

Large Peg-Army Maneuvers

Luciano Gualà¹, Stefano Leucci², Emanuele Natale², and Roberto Tauraso⁴

- 1 Università di Roma Tor Vergata, Italy
guala@mat.uniroma2.it
- 2 Sapienza Università di Roma, Italy
leucci@di.uniroma1.it
- 3 Sapienza Università di Roma, Italy
natale@di.uniroma1.it
- 4 Università di Roma Tor Vergata, Italy
guala@mat.uniroma2.it

Abstract

Despite its long history, the classical game of peg solitaire continues to attract the attention of the scientific community. In this paper, we consider two problems with an algorithmic flavour which are related with this game, namely Solitaire-Reachability and Solitaire-Army. In the first one, we show that deciding whether there is a sequence of jumps which allows a given initial configuration of pegs to reach a target position is NP-complete. Regarding Solitaire-Army, the aim is to successfully deploy an army of pegs in a given region of the board in order to reach a target position. By solving an auxiliary problem with relaxed constraints, we are able to answer some open questions raised by Csakany and Juhasz (Mathematics Magazine, 2000).

1998 ACM Subject Classification G.2.0 General

Keywords and phrases Complexity of Games, Solitaire Army

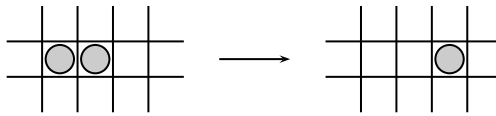
Digital Object Identifier 10.4230/LIPIcs.FUN.2016.18

1 Introduction

Not so very long ago there became widespread an excellent kind of game, called Solitaire, where I play on my own, but as if with a friend as witness and referee to see that I play correctly. A board is filled with stones set in holes, which are to be removed in turn, but none (except the first, which may be chosen for removal at will) can be removed unless you are able to jump another stone across it into an adjacent empty place, when it is captured as in Draughts. He who removes all the stones right to the end according to this rule, wins; but he who is compelled to leave more than one stone still on the board, yields the palm. This game can more elegantly be played backwards, after one stone has been put at will on an empty board, by placing the rest with it, but the same rule being observed for the addition of stones as was stated just above for their removal. Thus we can either fill the board, or, what would be more clever, shape a predetermined figure from the stones; perhaps a triangle, a quadrilateral, an octagon, or some other, if this be possible; but such a task is by no means always possible: and this itself would be a valuable art, to foresee what can be achieved; and to have some way, particularly geometrical, of determining this. – Gottfried Wilhelm Leibniz¹

¹ Original Latin text in *Miscellanea Berolinensia* 1 (1710) 24, the given translation is from [3].





■ **Figure 1** A peg solitaire move.

In this work we investigate some computational and mathematical aspects of the classical game known as *peg solitaire*. In peg solitaire, we have a grid graph (the *board*) on each of whose nodes (the *holes*) there may be at most one *peg*. The initial configuration of pegs evolves by performing one of the following four moves (the *jumps*): for each triple of horizontally or vertically adjacent nodes, if the first and the second nodes are occupied by pegs and there is no peg on the third one, then we can remove the two pegs and place a new one on the third node (see Figure 1). A *puzzle* of peg solitaire is defined by an initial and a final configuration, and consists of finding a sequence of moves that transforms the initial configuration into the final one.

Because of the complexity generated by such simple rules [12, 14, 9, 6], the game has attracted the attention of many mathematically-inclined minds over its long history (for which we refer the reader to Beasley’s writings on the topic [7, 4, 5]). In fact, we started with a quotation from Leibniz, and the origin of the game may well precede his time. Despite such a respectable age, the problem of deciding which peg solitaire puzzles can be solved is far from settled [4].

The present paper contributes to such investigation by considering two specific problems:

- **SOLITAIRE-REACHABILITY**: given a target position and an initial configuration of pegs on a finite board, we wish to determine whether there exists a sequence of moves that allows some peg to be placed in the given target position.
- **SOLITAIRE-ARMY**: given a target position and a region of an infinite board, find an initial configuration of pegs inside that region, and a finite sequence of moves that allows some peg to be placed in the given target position.

We study the former problem from a computational point of view and we prove that **SOLITAIRE-REACHABILITY** is NP-complete. This result, which we discuss in Section 2, is a significant step in understanding why peg solitaire puzzles are intrinsically difficult. They are actually still a good testbed for artificial intelligence techniques [19, 18]. Indeed, the first computational hardness result for this game was proved in 1990 in [22], but was limited to those peg solitaire puzzles in which the final configuration is required to have only one peg (and hence the goal was that of cleaning the entire board).

The **SOLITAIRE-ARMY** problem has received a lot of attention from the mathematical and game-connoisseur community [7, 1, 8]. In this body of works, the part of the board where no pegs are allowed in the initial configuration is called *desert*, and the typical goal is reaching the farthest distance inside the desert.

In the classical example, introduced by J. H. Conway, the desert is a half-plane. Conway devised an elegant potential function argument which shows that, for any initial configuration, and no matter what sequence of moves one may attempt, no peg can reach a distance larger than four in the desert [17].

Other shapes for the desert have been considered. For instance, in [11], the authors focus on square-shaped and rhombus-shaped deserts. Here the natural target position is the center of the square/rhombus, and the goal is that of finding the largest size of the desert for which the puzzle is solvable. Among other results, in [11] it is shown that the 9×9 square-shaped

and 13×13 rhombus-shaped deserts are solvable, while the 13×13 square-shaped and 17×17 rhombus-shaped are not. Therefore the problems whether the 11×11 square-shaped and 15×15 rhombus-shaped deserts are solvable were left open².

In Section 3 we discuss our contribution to the SOLITAIRE-ARMY problem. We develop a general approach that can be used to attack such kind of puzzles, and as a byproduct, we are able to show that both the 11×11 square-shaped and 15×15 rhombus-shaped deserts are actually solvable. The main idea underlying our technique is considering an auxiliary problem where the constraint that any node can have at most one peg is *relaxed* to allowing pegs to be stacked. We actually allow each node to have any integer number of pegs, including negative. This setting has the advantage that the order of the moves is immaterial. The auxiliary problem admits a natural compact Integer Linear Programming (ILP) formulation which, when the puzzle is simple enough, can be safely solved by means of an ILP solver. Once the relaxed problem is solved we provide also an efficient algorithm which *converts* the relaxed solution into a solution for the original problem. To appreciate the combinatorial beauty of our solutions, we recommend to visit the gallery of animations provided at <http://solitairearmy.isnphard.com>.

Interestingly enough, an analogous relaxation was considered in [10] for a similar problem where a different set of moves is used (the so-called *pebbling game*). The authors claim the equivalence between the relaxed and the original problem (see Lemma 3 in [10]) but the proof is omitted as an ‘easy induction argument’. Apparently, E. W. Dijkstra disagrees with such statement: “*it would have been nice if ‘the easy induction argument’ had been shown: a few colleagues and I spent $1\frac{1}{2}$ hours on not finding it*” [13]. Since our equivalence result holds for a large class of moves which includes both pegs and pebbles, we also obtain their result as a corollary.

Other related results. Our hardness reduction contributes to the line of research investigating the computational complexity of combinatorial games [16, 2, 15]. As for positive results regarding peg solitaire, it has been shown that, on a rectangular board of fixed height, the set of initial configurations that can be reduced to a single peg form a regular language [21, 20]. Finally, the idea of formulating peg solitaire puzzles as integer linear programs was already suggested in [3] and effectively used in [19].

