widehat{\mathsf{G}}[m]$ has the positive payoff property. We prove:

▶ **Lemma 4.** *Fix an odd integer $m \geq 3$. Then, $\widehat{\mathsf{G}}[m]$ has a unique Nash equilibrium $\boldsymbol{\sigma}$, which is fully mixed and has $\widehat{\mathsf{U}}_i(\boldsymbol{\sigma}) = \frac{1}{m}$ for each player $i \in [3]$.*

Proof. Fix a Nash equilibrium $\boldsymbol{\sigma}$ for $\widehat{\mathsf{G}}[m]$. To prove that $\boldsymbol{\sigma}$ is fully mixed, assume, by way of contradiction, that for a strategy $j \in [m]$, $\sigma_1(j) = 0$. This implies that $\sigma_3(j-1) = 0$ since otherwise ($\sigma_3(j-1) > 0$), Lemma 1 (Condition (1)) implies that $\widehat{\mathsf{U}}_3(\boldsymbol{\sigma}) = \widehat{\mathsf{U}}_3(\boldsymbol{\sigma}_{-3} \diamond (j-1)) = 0$, which contradicts Lemma 2. This implies that $\sigma_2(j-2) = 0$ since otherwise ($\sigma_2(j-2) > 0$), Lemma 1 (Condition (1)) implies that $\widehat{\mathsf{U}}_2(\boldsymbol{\sigma}) = \widehat{\mathsf{U}}_2(\boldsymbol{\sigma}_{-2} \diamond (j-2)) = 0$, which contradicts Lemma 2. This implies that $\sigma_1(j-2) = 0$ since otherwise ($\sigma_1(j-2) > 0$), Lemma 1 (Condition (1)) implies that $\widehat{\mathsf{U}}_1(\boldsymbol{\sigma}) = \widehat{\mathsf{U}}_1(\boldsymbol{\sigma}_{-1} \diamond (j-2)) = 0$, which contradicts Lemma 2. Since m is odd, a repeated application of the implication yields that $\sigma_1(1) = \ldots = \sigma_1(m) = 0$, $\sigma_2(1) = \ldots = \sigma_2(m) = 0$ and $\sigma_3(1) = \ldots = \sigma_3(m) = 0$. A contradiction. Hence, for each player $i \in [3]$, for each strategy $j \in [m]$, $\sigma_i(j) > 0$, or $\boldsymbol{\sigma}$ is fully mixed. By Lemma 1 (Condition (1)), this implies that for each player $i \in [3]$, for each strategy $j \in [m]$, $\widehat{\mathsf{U}}_i(\boldsymbol{\sigma}) = \widehat{\mathsf{U}}_i(\boldsymbol{\sigma}_{-i} \diamond j)$. By the definition of the payoff functions, for each strategy $j \in [m]$, $\widehat{\mathsf{U}}_1(\boldsymbol{\sigma}_{-1} \diamond j) = \sigma_2(j)$. $\widehat{\mathsf{U}}_2(\boldsymbol{\sigma}_{-2} \diamond j) = \sigma_3(j+1)$ and $\widehat{\mathsf{U}}_3(\boldsymbol{\sigma}_{-3} \diamond j) = \sigma_1(j+1)$. Hence, it follows that for each player $i \in [3]$ and for each strategy $j \in [m]$, $\sigma_i(j)$ is independent of j, which implies that $\sigma_i(j) = \frac{1}{m}$, so that $\boldsymbol{\sigma}$ is unique with $\mathsf{U}_i(\boldsymbol{\sigma}) = \frac{1}{m}$ for each player $i \in [3]$. ◀

3 The Game Reduction

Fix an arbitrary number $\delta > 0$. The game reduction takes as input a pair of games:

- A 3-player game $\widetilde{\mathsf{G}}$ with $\Sigma_i = [n]$ for each player $i \in [3]$, the *inbox game*.
- A 3-player game $\widehat{\mathsf{G}}$ with $\Sigma_i = [m]$ for each player $i \in [3]$, with $m \geq n$, the *gadget game*.

The game reduction constructs a 3-player game $\mathsf{G} = \mathsf{G}\left(\widetilde{\mathsf{G}}, \widehat{\mathsf{G}}\right)$; $\widetilde{\mathsf{G}}$ and $\widehat{\mathsf{G}}$ are the *subgames*.

3.1 Definition and Some Notation

Set $\phi^* := \min\left\{\underline{u}(\widetilde{\mathsf{G}}), \underline{u}(\widehat{\mathsf{G}})\right\} - 1$ and $\phi := \phi^* - 1$. We construct the game G as follows:

- For each player $i \in [r]$, $\Sigma_i = [p]$, with $p = n(n+1)+m$; $[p]$ is partitioned into $n+2$ blocks $\mathsf{B}_0, \mathsf{B}_1, \ldots, \mathsf{B}_{n+1}$, where for each index h with $0 \leq h \leq n$, $\mathsf{B}_h := \{hn+1, \ldots, (h+1)n\}$ and $\mathsf{B}_{n+1} := \{n(n+1)+1, \ldots, n(n+1)+m\}$; thus, $|\mathsf{B}_{n+1}| = m$, while $|\mathsf{B}_h| = n$ for each index h with $0 \leq h \leq n$. For each index h with $0 \leq h \leq n+1$ and $k \in [|\mathsf{B}_h|]$, denote as $\mathsf{B}_h(k)$ the order k strategy in B_h; thus, $\mathsf{B}_h(k)$ is the order $(hn+k)$ strategy in Σ_i.
- The payoff functions for G are given in Figure 1.

Clearly, G is constructed in time polynomial in the sizes of $\widetilde{\mathsf{G}}$ and $\widehat{\mathsf{G}}$. Note that by Case (1), $\widetilde{\mathsf{G}} + \delta$ is a subgame of G; by Case (2), $\widehat{\mathsf{G}}$ is a subgame of G. So, the blocks B_0 and B_{n+1} correspond to the input games $\widetilde{\mathsf{G}}$ and $\widehat{\mathsf{G}}$, respectively.