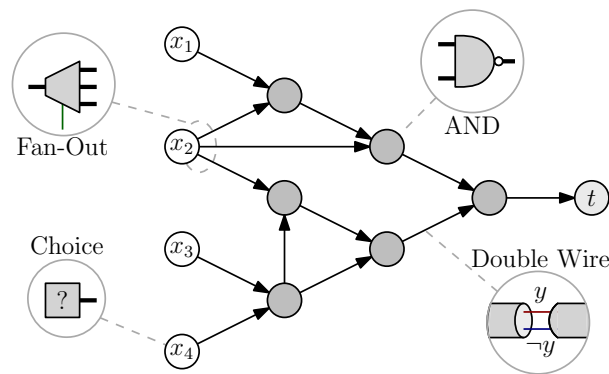
2 Our Hardness Reduction

2.1 Overview of the Reduction

Here we consider the problem SOLITAIRE-REACHABILITY: given an initial configuration of pegs on a finite board, it asks to decide whether there exists a sequence of moves that cause a peg to be placed in a given target position. We prove that SOLITAIRE-REACHABILITY is NP-complete.

Our reduction is from the *planar circuit satisfiability problem* (PCSAT for short): in PCSAT we are given a *boolean network* represented as a planar directed acyclic graph G having a single *sink* vertex $t \in V(G)$ (i.e., a vertex having out-degree 0). Each other vertex in $V(G)$ is either an *input* vertex or a *NAND* vertex. The vertex t is required to have in-degree 1, input vertices must be sources in G (i.e. their in-degree must be 0), while NAND vertices must have in-degree exactly 2 and out-degree at least 1. The problem consists in determining

² Note that the square/rhombus should have the side of odd length in order for the center of the desert to be well-defined.



■ **Figure 2** An instance of PCSAT corresponding to the formula $(\neg x_1 \vee \neg x_2) \wedge (x_2 \vee \neg x_3 \vee \neg x_4)$. Gadgets that we need to implement are highlighted.

whether it is possible to assign a truth value $\pi(u) \in \{\text{TRUE}, \text{FALSE}\}$ to each vertex $u \in G$ in order to satisfy the following properties:

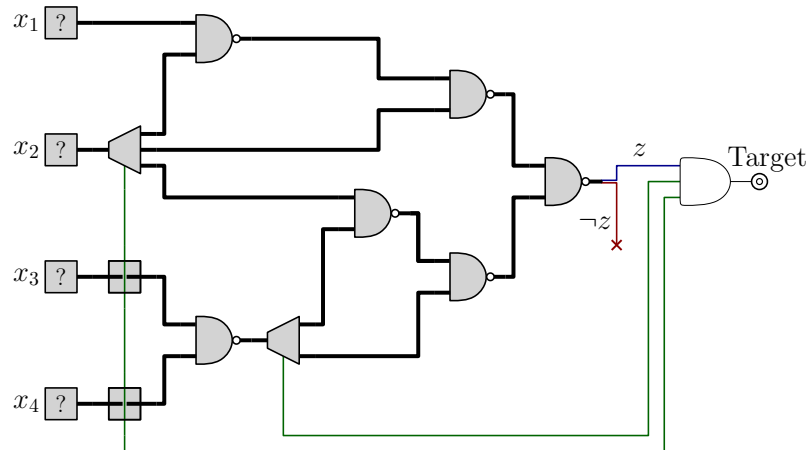
- the truth value assigned to a NAND vertex u is the NAND of the truth values of its two in-neighbors, i.e., if v_1 and v_2 are the in-neighbors of u , we have $\pi(u) = \neg(\pi(v_1) \wedge \pi(v_2))$;
- the value assigned to the sink vertex t coincides with the truth value of its only in-neighbor;
- $\pi(t) = \text{TRUE}$.

Notice that the assignment $\pi(\cdot)$ is completely determined by the truth values of the input vertices. It is well known that this problem is NP-hard.³ Clearly, the graph can be thought as a *boolean circuit* consisting of *links* and *gates* which computes a boolean output as a function of the circuit's inputs x_1, x_2, \dots .

Given any instance of PCSAT, we will build an instance of SOLITAIRE-REACHABILITY which simulates the behavior of such a circuit. We encode the circuit's mechanics using *dual-rail logic*: each edge of the network will be transformed into a *dual-rail wire* consisting of two *single wires* which will always “carry” opposite boolean values. In order to perform such a transformation we will make use of some gadgets: the *choice gadget* will be used to encode input vertices, the *NAND gadget* will represent a NAND vertex, and the *fan-out gadget* will allow to split a single double-wire into multiple double-wires to be fed as input into other gadgets. For technical reasons, the fan-out gadget will output an additional boolean *control signal* on a dedicated single wire, which we will call *control wire*. Intuitively, if we require this control signal to be TRUE, then the correct operation of the fan-out gadget is guaranteed. All these control wires can be safely brought outside of the area of the board that contains the gadgets and all the other wires. Once this has been done, we can finally AND together the boolean signals carried by all these control lines along with the signal carried by the double wire corresponding to the unique edge entering the output vertex t . Since G is planar we will have no intersections between double-wires. However, the same does not hold for control wires. Whenever a control wire intersects a double wire we will use a suitable additional gadget which we call *control-crossover* that ensure that, whenever the control signal is TRUE, the signal carried by the double wire will not be affected.

Figure 2 shows a possible instance of PCSAT and highlights the gadgets we need to implement, while Figure 3 is a high level picture of the associated instance of SOLITAIRE-

³ See, for example, the notes of lecture 6 of the course “Algorithmic Lower Bounds: Fun with Hardness Proofs” by Prof. Erik Demaine (<http://courses.csail.mit.edu/6.890/fall14/lectures/>).



■ **Figure 3** A high-level picture of the instance of SOLITAIRE-REACHABILITY corresponding to the instance of PCSAT shown in Figure 2. Double wires are bold while single wires are thin. Control wires (in green) are ANDed together with the single wire carrying the positive signal z of the circuit output. The negated signal $\neg z$ is ignored.

REACHABILITY. Clearly, the circuit shown in Figure 3 will be implemented as a certain configuration of pegs on a board. The output of the last AND gate will correspond to the target position of SOLITAIRE-REACHABILITY: a peg can be brought to the target position iff there is a truth assignment of the inputs that causes the circuit to output TRUE.

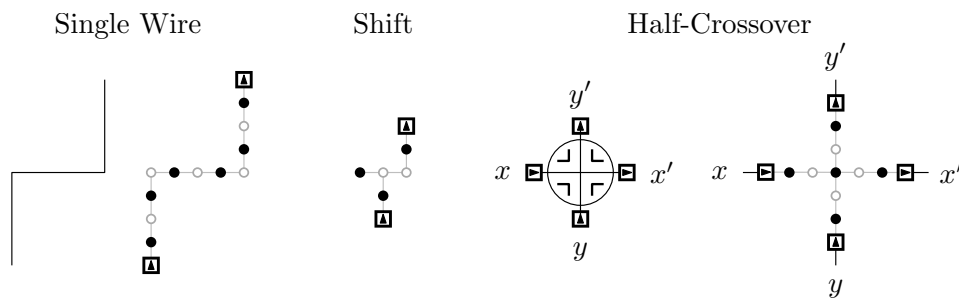
2.2 Description of the Gadgets

In this section we describe the gadgets used by our reduction. We will need to design gates for both binary and dual-rail logic. We will use the following color convention: the former will be colored white, while the latter will be in gray.

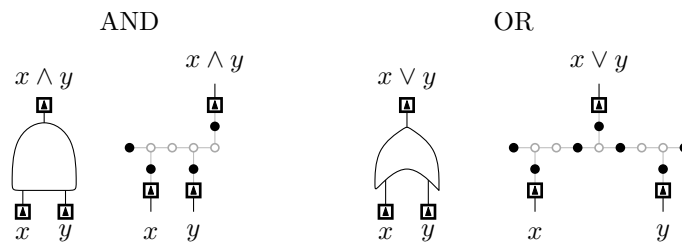
2.2.1 Binary Logic Gadgets

We start by describing the gadgets of our construction that deal with binary logic. We have single wires and binary logic gates. Intuitively a single wire “carries” a boolean signal from its first endpoint to the other. The truth value of the signal is encoded as follows: if there is a peg on the first endpoint then the signal is TRUE, and the wire will allow the peg to be moved to the other endpoint via a sequence of moves. On the converse, if there is no peg on the first endpoint then no move can be made and therefore no peg can be placed on the second endpoint, which encodes a FALSE signal. The implementation of a single wire is straightforward and it is shown in Figure 4. In order to deal with parity issues we can use the Shift gadget (also shown in Figure 4) that allows to shift a wire by one row/column of the board. This gadget can also be used to force the signal to flow in one direction.

As far as gates are concerned, each gate takes one or more boolean inputs (which are again encoded by the presence or absence of pegs in certain input positions), and has one or more boolean outputs which are a function of the inputs. If an output is TRUE, this means that there exists a sequence of moves which places a peg on the output position of the gate. Otherwise, if an output is FALSE, no sequence of moves can place a peg on the output position of the gate.



■ **Figure 4** Symbols and implementations of the wire (left), shift (center), and half-crossover (right). In the half-crossover, if the inputs x and y are not both true, then $x' = x$ and $y' = y$. Otherwise either x' or y' will be true (but not both).



■ **Figure 5** Implementation of the AND and OR binary logic gates.

Binary Logic Half-Crossover

Intuitively, this gadget allows for two single wires to safely cross each other if at least one of them is FALSE. When both signals are TRUE they cannot both cross, and one of them will become FALSE.

More formally, the half-crossover has two inputs x and y and two outputs x' and y' . If x and y are not both true, then x' is true iff x is true, and y' is true iff y is true. Otherwise, if both x and y are true, then only one of x' and y' can be set to true. The gadget is shown in Figure 4.

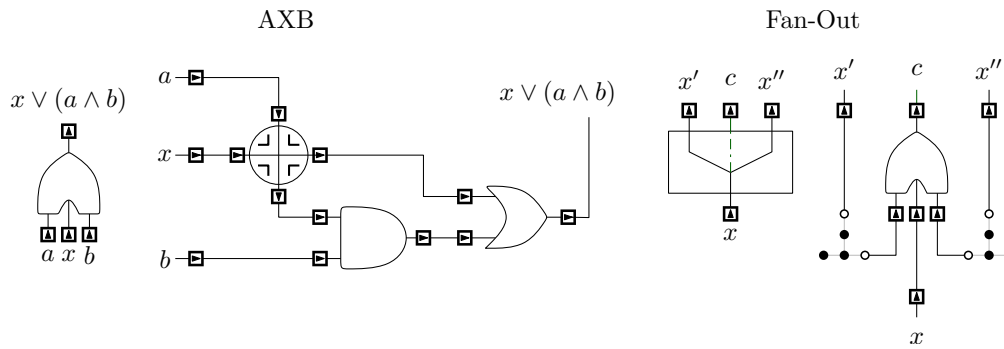
Binary Logic AND and Binary Logic OR

The binary logic AND and OR gadgets have two inputs and one output and their implementations are shown in Figure 5. In the AND gadget it is possible to place a peg on the output position only when both pegs are present on the input positions while, in the OR gadget, it is possible to output a peg iff at least one input peg is present. Clearly, multiple copies of both the AND and the OR gadgets can be chained together in order to simulate AND and OR gates with multiple inputs.

Binary Logic Fan-Out

This gadget allows to duplicate an input signal x . For technical reasons it also has an additional output signal c that we call control signal. Whenever c is TRUE, this gate acts as a classical fan-out: the values of the two outputs x' and x'' (see Figure 6) will coincide to the value of x . We will always require all the control lines to be TRUE. In order to implement this gate we need an additional gadget that we call AXB gate.

The AXB gate takes three inputs a, x, b (in order) and computes a single output whose value is TRUE iff x is TRUE or both a and b are TRUE. The implementation is given in



■ **Figure 6** Implementation of the AXB and fan-out binary logic gates.

Figure 6. Notice that if x is TRUE, then a TRUE signal can traverse the half-crossover and trigger the OR-gate, which outputs TRUE. If x is FALSE, then the only way for the gate to output TRUE is to have both a and b set to TRUE. Indeed, in this case, a can traverse the half-crossover and reach the AND gate together with b . Both this signals are needed to cause the AND and the OR gates to output TRUE.

Now we argue on the correctness of the fan-out gadget. Assume that the output control signal c is TRUE. This means that either x is TRUE and hence two pegs can be placed on the output positions of x' and x'' , or x is FALSE which means that the other two inputs of the AXB gate must both be true, which implies that no peg can reach x' or x'' .

Binary Logic Control-Crossover

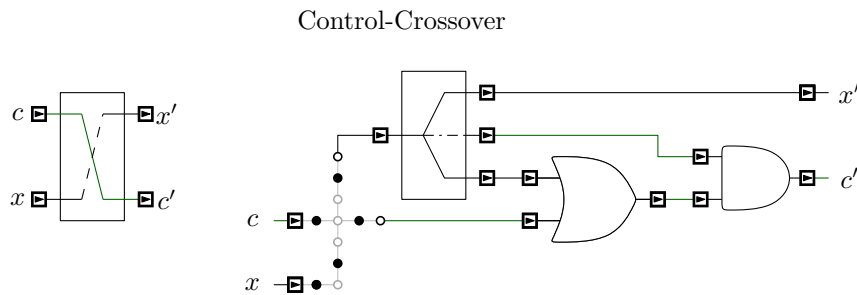
This gadget allows a control signal c to safely cross a the signal x carried by single wire. The gadget has two outputs c' and x' and its implementation is shown in Figure 7. If c' is required to be TRUE, then c must be also true and x' will be TRUE iff x is TRUE. Notice that if $c' = \text{FALSE}$ then x' can be freely set to TRUE or FALSE. However, we will always require c' to be set to TRUE.

We now argue on the correctness of the gadget. We only need to consider the case $c' = \text{TRUE}$. It is easy to see that, in order for c' to be TRUE, c must be TRUE as well. Indeed if $c = \text{FALSE}$, then at least one of the two inputs of the AND gate must be false. Let us assume $c = \text{TRUE}$. Either $x = \text{TRUE}$ or $x = \text{FALSE}$. In the first case, the gate can output $x' = \text{TRUE}$ and $c' = \text{TRUE}$: we can use the peg from c to allow the peg from x to reach the fan-out gate which can now output three TRUE signals. Notice that this requires the peg from x to jump over the peg from c . In the other case, i.e. $x = \text{FALSE}$, we have that the gate can output $x' = \text{FALSE}$ and $c' = \text{TRUE}$ by using the peg from c to activate the OR gate, and the control signal from the fan-out gate to activate the end gate. Now, we only need to show that the gate cannot output $x' = \text{TRUE}$ and $c' = \text{TRUE}$. Indeed, as $c' = \text{TRUE}$, the control signal from the fan-out gate must be true as well. The claim follows since, in this case, there is no way for a TRUE signal to reach the input of the fan-out gate.