For an n-dimensional vector $\boldsymbol{x} \in \mathbb{R}^n$, denote as $\overrightarrow{\boldsymbol{x}} \in \mathbb{R}^p$ the p-dimensional vector with $\overrightarrow{x}_j = x_j$ for $j \in [n]$ and $\overrightarrow{x}_j = 0$ for $n < j \leq p$. Similarly, for an m-dimensional vector $\boldsymbol{x} \in \mathbb{R}^m$, denote as $\overleftarrow{\boldsymbol{x}} \in \mathbb{R}^p$ the p-dimensional vector with $\overleftarrow{x}_j = 0$ for $i \in [n(n+1)]$ and $\overleftarrow{x}_j = x_j$ for $n(n+1) < j \leq p$. Hence, for a mixed profile $\boldsymbol{\sigma} \in \widetilde{\Sigma}$, $(\overrightarrow{\sigma_1}, \overrightarrow{\sigma_2}, \overrightarrow{\sigma_3}) \in \Sigma$; for a mixed profile $\boldsymbol{\sigma} \in \widehat{\Sigma}$, $(\overleftarrow{\sigma_1}, \overleftarrow{\sigma_2}, \overleftarrow{\sigma_3}) \in \Sigma$. For a given multidimensional space, $\mathcal{B}_\varrho$ denote the ball of radius ϱ. For a mixed profile $\boldsymbol{\sigma}$, we write $\boldsymbol{\sigma} \in \mathcal{B}_\varrho$ when $\sigma_i \in \mathcal{B}_\varrho$ for each player $i \in [3]$.

Case	Condition on the profile $\mathbf{s}$	Payoff vector $\mathsf{U}(\mathbf{s})$
(1)	$s_i \in \mathsf{B}_0$ for each player $i \in [3]$	$\left\langle \widetilde{\mathsf{U}}_1(\mathbf{s}) + \delta, \widetilde{\mathsf{U}}_2(\mathbf{s}) + \delta, \widetilde{\mathsf{U}}_3(\mathbf{s}) + \delta \right\rangle$
(2)	$s_i \in \mathsf{B}_{n+1}$ for each player $i \in [3]$	$\left\langle \widehat{\mathsf{U}}_1(\mathbf{s}), \widehat{\mathsf{U}}_2(\mathbf{s}), \widehat{\mathsf{U}}_3(\mathbf{s}) \right\rangle$
(3)	$s_i = \mathsf{B}_{n+1}(k)$ with $k \in [n]$ & $s_j \in \mathsf{B}_0$ for $j \neq i$	$\mathsf{U}_i(\mathbf{s}) = \mathsf{U}_i(\mathbf{s}_{-i} \diamond k)$ $\mathsf{U}_j(\mathbf{s}) = \phi$ for $j \neq i$
(4)	$s_i = \mathsf{B}_h(k)$ with $h, k \in [n]$ & $s_j \in \mathsf{B}_0$ for $j \neq i$ with $s_{i+1} = h$	$\mathsf{U}_i(\mathbf{s}) = \widetilde{\mathsf{U}}_1(\mathbf{s}_{-i} \diamond k) + 2\delta$ $\mathsf{U}_j(\mathbf{s}) = \phi$ for $j \neq i$
(5)	$s_i = \mathsf{B}_h(k)$ with $h, k \in [n]$ & $s_j \in \mathsf{B}_0$ for $j \neq i$ with $s_{i+1} \neq h$	$\mathsf{U}_i(\mathbf{s}) = \widetilde{\mathsf{U}}_i(\mathbf{s}_{-i} \diamond k)$ $\mathsf{U}_j(\mathbf{s}) = \phi$ for $j \neq i$
(6)	$\mathsf{P}(\mathbf{s}) \neq \emptyset$ & $\mathsf{P}(\mathbf{s}) \neq [3]$, with $\mathbf{s}$ not falling in Case (3)	$\mathsf{U}_i(\mathbf{s}) = \phi^*$ if $i \in \mathsf{P}(\mathbf{s})$ $\mathsf{U}_i(\mathbf{s}) = \phi$, if $i \notin \mathsf{P}(\mathbf{s})$
(7)	None of the above	$\langle \phi, \phi, \phi \rangle$

Figure 1 The payoff functions for the game G. Here $\mathsf{P}(\mathbf{s}) := \{i \in [3] \mid s_i \in \mathsf{B}_{n+1}\}$, the set of players choosing strategies from the block B_{n+1} in the profile $\mathbf{s}$.

3.2 Correspondences Between Nash Equilibria

We now establish certain correspondences between the Nash equilibria for the subgames $\widetilde{\mathsf{G}}$ and $\widehat{\mathsf{G}}$, respectively, and the Nash equilibria for the constructed game G.

3.2.1 Backward Correspondence: From the Game G to the Subgames

We shall prove that a Nash equilibrium for G is induced by a Nash equilibrium for either $\widetilde{\mathsf{G}}$ or $\widehat{\mathsf{G}}$. We start with two technical claims about a Nash equilibrium for G (Lemmas 5 and 6). We first prove that if some player is playing some strategy outside B_0, then none of the other two players is playing a strategy in B_0.