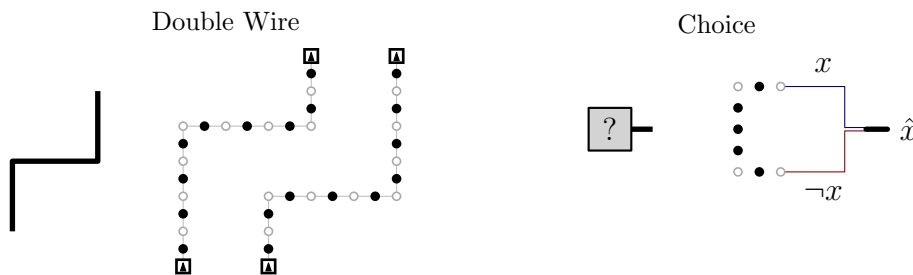
2.2.2 Dual-Rail Logic Gadgets

Double Wire and Dual-Rail Logic Choice

These two gadgets are straightforward and their implementation are shown in Figure 8. Double wires will be drawn in bold to distinguish them from single wires. Moreover we will refer to the pair of signals x and $\neg x$ as \hat{x} . In the choice gadget, the central peg must



■ **Figure 7** Implementation of the control-crossover gadget. If $c' = \text{TRUE}$, then $c = \text{TRUE}$ and $x' = x$. If $c' = \text{FALSE}$ then x' is unrelated to x and can be either TRUE or FALSE.



■ **Figure 8** Implementation of a double wire and of the dual-rail logic choice gadget.

jump over either the peg on its top (that corresponds to x) or the peg on its bottom (that corresponds to $\neg x$).

Dual-Rail Logic NAND

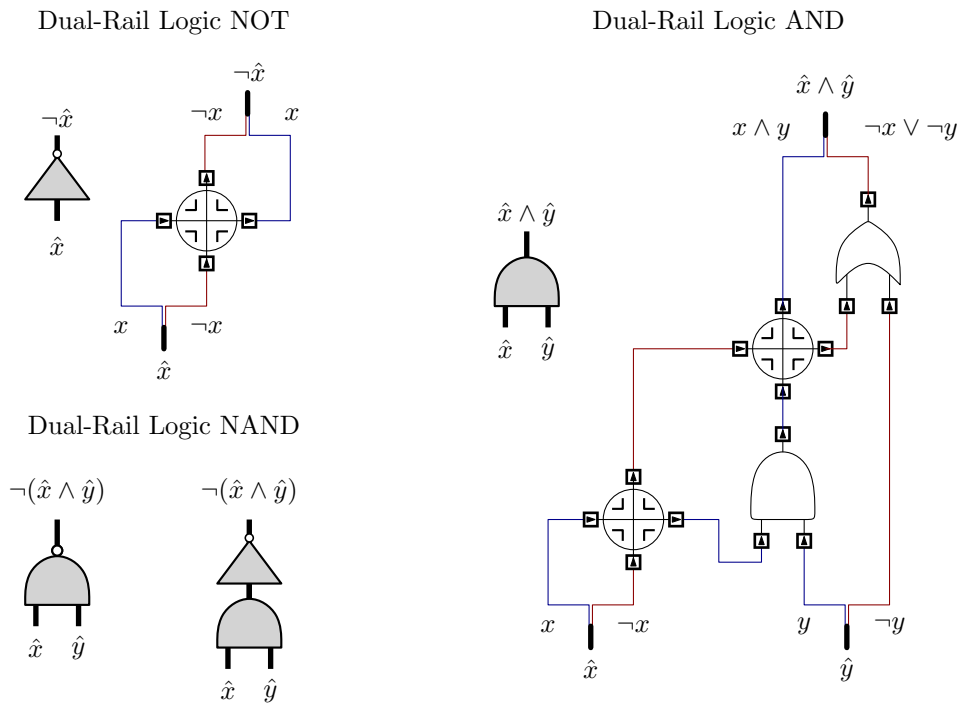
We implement this gate by separately building a dual-rail NOT gate and a dual-rail AND gate, see Figure 9. To compute $\neg \hat{x}$ from \hat{x} , we just need to exchange the roles of the single wires corresponding to x and $\neg x$, hence the NOT gate is actually just a half-crossover. The AND gate computes the logic AND of \hat{x} and \hat{y} by separately computing $x \wedge y$ and $\neg(x \wedge y) = \neg x \vee \neg y$. Notice that, every time we use a half-crossover, at least one of its inputs is FALSE.

Dual-Rail Logic Fan-Out

This gate is shown in Figure 10 and it is similar to the binary logic fan-out except that it works on double wires. The gadget makes use of two binary logic fan-outs whose control lines are crossed with the single wires using binary logic control crossovers. The additional control output c will be TRUE iff all the inner control lines are TRUE as well, hence ensuring the correct operation of the gadget. Notice that we also need to cross a single wire carrying the signal $\neg x$ with another single wire carrying $\neg x$ but this can be safely done by using an half-crossover.

Dual-Rail Logic Control-Crossover

This gadget allows for a control wire to safely cross a double wire: if the control output c' is TRUE, then the control input c is true as well and the output x' is equal to the input x



■ **Figure 9** Implementation of the dual-rail logic NOT, AND, and NAND gates.

(see Figure 11). Its implementation is straightforward as it suffices to use two binary logic control-crossovers to cross c with the single wires corresponding to x and $\neg x$, respectively.

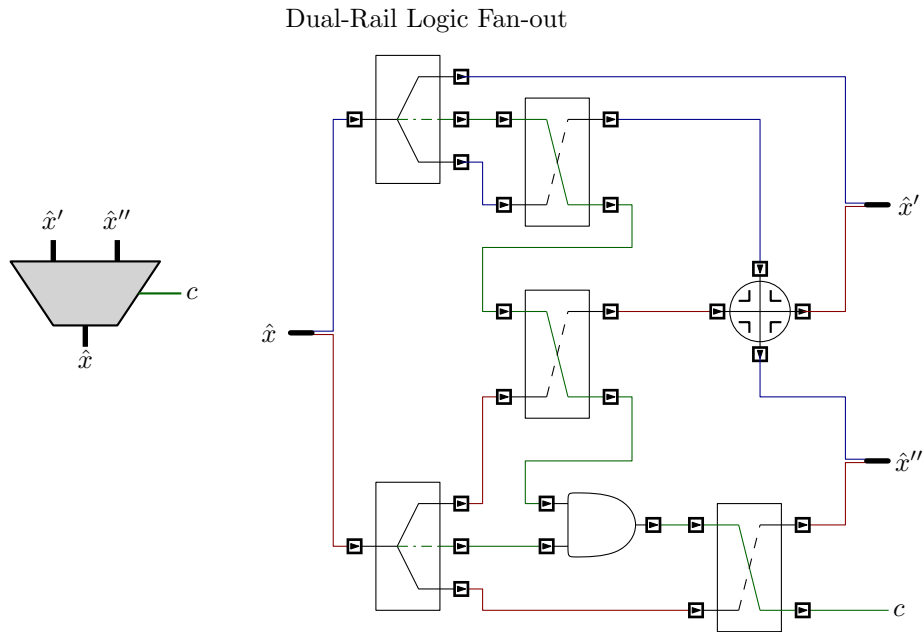
3 The Solitaire-Army problem

3.1 Overview

In this section we focus on the SOLITAIRE-ARMY problem which, given a target position g and a region R of an infinite board, requires finding an initial configuration of pegs inside R , and a finite sequence of moves such that a peg is finally placed in g . We will call *desert* the part of the board where no pegs are allowed in the initial configuration. Moreover, when we consider a configuration of the board, for any position p , we denote by $n(p)$ the number of pegs placed at p in the configuration⁴.