▶ **Lemma 5.** *Fix a Nash equilibrium $\boldsymbol{\sigma} \in \mathcal{NE}(\mathsf{G})$ for which there is a player i' such that $\mathsf{Support}(\sigma_{i'}) \setminus \mathsf{B}_0 \neq \emptyset$. Then, for every player $i \neq i'$, $\mathsf{Support}(\sigma_i) \cap \mathsf{B}_0 = \emptyset$.*

Proof. Assume, by way of contradiction, that there is a player $i \neq i'$ with $\mathsf{Support}(\sigma_i) \cap \mathsf{B}_0 \neq \emptyset$. Choose an arbitrary strategy $k \in \mathsf{Support}(\sigma_i) \cap \mathsf{B}_0$. Since $k \in \mathsf{Support}(\sigma_i)$, Lemma 1 (Condition (1)) implies that $\mathsf{U}_i(\boldsymbol{\sigma}) = \sum_{\mathbf{s}_{-i} \in \boldsymbol{\Sigma}_{-i}} \mathsf{U}_i(\mathbf{s}_{-i} \diamond k) \cdot \mathbb{P}_{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_{-i})$. Lemma 1 (Condition (2)) implies that

$$\mathsf{U}_i(\boldsymbol{\sigma}_{-i} \diamond \mathsf{B}_{n+1}(k)) \quad = \quad \sum_{\mathbf{s}_{-i} \in \boldsymbol{\Sigma}_{-i}} \mathsf{U}_i(\mathbf{s}_{-i} \diamond \mathsf{B}_{n+1}(k)) \cdot \mathbb{P}_{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_{-i}) \quad \leq \quad \mathsf{U}_i(\boldsymbol{\sigma})\,.$$

Fix a partial profile $\mathbf{s}_{-i} \in \boldsymbol{\Sigma}_{-i}$. There are six possible cases for the two players other than i.

1. *Both players choose a strategy from B_0.* Then, $\mathbf{s}_{-i} \diamond k$ falls into Case (1) of the payoff table, while $\mathbf{s}_{-i} \diamond \mathsf{B}_{n+1}(k)$ falls into Case (3), so that $\mathsf{U}_i(\mathbf{s}_{-i} \diamond k) = \mathsf{U}_i(\mathbf{s}_{-i} \diamond \mathsf{B}_{n+1}(k))$.
2. *Both players choose a strategy from B_{n+1}.* Then, $\mathbf{s}_{-i} \diamond k$ falls into Case (6) of the payoff table with $\mathsf{U}_i(\mathbf{s}_{-i} \diamond k) = \phi$, while $\mathbf{s}_{-i} \diamond \mathsf{B}_{n+1}(k)$ falls into Case (2) with $\mathsf{U}_i(\mathbf{s}_{-i} \diamond \mathsf{B}_{n+1}(k)) \geq \underline{u}(\widehat{\mathsf{G}}) > \phi$.
3. *Both players choose a strategy outside $\mathsf{B}_0 \cup \mathsf{B}_{n+1}$.* Then, $\mathbf{s}_{-i} \diamond k$ falls into Case (7) of the payoff table with $\mathsf{U}_i(\mathbf{s}_{-i} \diamond k) = \phi$, while $\mathbf{s}_{-i} \diamond \mathsf{B}_{n+1}(k)$ falls into Case (6) with $\mathsf{U}_i(\mathbf{s}_{-i} \diamond \mathsf{B}_{n+1}(k)) = \phi^* > \phi$.
4. *One player chooses a strategy from B_0 and the other chooses one from B_{n+1}.* Then, $\mathbf{s}_{-i} \diamond k$ falls into Case (3) of the payoff table with $\mathsf{U}_i(\mathbf{s}_{-i} \diamond k) = \phi$, while $\mathbf{s}_{-i} \diamond \mathsf{B}_{n+1}(k)$ falls into Case (6) with $\mathsf{U}_i(\mathbf{s}_{-i} \diamond \mathsf{B}_{n+1}(k)) = \phi^* > \phi$.

5. *One player chooses a strategy from* B_0 *and the other chooses one outside* $\mathsf{B}_0 \cup \mathsf{B}_{n+1}$*.* Then, $\mathbf{s}_{-i} \diamond k$ falls into either Case (4) or (5) of the payoff table with $\mathsf{U}_i(\mathbf{s}_{-i} \diamond k) = \phi$, while $\mathbf{s}_{-i} \diamond \mathsf{B}_{n+1}(k)$ falls into Case (6) with $\mathsf{U}_i(\mathbf{s}_{-i} \diamond \mathsf{B}_{n+1}(k)) = \phi^* > \phi$.
6. *One player chooses a strategy from* B_{n+1} *and the other chooses one outside* $\mathsf{B}_0 \cup \mathsf{B}_{n+1}$*.* Then, both $\mathbf{s}_{-i} \diamond k$ and $\mathbf{s}_{-i} \diamond \mathsf{B}_{n+1}(k)$ fall into Case (6) of the payoff table with $\mathsf{U}_i(\mathbf{s}_{-i} \diamond k) = \phi$ and $\mathsf{U}_i(\mathbf{s}_{-i} \diamond \mathsf{B}_{n+1}(k)) = \phi^* > \phi$.

Note that in the first of the six cases, player i has the same payoff for the two strategies k and $\mathsf{B}_{n+1}(k)$, while in all other cases, player i improves her payoff when switching to $\mathsf{B}_{n+1}(k)$. Hence, there is no profile supported in $\boldsymbol{\sigma}_{-i} \diamond \mathsf{B}_{n+1}(k)$ in which player i has a smaller payoff. The assumption implies that $\mathsf{Support}(\sigma_j) \setminus \mathsf{B}_0 \neq \emptyset$; it follows that there is at least one profile supported in $\boldsymbol{\sigma}_{-i} \diamond \mathsf{B}_{n+1}(k)$ for which player i has a larger payoff. A contradiction. ◀

We continue to prove that if some player is playing no strategy from B_0, then the other two players are playing only strategies from B_{n+1}.