When trying to solve an instance of SOLITAIRE-ARMY, it is convenient to consider the *reversed* version of the game, as if we were rewinding a video of someone trying to solve the puzzle. More formally, in the reverse SOLITAIRE-ARMY, a move is defined as follows. Let $\mathbf{p} = (p_1, p_2, p_3)$ be a generic triple of vertically or horizontally adjacent positions p_1 , p_2 and p_3 . A move consists of decreasing by one $n(p_3)$ and increasing by one both $n(p_1)$ and $n(p_2)$, provided that $n(p_1) = n(p_2) = 0$ and $n(p_3) = 1$. Clearly, an instance of SOLITAIRE-ARMY is solvable if and only if, in the reversed version of the game, it is possible to reach a configuration with no peg in the desert, starting from an initial configuration with a single peg placed in g . In the rest of this section, we will consider always the reversed version of SOLITAIRE-ARMY and we will refer to it as simply SOLITAIRE-ARMY.

⁴ We remark that in SOLITAIRE-ARMY it always holds $n(p) \in \{0, 1\}$.



■ **Figure 10** Implementation of the dual-rail logic fan-out. Whenever $c' = \text{TRUE}$, the gadget duplicates \hat{x} into \hat{x}' and \hat{x}'' .

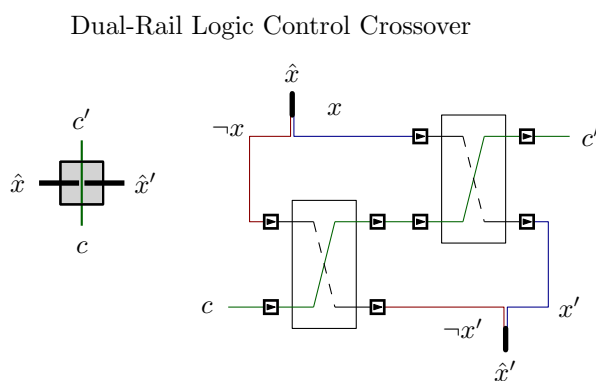
We are now ready to describe a general approach that can be conveniently used to attack an instance of SOLITAIRE-ARMY. One of the most annoying issue when trying to solve the problem is the fact that the order of the moves does matter. To circumvent this difficulty, we simplify the game by defining a *relaxed* version of SOLITAIRE-ARMY that we call R-SOLITAIRE-ARMY. In this variant, given a triple $\mathbf{p} = (p_1, p_2, p_3)$ on the board, a move consists of decreasing $n(p_3)$ by one and increasing both $n(p_1)$ and $n(p_2)$ by one, with no constraints on the values of $n(p_1)$, $n(p_2)$ and $n(p_3)$. Therefore we allow each position to have any integer number of pegs, including negative. Here the goal is to reach a configuration in which there are zero pegs in the desert and every position in the region R is occupied by one or zero pegs.

Our main contribution of this section is the following theorem, which shows the equivalence among the two versions of the game. The proof is deferred to the next Section.

► **Theorem 1.** *R-SOLITAIRE-ARMY is solvable if and only if SOLITAIRE-ARMY is solvable. Moreover, any solution for R-SOLITAIRE-ARMY can be transformed in polynomial time into a solution for SOLITAIRE-ARMY (with at most the same number of moves).*

We emphasize that Theorem 1 actually holds for a wider class of games (including the *pebbling* game [10]). More precisely, we can define a generalized (reversed) SOLITAIRE-ARMY problem in which the set of moves is specified by a collection of tuples $\mathbf{p}^{(i)} = (p_1^{(i)}, \dots, p_{\ell_i}^{(i)})$ of vertices of the board. In this game a move consists of removing a peg from the vertex $p_{\ell_i}^{(i)}$ and of adding a peg in the remaining vertices of the tuple, provided that $n(p_{\ell_i}^{(i)}) = 1$ and $n(p_1^{(i)}) = \dots = n(p_{\ell_i-1}^{(i)}) = 0$.

Since it turns out that R-SOLITAIRE-ARMY admits a compact integer linear programming formulation (ILP), then several SOLITAIRE-ARMY puzzles can be solved by using a good



■ **Figure 11** Implementation of the dual-rail logic control-crossover.

■ **Table 1** Solutions to SOLITAIRE-ARMY of particular interest.

Desert	Goal vertex	Number of moves
Square 7×7	Center of the desert	15
Square 9×9	Center of the desert	39
Square 11×11	Center of the desert	212
Square 12×12	One of the four centers of the desert	301
Square 11×11 at the border of the board, when the board is a half-plane	Center of the desert	246
Square 11×11 , when the board is the union of three half-planes tangent to the desert	Center of the desert	241
Rhombus 15×15 (i.e. with axes of length 15)	Center of the desert	176
Rhombus 15×15 at the border of the board, when the board is a diagonal half-plane	Center of the desert	202
Rhombus 15×15 , when the board is the union of three half-planes tangent to the desert	Center of the desert	183

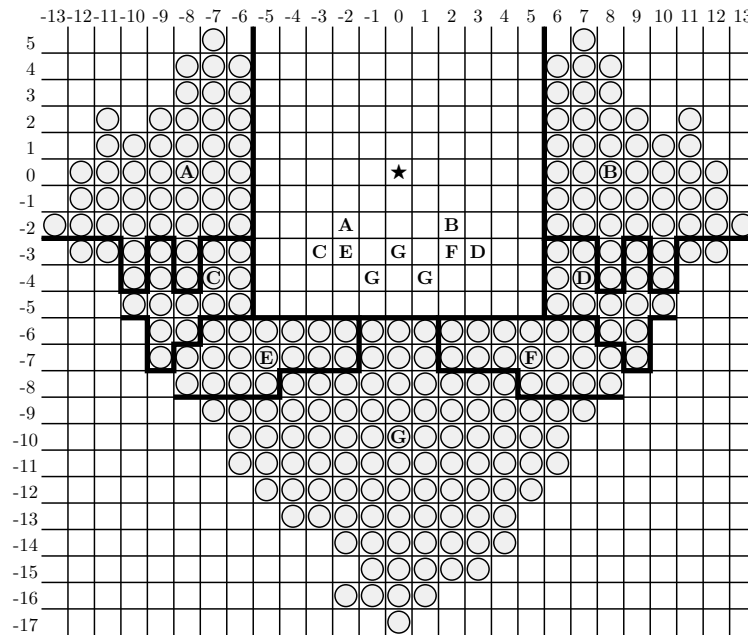
ILP solver⁵. Table 1 summarizes some of the results that we obtained by using our general approach. We also provide a gallery of *animated* solutions at <http://solitairearmy.isnphard.com>. As an example, in Figure 12, we illustrate a final configuration for the SOLITAIRE-ARMY in which the initial target position is the center of a 11×11 square-shaped desert.