▶ **Lemma 6.** *Fix a Nash equilibrium* $\boldsymbol{\sigma} \in \mathcal{NE}(\mathsf{G})$ *for which there is a player* i' *with* $\mathsf{Support}(\sigma_{i'}) \cap \mathsf{B}_0 = \emptyset$*. Then, for each player* $i \neq i'$, $\mathsf{Support}(\sigma_i) \setminus \mathsf{B}_{n+1} = \emptyset$*.*

Proof. Assume, by way of contradiction, that there is a player $i \neq i'$ with $\mathsf{Support}(\sigma_i) \setminus \mathsf{B}_{n+1} \neq \emptyset$. Choose an arbitrary strategy $k \in \mathsf{Support}(\sigma_i) \setminus \mathsf{B}_{n+1}$ and an arbitrary strategy $h \in \mathsf{B}_{n+1}$. Since $k \in \mathsf{Support}(\sigma_i)$, Lemma 1 (Condition (1)) implies that $\mathsf{U}_i(\boldsymbol{\sigma}) = \sum_{\mathbf{s}_{-i} \in \boldsymbol{\Sigma}_{-i}} \mathsf{U}_i(\mathbf{s}_{-i} \diamond k) \cdot \mathbb{P}_{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_{-i})$. Lemma 1 (Condition (2)) implies that

$$\mathsf{U}_i(\boldsymbol{\sigma}_{-i} \diamond h) \quad = \sum_{\mathbf{s}_{-i} \in \boldsymbol{\Sigma}_{-i}} \mathsf{U}_i(\mathbf{s}_{-i} \diamond h) \cdot \mathbb{P}_{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_{-i}) \quad \leq \quad \mathsf{U}_i(\boldsymbol{\sigma})\,.$$

Fix now a profile $\mathbf{s}_{-i} \in \Sigma_{-i}$. There are five possible cases for the two players other than i.

1. *Both players play a strategy from* B_{n+1}*.* Then, $\mathbf{s}_{-i} \diamond k$ falls into Case (6) of the payoff table with $\mathsf{U}_i(\mathbf{s}_{-i} \diamond k) = \phi$, while $\mathbf{s}_{-i} \diamond h$ falls into Case (2) with $\mathsf{U}_i(\mathbf{s}_{-i} \diamond h) \geq \underline{u}(\widehat{\mathsf{G}}) > \phi$.
2. *Both players choose a strategy outside* $\mathsf{B}_0 \cup \mathsf{B}_{n+1}$*.* Then, $\mathbf{s}_{-i} \diamond k$ falls into Case (7) of the payoff table with $\mathsf{U}_i(\mathbf{s}_{-i} \diamond k) = \phi$, while $\mathbf{s}_{-i} \diamond h$ falls into Case (6) with $\mathsf{U}_i(\mathbf{s}_{-i} \diamond h) = \phi^* > \phi$.
3. *One player chooses a strategy from* B_0 *and the other chooses one from* B_{n+1}*.* Then, $\mathbf{s}_{-i} \diamond k$ falls into either Case (3) or (7) of the payoff table with $\mathsf{U}_i(\mathbf{s}_{-i} \diamond k) = \phi$, while $\mathbf{s}_{-i} \diamond h$ falls into Case (6) with $\mathsf{U}_i(\mathbf{s}_{-i} \diamond h) = \phi^* > \phi$.
4. *One player chooses a strategy from* B_0 *and the other chooses one outside* $\mathsf{B}_0 \cup \mathsf{B}_{n+1}$*.* Then, $\mathbf{s}_{-i} \diamond k$ falls into either Case (4) or (5) or (7) of the payoff table with $\mathsf{U}_i(\mathbf{s}_{-i} \diamond k) = \phi$, while $\mathbf{s}_{-i} \diamond h$ falls into Case (6) with $\mathsf{U}_i(\mathbf{s}_{-i} \diamond h) = \phi^* > \phi$.
5. *One player chooses a strategy from* B_{n+1} *and the other chooses one outside* $\mathsf{B}_0 \cup \mathsf{B}_{n+1}$*.* Then, both $\mathbf{s}_{-i} \diamond k$ and $\mathbf{s}_{-i} \diamond h$ fall into Case (6) of the payoff table, so that $\mathsf{U}_i(\mathbf{s}_{-i} \diamond k) = \phi$ and $\mathsf{U}_i(\mathbf{s}_{-i} \diamond h) = \phi^* > \phi$.

Thus, in all profiles supported in $\boldsymbol{\sigma}_{-i} \diamond \mathsf{B}_{n+1}(k)$, player i improves her payoff by switching to strategy h. A contradiction. ◀

We are now ready to prove:

▶ **Lemma 7.** *Fix a Nash equilibrium* $\boldsymbol{\sigma} \in \mathcal{NE}(\mathsf{G})$*. Then, there are only two possible cases:*

- $\boldsymbol{\sigma} = (\overrightarrow{\tau_1}, \overrightarrow{\tau_2}, \overrightarrow{\tau_3})$ *for some Nash equilibrium* $\boldsymbol{\tau} \in \mathcal{NE}(\widetilde{\mathsf{G}})$*.*
- $\boldsymbol{\sigma} = (\overleftarrow{\tau_1}, \overleftarrow{\tau_2}, \overleftarrow{\tau_3})$ *for some Nash equilibrium* $\boldsymbol{\tau} \in \mathcal{NE}(\widehat{\mathsf{G}})$*.*

Proof. By Lemmas 5 and 6, either *(i)* $\sigma_1 = \overrightarrow{\tau_1}$, $\sigma_2 = \overrightarrow{\tau_2}$ and $\sigma_3 = \overrightarrow{\tau_3}$ for some mixed profile $\boldsymbol{\tau} \in \widetilde{\Sigma}$ or *(ii)* $\sigma_1 = \overleftarrow{\tau_1}$, $\sigma_2 = \overleftarrow{\tau_2}$, $\sigma_3 = \overleftarrow{\tau_3}$ for some mixed profile $\boldsymbol{\tau} \in \widehat{\Sigma}$. The two properties that $\boldsymbol{\tau} \in \mathcal{NE}(\widetilde{\mathsf{G}})$ and $\boldsymbol{\tau} \in \mathcal{NE}(\widehat{\mathsf{G}})$ follow from the facts that both $\widetilde{\mathsf{G}} + \delta$ and $\widehat{\mathsf{G}}$ are subgames of G, and from Lemma 3, by which $\widetilde{\mathsf{G}}$ and $\widetilde{\mathsf{G}} + \delta$ have the same set of Nash equilibria. ◀

3.2.2 Forward Correspondence: From the Subgames to the Game G

We now characterize the Nash equilibria for the two subgames $\widetilde{\mathsf{G}}$ and $\widehat{\mathsf{G}}$ that induce corresponding Nash equilibria for G. Specifically, these are all the Nash equilibria for $\widehat{\mathsf{G}}$ (Lemma 8) and every Nash equilibrium in $\mathcal{B}_{1/2}$ for $\widetilde{\mathsf{G}}$ (Lemma 9), respectively. We first prove:

▶ **Lemma 8.** *Fix a Nash equilibrium* $\boldsymbol{\sigma} \in \mathcal{NE}(\widehat{\mathsf{G}})$. *Then,* $(\overleftarrow{\sigma_1}, \overleftarrow{\sigma_2}, \overleftarrow{\sigma_3}) \in \mathcal{NE}(\mathsf{G})$.

Proof. By Case (2) of the payoff table, for each player $i \in [3]$, $\mathsf{U}_i\left(\overleftarrow{\sigma_1}, \overleftarrow{\sigma_2}, \overleftarrow{\sigma_3}\right) = \widehat{\mathsf{U}}_i(\boldsymbol{\sigma}) \geq \underline{u}(\widehat{\mathsf{G}})$. Since $\boldsymbol{\sigma} \in \mathcal{NE}(\widehat{\mathsf{G}})$, no player could improve her payoff by deviating to a strategy from B_{n+1}. A deviation to a strategy outside B_{n+1} gives rise to a mixed profile for which only profiles from Case (6) are supported, and the expected payoff of the deviating player is $\phi < \underline{u}(\widehat{\mathsf{G}})$. ◀

We continue to prove:

▶ **Lemma 9.** *Fix a Nash equilibrium* $\boldsymbol{\sigma} \in \mathcal{NE}(\widetilde{\mathsf{G}})$. *Then,* $(\overrightarrow{\sigma_1}, \overrightarrow{\sigma_2}, \overrightarrow{\sigma_3}) \in \mathcal{NE}(\mathsf{G})$ *if and only if* $\boldsymbol{\sigma} \in \mathcal{B}_{1/2}$.

Proof. Fix a player $i \in [3]$. By Lemma 1 (Condition (1)), for each strategy $j \in \mathsf{Support}(\sigma_i)$,

$$\begin{aligned}
\mathsf{U}_i\left(\overrightarrow{\sigma_1}, \overrightarrow{\sigma_2}, \overrightarrow{\sigma_3}\right) &= \sum_{\mathbf{s}_{-i} \in \boldsymbol{\Sigma}_{-i}} \mathsf{U}_i\left(\mathbf{s}_{-i} \diamond j\right) \cdot \mathbb{P}_{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_{-i}) \\
&= \sum_{\mathbf{s}_{-i} \in \widetilde{\boldsymbol{\Sigma}}_{-i}} \mathsf{U}_i(\mathbf{s}_{-i} \diamond j) \cdot \mathbb{P}_{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_{-i}) \\
&= \sum_{\mathbf{s}_{-i} \in \widetilde{\boldsymbol{\Sigma}}_{-i}} \left(\widetilde{\mathsf{U}}_i\left(\mathbf{s}_{-i} \diamond j\right) + \delta\right) \cdot \mathbb{P}_{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_{-i}) \\
&= \delta + \sum_{\mathbf{s}_{-i} \in \widetilde{\boldsymbol{\Sigma}}_{-i}} \widetilde{\mathsf{U}}_i\left(\mathbf{s}_{-i} \diamond j\right) \cdot \mathbb{P}_{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_{-i}) \\
&> \underline{u}(\widetilde{\mathsf{G}})\,.
\end{aligned}$$

By Lemma 3, $\mathcal{NE}(\widetilde{\mathsf{G}}) = \mathcal{NE}(\widetilde{\mathsf{G}} + \delta)$; thus, player i cannot improve her payoff by switching to a strategy from B_0. Also, by Case (3), player i cannot improve her payoff by switching to any of the first n strategies in B_{n+1}; by Case (6), player i cannot improve her payoff by switching to any of the last $m - n$ strategies in B_{n+1} since $\underline{u}(\widetilde{\mathsf{G}}) > \phi^*$. So, it remains to consider deviations to strategies outside $\mathsf{B}_0 \cup \mathsf{B}_{n+1}$. We proceed by case analysis.

1. Assume first that $\boldsymbol{\sigma} \notin \mathcal{B}_{1/2}$. Hence, there is a player $h = i + 1$ with $\sigma_h \notin \mathcal{B}_{1/2}$. (This assumption is without loss of generality since i is arbitrary.) Consider the third player $h' = i + 2$. Denote as $k \in \mathsf{B}_0$ the strategy with $\sigma_h(k) > \frac{1}{2}$. Consider player i switching to the strategy $\mathsf{B}_k(j)$. By Cases (4) and (5) of the payoff table, her expected payoff is

$$\begin{aligned}
&\sum_{\mathbf{s}_{-i} \in \boldsymbol{\Sigma}_{-i}} \mathsf{U}_i\left(\mathbf{s}_{-i} \diamond \mathsf{B}_k(j)\right) \cdot \mathbb{P}_{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_{-i}) \\
= &\sum_{\mathbf{s}_{-i} \in \widetilde{\boldsymbol{\Sigma}}_{-i}} \mathsf{U}_i\left(\mathbf{s}_{-i} \diamond \mathsf{B}_k(j)\right) \cdot \mathbb{P}_{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_{-i}) \\
= &\sum_{\mathbf{s}_{-i} \in \widetilde{\boldsymbol{\Sigma}}_{-i} : s_h = k} \left(\widetilde{\mathsf{U}}_i(\mathbf{s}_{-i} \diamond j) + 2\delta\right) \mathbb{P}_{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_{-i}) + \sum_{\mathbf{s}_{-i} \in \widetilde{\boldsymbol{\Sigma}}_{-i} : s_h \neq k} \widetilde{\mathsf{U}}_i(\mathbf{s}_{-i} \diamond j) \mathbb{P}_{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_{-i}) \\
= &\ 2\delta \sum_{\mathbf{s}_{-i} \in \widetilde{\boldsymbol{\Sigma}}_{-i} : s_h = k} \mathbb{P}_{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_{-i}) + \sum_{\mathbf{s}_{-i} \in \widetilde{\boldsymbol{\Sigma}}_{-i}} \widetilde{\mathsf{U}}_i(\mathbf{s}_{-i} \diamond j) \mathbb{P}_{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_{-i})
\end{aligned}$$