3.2 Proof of Theorem 1

Recall that we are considering the reversed version of SOLITAIRE-ARMY. For any finite board, let n be the number of positions. Let $\mathbf{0}$ be the $n \times 1$ vector whose entries are all 0s, and $\mathbf{1}$ be the $n \times 1$ vector whose entries are all 1s. Let \mathbf{b} be a $n \times 1$ integer vector with $\mathbf{0} \leq \mathbf{b} \leq \mathbf{1}$ which represents the initial configuration of pegs. Let \mathbf{c} be a $n \times 1$ integer vector with $\mathbf{0} \leq \mathbf{c} \leq \mathbf{1}$ which represents the constraints on the final configuration. In particular the zero entries of \mathbf{c} specify the region that at the end should be cleared of pegs (the desert).

⁵ We used Gurobi Optimizer, that Gurobi Optimization Inc. has gently made freely available to academic users for research purposes.

18:12 Large Peg-Army Maneuvers



■ **Figure 12** The initial deployment of the SOLITAIRE-ARMY which sends a peg to the center of a square 11×11 desert (this problem was left open in [11]). The army is divided into seven platoons each denoted by a letter. The letters inside the square represent the positions reached by the corresponding platoons. When those positions are taken, it is easy to conquer the center denoted by a star. The complete sequence of moves is given in the Section 3.3.

An $n \times m$ matrix $A = (a_{ij})$ defines the set of moves on the board: for $j = 1, \dots, m$, the j th move adds $a_{ij} \in \{0, 1, -1\}$ pegs to the i th position. The *fundamental assumption* on this matrix is that each column of A , a move, has at most a -1 entry. No further assumption on the structure of matrix A is made.

According of this notation, we state the relaxed problem (R-SOLITAIRE-ARMY): find a $m \times 1$ integer vector $\mathbf{x} \geq \mathbf{0}$ such that

$$\mathbf{0} \leq A\mathbf{x} + \mathbf{b} \leq \mathbf{c}.$$

Note that the component $(A\mathbf{x} + \mathbf{b})_i$ gives the number of pegs at the i th position after we applied $(\mathbf{x})_j$ times the j th move for $j = 1, \dots, m$ to the initial configuration \mathbf{b} .

On the other hand, the original problem SOLITAIRE-ARMY can be rephrased as follows. Find $m \times 1$ unit vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_N$ (a *unit vector* is a vector which has an entry 1 and 0s elsewhere) such that

$$\mathbf{0} \leq A\mathbf{x}_k + \mathbf{b} \leq \mathbf{1} \quad \text{for } 1 \leq k \leq N - 1 \quad \text{and} \quad \mathbf{0} \leq A\mathbf{x}_N + \mathbf{b} \leq \mathbf{c}.$$

where $\mathbf{x}_k := \sum_{t=1}^k \mathbf{u}_t$. Hence if a solution \mathbf{x} of R-SOLITAIRE-ARMY is available, we need to find a decomposition of \mathbf{x} into an ordered sum of unit vectors which represents a specific sequence of feasible moves for SOLITAIRE-ARMY.

For a generic vector \mathbf{x} , we denote by $|\mathbf{x}|$ the sum of the entries of \mathbf{x} . The following preliminary lemma describes a crucial property we will need later.

► **Lemma 2.** *Let \mathbf{x} be a solution of R-SOLITAIRE-ARMY and let \mathbf{y} be a $m \times 1$ integer vector such that*

$$\mathbf{0} \leq \mathbf{y} \leq \mathbf{x} \quad \text{and} \quad A\mathbf{y} \geq \mathbf{0}.$$

If \mathbf{y} is maximal, i. e. there is no unit vector $\mathbf{u} \leq \mathbf{x} - \mathbf{y}$ such that $\mathbf{y}' := \mathbf{y} + \mathbf{u}$ satisfies the above inequalities, then $\mathbf{x}' := \mathbf{x} - \mathbf{y}$ is another solution of R-SOLITAIRE-ARMY.

Proof. We have to show that $\mathbf{0} \leq A\mathbf{x}' + \mathbf{b} \leq \mathbf{c}$. The inequality on the right is trivial:

$$A\mathbf{x}' + \mathbf{b} \leq (A\mathbf{x}' + \mathbf{b}) + A\mathbf{y} = A\mathbf{x} + \mathbf{b} \leq \mathbf{c}.$$

If the remaining inequality $A\mathbf{x}' + \mathbf{b} \geq \mathbf{0}$ does not hold then there is some $i \in \{1, \dots, n\}$ such that $(A\mathbf{x}' + \mathbf{b})_i \leq -1$ which implies

$$(A\mathbf{x}')_i \leq -1 - (\mathbf{b})_i \leq -1.$$

Hence there exists a unit vector $\mathbf{u} \leq \mathbf{x}'$ such that $(A\mathbf{u})_i = -1$. We claim that $A(\mathbf{y} + \mathbf{u}) \geq \mathbf{0}$, which contradicts the maximality of \mathbf{y} . As regards the i th entry,

$$(A(\mathbf{y} + \mathbf{u}))_i = (A\mathbf{x} + \mathbf{b})_i - (A\mathbf{x}' + \mathbf{b})_i + (A\mathbf{u})_i \geq 0 + 1 - 1 = 0.$$

For $l \neq i$, due to the fundamental assumption on the matrix A , we have $(A\mathbf{u})_l \geq 0$ and the proof is complete. \blacktriangleleft

In order to prove Theorem 1, it suffices to show that if R-SOLITAIRE-ARMY is solvable then SOLITAIRE-ARMY is solvable as well. The proof is by induction on the the number of moves $|\mathbf{x}|$ of the solution of R-SOLITAIRE-ARMY \mathbf{x} . If $|\mathbf{x}| = 1$ then we set $\mathbf{u}_1 := \mathbf{x}$ and we are done. Assume that $|\mathbf{x}| > 1$ and any \mathbf{x}' solution of R-SOLITAIRE-ARMY with $|\mathbf{x}'| < |\mathbf{x}|$ can be transformed into a solution of SOLITAIRE-ARMY.

Let $\mathbf{x}_0 = \mathbf{0}$. We need to show that if we are at the step k with $0 \leq k < |\mathbf{x}|$ and $\mathbf{0} \leq A\mathbf{x}_k + \mathbf{b} \leq \mathbf{1}$ then we are able to move forward to step $k + 1$: there exists a unit vector $\mathbf{u} \leq \mathbf{x} - \mathbf{x}_k$ such that $\mathbf{0} \leq A\mathbf{x}_{k+1} + \mathbf{b} \leq \mathbf{1}$ where $\mathbf{x}_{k+1} := \mathbf{x}_k + \mathbf{u}$.

We divide the proof of this fact into two claims.

► **Claim 3.** *There is a unit vector \mathbf{u} such that $\mathbf{u} \leq \mathbf{x} - \mathbf{x}_k$ and $A(\mathbf{x}_k + \mathbf{u}) + \mathbf{b} \geq \mathbf{0}$.*

Proof. Let $\mathbf{z} := \mathbf{x} - \mathbf{x}_k$, then $\mathbf{z} \geq \mathbf{0}$ and $\mathbf{z} \neq \mathbf{0}$.