$$
\begin{aligned}
&> \delta + \sum_{\mathbf{s}_{-i} \in \widetilde{\Sigma}_{-i}} \widetilde{\mathsf{U}}_i(\mathbf{s}_{-i} \diamond j) \mathbb{P}_{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_{-i}) \\
&= \mathsf{U}_i \left(\overrightarrow{\sigma_1}, \overrightarrow{\sigma_2}, \overrightarrow{\sigma_3}\right) ,
\end{aligned}
$$

which implies that $(\overrightarrow{\sigma_1}, \overrightarrow{\sigma_2}, \overrightarrow{\sigma_3}) \notin \mathcal{NE}(\mathsf{G})$.

2. Assume now that $\boldsymbol{\sigma} \in \mathcal{B}_{1/2}$. Consider player i switching to an arbitrary strategy $\mathsf{B}_k(j)$ outside $\mathsf{B}_0 \cup \mathsf{B}_{n+1}$. Set $h := i + 1$ and $h' := i + 2$. By Cases (4) and (5), her expected payoff is

$$
\begin{aligned}
& \sum_{\mathbf{s}_{-i} \in \Sigma_{-i}} \mathsf{U}_i \left(\mathbf{s}_{-i} \diamond \mathsf{B}_k(j)\right) \cdot \mathbb{P}_{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_{-i}) \\
= & \sum_{\mathbf{s}_{-i} \in \widetilde{\Sigma}_{-i}} \mathsf{U}_i \left(\mathbf{s}_{-i} \diamond \mathsf{B}_k(j)\right) \cdot \mathbb{P}_{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_{-i}) \\
= & \sum_{\mathbf{s}_{-i} \in \widetilde{\Sigma}_{-i} | s_h = k} \left(\widetilde{\mathsf{U}}_i \left(\mathbf{s}_{-i} \diamond j\right) + 2\,\delta\right) \mathbb{P}_{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_{-i}) + \sum_{\mathbf{s}_{-i} \in \widetilde{\Sigma}_{-i} | s_h \neq k} \widetilde{\mathsf{U}}_i \left(\mathbf{s}_{-i} \diamond j\right) \cdot \mathbb{P}_{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_{-i}) \\
= & \; 2\,\delta \sum_{\mathbf{s}_{-i} \in \widetilde{\Sigma}_{-i} | s_h = k} \mathbb{P}_{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_{-i}) + \sum_{\mathbf{s}_{-i} \in \widetilde{\Sigma}_{-i}} \widetilde{\mathsf{U}}_i \left(\mathbf{s}_{-i} \diamond j\right) \cdot \mathbb{P}_{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_{-i}) \\
\leq & \; \delta + \sum_{\mathbf{s}_{-i} \in \widetilde{\Sigma}_{-i}} \widetilde{\mathsf{U}}_i \left(\mathbf{s}_{-i} \diamond j\right) \cdot \mathbb{P}_{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_{-i}) \\
= & \; \mathsf{U}_i \left(\overrightarrow{\sigma_1}, \overrightarrow{\sigma_2}, \overrightarrow{\sigma_3}\right) ,
\end{aligned}
$$

which implies that $(\overrightarrow{\sigma_1}, \overrightarrow{\sigma_2}, \overrightarrow{\sigma_3}) \in \mathcal{NE}(\mathsf{G})$.

The proof is now complete. ◀

4 The ∃ℝ-Complete Decision Problems

We now present the ∃ℝ-completeness results. We show:

▶ **Theorem 10.** *Restricted to r-player games with $r \geq 3$, the following decision problems are ∃ℝ-complete:*

Group I	*Group II*
∃ SECOND NASH	∃ NASH WITH LARGE PAYOFFS
∃ NASH WITH SMALL PAYOFFS	∃ NASH WITH LARGE TOTAL PAYOFF
∃ NASH WITH SMALL TOTAL PAYOFF	∃ WITH SMALL SUPPORTS
∃ NASH WITH LARGE SUPPORTS	
∃ NASH WITH RESTRICTING SUPPORTS	
∃ NASH WITH RESTRICTED SUPPORTS	
∃ NON-PARETO OPTIMAL NASH	
∃ NON-STRONGLY-PARETO-OPTIMAL NASH	

Membership of the decision problems (for r-player games with $r \geq 3$) in ∃ℝ is established with standard techniques, employing simple formulas to define their properties (cf. [9]).