If $A\mathbf{z} \geq \mathbf{0}$ then we extend \mathbf{z} to a maximal vector \mathbf{y} such that $\mathbf{z} \leq \mathbf{y} \leq \mathbf{x}$ and $A\mathbf{y} \geq \mathbf{0}$. By Lemma 2, $\mathbf{x}' := \mathbf{x} - \mathbf{y}$ is another solution for R-SOLITAIRE-ARMY with $|\mathbf{x}'| < |\mathbf{x}|$ and, by the induction hypothesis, it can be transformed into a solution of SOLITAIRE-ARMY.

On the other hand, if $A\mathbf{z} \geq \mathbf{0}$ does not hold, then there is some $i \in \{1, \dots, n\}$ such that $(A\mathbf{z})_i \leq -1$ and there is a unit vector $\mathbf{u} \leq \mathbf{z}$ such that $(A\mathbf{u})_i = -1$. If $l \neq i$ it follows that

$$(A(\mathbf{x}_k + \mathbf{u}) + \mathbf{b})_l = (A\mathbf{x}_k + \mathbf{b})_l + (A\mathbf{u})_l \geq (A\mathbf{x}_k + \mathbf{b})_l \geq 0.$$

As regards the i th entry, $A\mathbf{x} + \mathbf{b} \geq \mathbf{0}$ implies

$$(A\mathbf{x}_k + \mathbf{b})_i = (A\mathbf{x} + \mathbf{b})_i - (A\mathbf{z})_i \geq 0 + 1 = 1$$

and therefore

$$(A(\mathbf{x}_k + \mathbf{u}) + \mathbf{b})_i = (A\mathbf{x}_k + \mathbf{b})_i + (A\mathbf{u})_i = (A\mathbf{x}_k + \mathbf{b})_i - 1 \geq 1 - 1 = 0.$$

Hence $A(\mathbf{x}_k + \mathbf{u}) + \mathbf{b} \geq \mathbf{0}$. \blacktriangleleft

► **Claim 4.** *There is a unit vector \mathbf{u} such that $\mathbf{u} \leq \mathbf{x} - \mathbf{x}_k$ and $\mathbf{0} \leq A(\mathbf{x}_k + \mathbf{u}) + \mathbf{b} \leq \mathbf{1}$.*

18:14 Large Peg-Army Maneuvers

Proof. Let \mathcal{U} be the set of unit vectors $\mathbf{u} \leq \mathbf{x} - \mathbf{x}_k$ such that $A(\mathbf{x}_k + \mathbf{u}) + \mathbf{b} \geq \mathbf{0}$. The set \mathcal{U} is not empty by Claim 3. If Claim 4 holds then the proof is complete. Otherwise given any $\mathbf{u} \in \mathcal{U}$ there is some $i \in \{1, \dots, n\}$ such that

$$(A\mathbf{x}_k + \mathbf{b})_i = 1 \quad \text{and} \quad (A\mathbf{u})_i = 1.$$

On the other hand, $\mathbf{0} \leq A\mathbf{x} + \mathbf{b} \leq \mathbf{c}$ implies

$$(A(\mathbf{x} - \mathbf{x}_k - \mathbf{u}))_i = (A\mathbf{x} + \mathbf{b})_i - (A\mathbf{x}_k + \mathbf{b})_i - (A\mathbf{u})_i \leq 1 - 1 - 1 = -1,$$

and there is a unit vector $\mathbf{v} \leq \mathbf{x} - \mathbf{x}_k - \mathbf{u}$ such that $(A\mathbf{v})_i = -1$. Hence

$$(A(\mathbf{x}_k + \mathbf{v}) + \mathbf{b})_i = (A\mathbf{x}_k + \mathbf{b})_i + (A\mathbf{v})_i = 1 - 1 = 0$$

which means that $\mathbf{v} \in \mathcal{U}$.

By iterating this process, we generate a sequence of elements of \mathcal{U} . Since \mathcal{U} is finite, this sequence has a minimal cycle, $\mathbf{u}_1, \dots, \mathbf{u}_{r-1}, \mathbf{u}_r = \mathbf{u}_1$ and, by construction, for $2 \leq t \leq r$, $(A\mathbf{u}_t)_i = -1$ implies that $(A\mathbf{u}_{t-1})_i = 1$. Setting $\mathbf{z} := \sum_{t=2}^r \mathbf{u}_t$, we have $\mathbf{0} \leq \mathbf{z} \leq \mathbf{x}$, $A\mathbf{z} \geq \mathbf{0}$ and $\mathbf{z} \neq \mathbf{0}$. Now, as in Claim 3, we extend \mathbf{z} to a maximal vector \mathbf{y} . Then, by Lemma 2, $\mathbf{x}' := \mathbf{x} - \mathbf{y}$ is another solution for R-SOLITAIRE-ARMY with $|\mathbf{x}'| < |\mathbf{x}|$ and finally we apply the induction hypothesis. \blacktriangleleft

Finally, observe that the proof is constructive and can be easily turned into a polynomial-time algorithm that performs the transformation.

3.3 Explicit Solution to the Solitaire-Army Problem of Figure 12

Given a configuration of pegs on a board, we denote a jump with (x, y, d) , where $d \in \{\uparrow, \downarrow, \leftarrow, \rightarrow\}$ indicates the direction of the jump that has to be made by the peg in position (x, y) of the board. The moves that allows the peg army in Figure 12 to reach the center are listed below.

Platoon A: $(-7, -2, \rightarrow), (-9, -2, \rightarrow), (-11, -2, \rightarrow), (-13, -2, \rightarrow), (-10, -4, \uparrow), (-10, -2, \rightarrow), (-10, 0, \downarrow), (-12, 0, \rightarrow), (-11, 2, \downarrow), (-12, -1, \rightarrow), (-11, -2, \rightarrow), (-8, -2, \rightarrow), (-8, -4, \uparrow), (-9, -2, \rightarrow), (-6, -2, \rightarrow), (-6, 0, \downarrow), (-6, 2, \downarrow), (-6, 4, \downarrow), (-8, 1, \rightarrow), (-10, 1, \rightarrow), (-7, 3, \downarrow), (-7, 5, \downarrow), (-9, 2, \rightarrow), (-8, 4, \downarrow), (-8, -1, \rightarrow), (-10, -1, \rightarrow), (-7, 1, \downarrow), (-7, 3, \downarrow), (-9, 0, \rightarrow), (-11, 0, \rightarrow), (-8, 2, \downarrow), (-7, -2, \rightarrow), (-5, -2, \rightarrow), (-7, -1, \rightarrow), (-7, 1, \downarrow), (-9, 0, \rightarrow), (-8, -1, \rightarrow), (-6, -1, \rightarrow), (-7, 0, \rightarrow), (-6, 2, \downarrow), (-6, 0, \rightarrow), (-4, 0, \downarrow), (-4, -2, \rightarrow)$.

Platoon B: symmetric moves of **Platoon A**.

Platoon C: $(-8, -6, \uparrow), (-10, -5, \rightarrow), (-12, -3, \rightarrow), (-10, -3, \rightarrow), (-9, -7, \uparrow), (-9, -5, \uparrow), (-7, -3, \rightarrow), (-6, -5, \uparrow), (-7, -5, \uparrow), (-6, -3, \rightarrow), (-8, -3, \rightarrow), (-8, -5, \uparrow), (-9, -3, \rightarrow), (-7, -3, \rightarrow), (-5, -3, \rightarrow)$.

Platoon D: symmetric moves of **Platoon C**.

Platoon E: $(-2, -7, \uparrow), (-3, -7, \uparrow), (-4, -7, \uparrow), (-6, -6, \rightarrow), (-5, -8, \uparrow), (-5, -6, \rightarrow), (-6, -8, \uparrow), (-7, -6, \rightarrow), (-7, -8, \uparrow), (-8, -8, \uparrow), (-8, -6, \rightarrow), (-6, -6, \rightarrow), (-3, -6, \uparrow), (-4, -6, \uparrow), (-4, -4, \rightarrow), (-2, -5, \uparrow)$.

Platoon F: symmetric moves of **Platoon E**.

Platoon G: $(-3, -9, \uparrow), (-5, -9, \rightarrow), (-7, -9, \rightarrow), (-6, -11, \uparrow), (-6, -9, \rightarrow), (-3, -10, \uparrow), (-5, -10, \rightarrow), (-5, -12, \uparrow), (-4, -12, \uparrow), (-3, -11, \uparrow), (-5, -10, \rightarrow), (-4, -9, \uparrow), (-4, -7, \rightarrow), (-3, -9, \uparrow), (1, -7, \uparrow), (1, -9, \uparrow), (3, -9, \leftarrow), (5, -9, \leftarrow), (7, -9, \leftarrow), (6, -11, \uparrow), (6, -9, \leftarrow), (4, -9, \leftarrow), (3, -11, \uparrow), (5, -10, \leftarrow), (5, -12, \uparrow), (4, -12, \uparrow), (4, -14, \uparrow), (3, -8, \leftarrow), (3, -10, \uparrow), (5, -10, \leftarrow), (4, -8, \leftarrow), (1, -8, \uparrow), (1, -10, \uparrow), (3, -10, \leftarrow), (2, -12, \uparrow), (4, -12, \leftarrow), (3, -14, \uparrow), (1, -11, \uparrow), (1, -13, \uparrow), (1, -15, \uparrow), (3, -15, \leftarrow), (1, -16, \uparrow), (3, -12, \leftarrow), (1, -12, \uparrow), (0, -9, \uparrow), (0, -9, \uparrow), (-2, -9, \rightarrow), (-1, -11, \uparrow), (-1, -13, \uparrow), (-1, -15, \uparrow), (-3, -12, \rightarrow), (-2, -14, \uparrow), (-4, -13, \rightarrow), (-3, -10, \rightarrow), (2, -8, \leftarrow), (2, -10, \uparrow), (1, -10, \uparrow), (0, -8, \uparrow), (2, -8, \leftarrow), (0, -9, \uparrow), (0, -11, \uparrow)$.

$(-2,-11,\rightarrow), (0,-12,\uparrow), (-2,-12,\rightarrow), (0,-13,\uparrow), (2,-13,\leftarrow), (0,-14,\uparrow), (2,-14,\leftarrow), (-2,-13,\rightarrow),$
 $(-2,-8,\rightarrow), (-1,-10,\uparrow), (-1,-7,\uparrow), (-3,-7,\rightarrow), (-1,-8,\uparrow), (1,-6,\uparrow), (0,-6,\uparrow), (0,-8,\uparrow), (0,-10,\uparrow),$
 $(0,-12,\uparrow), (0,-14,\uparrow), (0,-16,\uparrow), (-2,-16,\rightarrow), (0,-17,\uparrow), (0,-15,\uparrow), (0,-13,\uparrow), (0,-11,\uparrow), (0,-9,\uparrow),$
 $(0,-7,\uparrow), (-1,-6,\uparrow), (0,-5,\uparrow).$

Finale: $(-3,-3,\rightarrow), (3,-3,\leftarrow), (-1,-4,\uparrow), (1,-4,\uparrow), (-2,-2,\rightarrow), (0,-3,\uparrow), (2,-2,\leftarrow), (0,-2,\uparrow).$

References

- 1 M. Aigner. Moving into the Desert with Fibonacci. *Math. Magazine*, 70(1):11–21, 1997.
- 2 G. Aloupis, E. D. Demaine, A. Guo, and G. Viglietta. Classic Nintendo games are (computationally) hard. *Theoretical Computer Science*, 586:135–160, 2015.
- 3 J. D. Beasley. *The Ins and Outs of Peg Solitaire*. Oxford University Press, 1985.
- 4 J. D. Beasley. Solitaire: Recent Developments. *arXiv:0811.0851*, 2008.
- 5 J. D. Beasley. John and Sue Beasley’s Webpage on Peg Solitaire, 2015. URL: <http://jsbeasley.co.uk/pegsol.htm>.
- 6 G. I. Bell. A Fresh Look at Peg Solitaire. *Mathe.Magazine*, 80(1):16–28, 2007.
- 7 G. I. Bell, D. S. Hirschberg, and P. Guerrero-Garcia. The minimum size required of a solitaire army. *arXiv/0612612*, 2006.
- 8 E. R. Berlekamp, J. H. Conway, and R. K. Guy. *Winning Ways for Your Mathematical Plays, Volume 2*. AK Peters, 2002.
- 9 A. Bialostocki. An application of elementary group theory to central solitaire. *The College Mathematics Journal*, 29(3):208, 1998.
- 10 F. Chung, R. Graham, J. Morrison, and A. Odlyzko. Pebbling a Chessboard. *The American Mathematical Monthly*, 102(2):113–123, 1995.
- 11 B. Csakany and R. Juhasz. The Solitaire Army Reinspected. *Math. Magazine*, 73(5):354–362, 2000.
- 12 E. W. Dijkstra. The checkers problem told to me by M.O. Rabin, 1992. URL: <http://www.cs.utexas.edu/users/EWD/ewd11xx/EWD1134.PDF>.
- 13 E. W. Dijkstra. Only a matter of style?, 1995. URL: <http://www.cs.utexas.edu/users/EWD/ewd12xx/EWD1200.PDF>.
- 14 M. Gardner. *The Unexpected Hanging and Other Mathematical Diversions*. University of Chicago Press, 1991.
- 15 L. Gualà, S. Leucci, and E. Natale. Bejeweled, Candy Crush and other match-three games are (NP-)hard. In *CIG 2014*, pages 1–8, 2014.
- 16 R. A. Hearn and E. D. Demaine. *Games, puzzles, and computation*. AK Peters, 2009.
- 17 R. Honsberger. A problem in checker jumping. *Mathematical Gems II*, pages 23–28, 1976.
- 18 C. Jefferson, A. Miguel, I. Miguel, and S. A. Tarim. Modelling and solving English Peg Solitaire. *Computers & Operations Research*, 33(10):2935–2959, 2006.
- 19 M. Kiyomi and T. Matsui. Integer Programming Based Algorithms for Peg Solitaire Problems. In *Computers and Games*, number 2063 in LNCS, pages 229–240. Springer, 2000.
- 20 C. Moore and D. Eppstein. One-Dimensional Peg Solitaire, and Duotaire. *arXiv/0008172*, 2000.
- 21 B. Ravikumar. Peg-solitaire, string rewriting systems and finite automata. In *Algorithms and Computation*, volume 1350, pages 233–242. Springer, 1997.
- 22 R. Uehara and S. Iwata. Generalized Hi-Q is NP-Complete. *IEICE TRANSACTIONS*, E73-E(2):270–273, 1990.