Proof. Assume first that $r = 3$. We use a polynomial time, many-to-one reduction from ∃ NASH IN A BALL. Consider an instance $\widetilde{\mathsf{G}}$ of ∃ NASH IN A BALL with $\varrho = \frac{1}{2}$, called the *inbox game.* Assume, without loss of generality, that $\widetilde{\Sigma}_i = [n]$ for each player $i \in [3]$. We start with an outline of the proof. The reduction will be the composition of the construction of a gadget game and the game reduction from Section 3. For each of *Group I* and *Group II,* we shall employ a suitable game $\widehat{\mathsf{G}}$, called the *gadget game,* which may be constructed from the inbox game $\widetilde{\mathsf{G}}$. Then, we shall apply the game reduction from Section 3 with $\widetilde{\mathsf{G}}$

Case	Condition on the profile $\mathbf{s}$	Payoff vector $\widehat{\mathsf{U}}(\mathbf{s})$
(1)	$\mathsf{P}(\mathbf{s}) = [3]$	$\langle u^*, u^*, u^* \rangle$
(2)	$\mathsf{P}(\mathbf{s}) \neq \emptyset \wedge \mathsf{P}(\mathbf{s}) \neq [3]$	$\widehat{\mathsf{U}}_i(\mathbf{s}) = u^*$ if $i \in \mathsf{P}(\mathbf{s})$ $\widehat{\mathsf{U}}_i(\mathbf{s}) = u^* - 1$ if $i \notin \mathsf{P}(\mathbf{s})$
(3)	$\mathsf{P}(\mathbf{s}) = \emptyset$	$\langle u^* - 2, u^* - 2, u^* 2 \rangle$

Figure 2 The payoff functions for the game $\widehat{\mathsf{G}}$. Here, $\mathsf{P}(\mathbf{s}) := \{i \in [3] \mid s_i = 1\}$, and $u^* := \overline{u}(\widetilde{\mathsf{G}}) + 1$.

and $\widehat{\mathsf{G}}$ as the subgames to obtain the game $\mathsf{G} = \mathsf{G}\left\langle \widetilde{\mathsf{G}}, \widehat{\mathsf{G}} \right\rangle$; G is the instance of the decision problem (from the corresponding *Group*) associated with some particular property of Nash equilibria; to prove that the decision problem is $\exists\mathbb{R}$-hard, we need to establish: The game $\widetilde{\mathsf{G}}$ has a Nash equilibrium in the ball $\mathcal{B}_{1/2}$ if and only if the game G has a Nash equilibrium with the property (respectively, the set of Nash equilibria for G has the property, as for the decision problem $\exists$ SECOND NASH).
We continue with the formal proof. We treat separately each of *Group I* and *Group II*.
Group I: Construct the 3-player gadget game $\widehat{\mathsf{G}}$ where for each player $i \in [3]$, $\widehat{\Sigma}_i = [n]$. The payoff functions are given in Figure 2: Clearly, the gadget game $\widehat{\mathsf{G}}$ is constructed in time polynomial in the size of the inbox game $\widetilde{\mathsf{G}}$. Note that $\widehat{\mathsf{G}}$ has a unique Nash equilibrium which is the profile $\mathbf{s} = (1, 1, 1)$, in which each player has payoff u^*.
Apply now the game reduction from Section 3 to construct the game G from the subgames $\widetilde{\mathsf{G}}$ and $\widehat{\mathsf{G}}$ (and an arbitrary $\delta > 0$). Since *(i)* G is constructed in time polynomial in the sizes of $\widetilde{\mathsf{G}}$ and $\widehat{\mathsf{G}}$, and *(ii)* $\widehat{\mathsf{G}}$ is constructed in time polynomial in the size of $\widetilde{\mathsf{G}}$, it follows that G is constructed in time polynomial in the size of $\widetilde{\mathsf{G}}$. Lemmas 7, 8 and 9 immediately imply:

▶ **Lemma 11.** *Assume that the inbox game $\widetilde{\mathsf{G}}$ has no Nash equilibrium in $\mathcal{B}_{1/2}$. Then, G has a unique Nash equilibrium $(\overleftarrow{\sigma_1}, \overleftarrow{\sigma_2}, \overleftarrow{\sigma_3})$ with the following properties:*
1. *For each player $i \in [3]$, $\mathsf{U}_i\left(\overleftarrow{\sigma_1}, \overleftarrow{\sigma_2}, \overleftarrow{\sigma_3}\right) = u^*$.*
2. *For each player $i \in [3]$, $\mathsf{Support}(\overleftarrow{\sigma_i}) = \{n(n+1)+1\}$.*
3. *$(\overleftarrow{\sigma_1}, \overleftarrow{\sigma_2}, \overleftarrow{\sigma_3})$ is Pareto-Optimal.*
4. *$(\overleftarrow{\sigma_1}, \overleftarrow{\sigma_2}, \overleftarrow{\sigma_3})$ is Strongly-Pareto-Optimal.*

On the other hand, Lemma 9 immediately implies:

▶ **Lemma 12.** *Assume that the inbox game $\widetilde{\mathsf{G}}$ has a Nash equilibrium in $\mathcal{B}_{1/2}$. Then, G has a Nash equilibrium $\boldsymbol{\tau}$ with the following properties:*
1. *For each player $i \in [3]$, $\mathsf{U}_i(\boldsymbol{\tau}) \leq \overline{u}(\widetilde{\mathsf{G}}) < u^*$.*
2. *For each player $i \in [3]$, $\mathsf{Support}(\tau_i) \in [n]$ with $|\mathsf{Support}(\tau_i)| \geq h$ for some integer $h \geq 2$,*
3. *$\boldsymbol{\tau}$ is not Pareto-Optimal.*
4. *$\boldsymbol{\tau}$ is not Strongly Pareto-Optimal.*

Now, combining the two families of properties in Lemmas 11 and 12 immediately yields the $\exists\mathbb{R}$-hardness of the following decision problems:
- $\exists$ SECOND NASH;
- $\exists$ NASH WITH SMALL PAYOFFS, taking u with $\overline{u}(\widetilde{\mathsf{G}}) < u \leq u^*$;
- $\exists$ NASH WITH SMALL TOTAL PAYOFF, taking u with $r \cdot \overline{u}(\widetilde{\mathsf{G}}) < u \leq r \cdot u^*$;
- $\exists$ NASH WITH LARGE SUPPORTS, taking k with $2 \leq k \leq h$;
- $\exists$ NASH WITH RESTRICTING SUPPORTS, taking a triple $(\mathsf{T}_1, \mathsf{T}_2, \mathsf{T}_3) = (\{i\}, \{j\}, \{k\})$ with arbitrary strategies $i, j, k \in [n]$;
- $\exists$ NASH WITH RESTRICTED SUPPORTS, taking, for each player $i \in [3]$, T_i such that $[n] \subseteq \mathsf{T}_i \subseteq [p] \setminus \{n(n+1)+1\}$;

- ∃ NON-PARETO-OPTIMAL NASH;
- ∃ NON-STRONGLY PARETO-OPTIMAL NASH.

Group II: Construct the 3-player gadget game $\widehat{\mathsf{G}} := \mathsf{G}[m] + \underline{u}(\widetilde{\mathsf{G}}) - 1$, where $m \geq 3$ is an odd integer with size polynomial in the size of n, and $\mathsf{G}[m]$ is the gadget game from Section 2.4. Clearly, the game $\widehat{\mathsf{G}}$ is constructed in time polynomial in the size of $\widetilde{\mathsf{G}}$. By Lemmas 3 and 4, $\widehat{\mathsf{G}}$ has a unique Nash equilibrium $\boldsymbol{\sigma}$ which is fully mixed and has $\widehat{\mathsf{U}}_i(\boldsymbol{\sigma}) = \underline{u}(\widetilde{\mathsf{G}}) - 1 + \frac{1}{m}$. Apply now the game reduction from Section 3 to construct the game G from the subgames $\widetilde{\mathsf{G}}$ and $\widehat{\mathsf{G}}$. As in *Group I,* and using the fact that m has size polynomial in that of n, G is constructed in time polynomial in the size of $\widetilde{\mathsf{G}}$. Lemmas 7, 8 and 9 immediately imply:

▶ **Lemma 13.** *Assume that the inbox game $\widetilde{\mathsf{G}}$ has no Nash equilibria in $\mathcal{B}_{1/2}$. Then, G has a unique Nash equilibrium $(\overleftarrow{\sigma_1}, \overleftarrow{\sigma_2}, \overleftarrow{\sigma_3})$ with the following properties:*
1. *For each player $i \in [3]$, $\mathsf{U}_i\left(\overleftarrow{\sigma_1}, \overleftarrow{\sigma_2}, \overleftarrow{\sigma_3}\right) = \underline{u}\left(\widetilde{\mathsf{G}}\right) - 1 + \frac{1}{m}$.*
2. *For each player $i \in [3]$, $|\mathsf{Support}(\overleftarrow{\sigma_i})| = m$.*

On the other hand, Lemma 9, immediately implies:

▶ **Lemma 14.** *Assume that the inbox game $\widetilde{\mathsf{G}}$ has a Nash equilibrium in $\mathcal{B}_{1/2}$. Then, G has a Nash equilibrium $\boldsymbol{\tau}$ with the following properties:*
1. *For each player $i \in [3]$, $\mathsf{U}_i(\boldsymbol{\tau}) \geq \underline{u}(\widetilde{\mathsf{G}}) - 1 + \frac{1}{m}$.*
2. *For each player $i \in [3]$, $|\mathsf{Support}(\tau_i)| \leq n$.*

Now, combining the two families of properties in Lemmas 13 and 14 immediately yields the ∃ℝ-hardness of the following decision problems:
- ∃ NASH WITH LARGE PAYOFFS, taking u with $\underline{u}(\widetilde{\mathsf{G}}) - 1 + \frac{1}{m} < u \leq \underline{u}(\widetilde{\mathsf{G}})$.
- ∃ NASH WITH LARGE TOTAL PAYOFF, taking u with $r \cdot \left(\underline{u}(\widetilde{\mathsf{G}}) - 1 + \frac{r}{m}, \right) < u \leq r \cdot \underline{u}(\widetilde{\mathsf{G}})$.
- ∃ NASH WITH LARGE SUPPORTS, taking k with $n \leq k < m$.

Each ∃ℝ-hardness result can be extended to r-player games with $r > 3$ using a trivial technique, also used in [9]. Specifically, to prove that a particular decision problem is ∃ℝ-hard for r-player games with $r > 3$ given that it is ∃ℝ-hard for 3-player games, we reduce from 3-player games to r-player games: We add $r - 3$ dummy players; each comes with a suitable payoff function so that the property associated with the decision problem is either satisfied vacuously by the dummy players, or its satisfaction is not affected by the dummy players. For example, for ∃ NASH WITH LARGE PAYOFFS, each dummy player comes with a single strategy 1 and the payoff of each dummy player is always u no matter what the other players choose; for ∃ NASH WITH RESTRICTED SUPPORTS, each dummy player comes with a single strategy 1 and the set T_i for each dummy player $i \in [r] \setminus [3]$ is taken as $\{1\}$. For ∃ NASH WITH LARGE SUPPORTS, each dummy player comes with k strategies, each yielding the same payoff no matter what the other players choose; thus, a mixed profile where each dummy player plays each of her k strategies with probability $\frac{1}{k}$ is a Nash equilibrium when restricted to the dummy players. This implies that there is a Nash equilibrium such that for each player $i \in [r]$, $|\mathsf{Support}(\sigma_i)| \geq k$ if and only if there is a Nash equilibrium such that for each non-dummy player i, $|\mathsf{Support}(\sigma_i)| \geq k$. Further details are omitted as trivial. ◀

5 Epilogue

The extensive catalog of ∃ℝ-complete decision problems about Nash equilibria in r-player games with $r \geq 3$ we presented extends significantly the corresponding ∃ℝ-completeness results from [9, 17] and completes the picture for the complexity characterization of such problems. Deciding any of these problems in $\mathcal{NP}$ (for r-player games with $r \geq 3$) is as hard as deciding ETR in $\mathcal{NP}$, which is considered very unlikely. The presented catalog seconds the corresponding one of $\mathcal{NP}$-complete decision problems about Nash equilibria in 2-player games from [1, 6, 10]. It remains open whether or not the established $\exists\mathbb{R}$-hardness survives the restriction to r-player *win-lose* games with $r \geq 3$; we note that the corresponding $\mathcal{NP}$-hardness for 2-player games [6, 10] was extended to 2-player win-lose games in [1].

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