

# The Planar Tree Packing Theorem

Markus Geyer<sup>1</sup>, Michael Hoffmann<sup>\*2</sup>, Michael Kaufmann<sup>3</sup>,  
Vincent Kusters<sup>†4</sup>, and Csaba D. Tóth<sup>‡5</sup>

- 1 Wilhelm-Schickard-Institut für Informatik, Universität Tübingen, Tübingen, Germany  
geyer, @informatik.uni-tuebingen.de
- 2 Department of Computer Science, ETH Zürich, Zürich, Switzerland  
hoffmann@inf.ethz.ch
- 3 Wilhelm-Schickard-Institut für Informatik, Universität Tübingen, Tübingen, Germany  
mk@informatik.uni-tuebingen.de
- 4 Department of Computer Science, ETH Zürich, Zürich, Switzerland  
vincent.kusters@inf.ethz.ch
- 5 California State University Northridge, Los Angeles, USA  
cdtoth@acm.org

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## Abstract

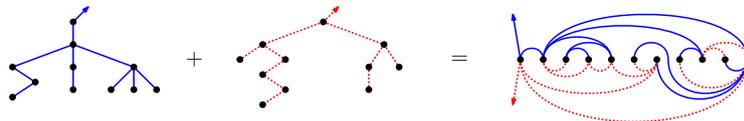
Packing graphs is a combinatorial problem where several given graphs are being mapped into a common host graph such that every edge is used at most once. In the planar tree packing problem we are given two trees  $T_1$  and  $T_2$  on  $n$  vertices and have to find a planar graph on  $n$  vertices that is the edge-disjoint union of  $T_1$  and  $T_2$ . A clear exception that must be made is the star which cannot be packed together with any other tree. But according to a conjecture of García et al. from 1997 this is the only exception, and all other pairs of trees admit a planar packing. Previous results addressed various special cases, such as a tree and a spider tree, a tree and a caterpillar, two trees of diameter four, two isomorphic trees, and trees of maximum degree three. Here we settle the conjecture in the affirmative and prove its general form, thus making it the planar tree packing theorem. The proof is constructive and provides a polynomial time algorithm to obtain a packing for two given nonstar trees.

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## 1 Introduction



The *packing problem* is to find a graph  $G$  on  $n$  vertices that contains a given collection  $G_1, \dots, G_k$  of graphs on  $n$  vertices each as edge-disjoint subgraphs. This problem has been

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studied in a wide variety of scenarios (see, e.g., [1, 3, 6]). Much attention has been devoted to the packing of trees (e.g., tree packing conjectures by Gyárfas [12] and by Erdős and Sós [5]). Hedetniemi [13] proved that any two nonstar trees can be packed into  $K_n$ . Teo and Yap [19] showed that *any* two graphs of maximum degree at most  $n - 1$  with a total of at most  $2n - 2$  edges pack into  $K_n$  unless they are one of thirteen specified pairs of graphs. Maheo et al. [14] characterized triples of trees that can be packed into  $K_n$ .

In the *planar packing* problem the graph  $G$  is required to be planar. García et al. [9] conjectured in 1997 that there exists a planar packing for any two nonstar trees, that is, for any two trees with diameter greater than two. The assumption that none of the trees is a star is necessary, since a star uses all edges incident to one vertex and so there is no edge left to connect that vertex in the other tree. García et al. proved their conjecture when one of the trees is a path and when the two trees are isomorphic. Oda and Ota [17] addressed the case that one of the trees is a caterpillar or that one of the trees is a spider of diameter at most four. A *caterpillar* is a tree that becomes a path when all leaves are deleted and a *spider* is a tree with at most one vertex of degree greater than two. Frati et al. [8] gave an algorithm to construct a planar packing of any spider with any tree. Frati [7] proved the conjecture for the case that both trees have diameter at most four. Finally, Geyer et al. [10] proved the conjecture for binary trees (maximum degree three). In this paper we settle the general conjecture in the affirmative:

► **Theorem 1.** *Every two nonstar trees of the same size admit a planar packing.*

**Related work.** Determining relationships between a graph and its subgraphs is one of the most studied topics in graph theory. The *subgraph isomorphism* problem asks to find a subgraph  $H$  in a graph  $G$ . The *graph thickness* problem [15] asks for the minimum number of planar subgraphs which the edges of a graph can be partitioned into. The *arboricity* problem [4] asks to determine the minimum number of forests which a graph can be partitioned into. Another related classical combinatorial problem is the  $k$  edge-disjoint spanning trees problem which dates back at least to Tutte [20] and Nash-Williams [16], who gave necessary and sufficient conditions for the existence of  $k$  edge-disjoint spanning trees in a graph. The interior edges of every maximal planar graph can be partitioned into three edge-disjoint trees, known as a *Schnyder wood* [18]. Gonçalves [11] proved that every planar graph can be partitioned in two edge-disjoint outerplanar graphs.

The study of relationships between a graph and its subgraphs can also be done the other way round. Instead of decomposing a graph, one can ask for a graph  $G$  that encompasses a given set of graphs  $G_1, \dots, G_k$  and satisfies some additional properties. This topic occurs with different flavors in the computational geometry and graph drawing literature. It is motivated by applications in visualization, such as the display of networks evolving over time and the simultaneous visualization of relationships involving the same entities. In the *simultaneous embedding* problem [2] the graph  $G = \bigcup G_i$  is given and the goal is to draw it so that the drawing of each  $G_i$  is plane. The *simultaneous embedding without mapping* problem [2] is to find a graph  $G$  on  $n$  vertices such that: (i)  $G$  contains all  $G_i$ 's as subgraphs, and (ii)  $G$  can be drawn with straight-line edges so that the drawing of each  $G_i$  is plane.

## 2 Notation and Overview

A *rooted tree* is a directed tree  $T$  with exactly one vertex of outdegree zero: its root, denoted  $\uparrow(T)$ . Every vertex  $v \neq \uparrow(T)$  has exactly one outgoing edge  $(v, p_T(v))$ . The target  $p_T(v)$  is the *parent* of  $v$  in  $T$ , and conversely  $v$  is a *child* of  $p_T(v)$ . In figures we denote the root

of a tree by an outgoing vertical arrow. For a vertex  $v$  of a rooted tree  $T$ , denote by  $t_T(v)$  the *subtree rooted at  $v$* , that is, the subtree of  $T$  induced by the vertices from which  $v$  can be reached on a directed path. The subscript is sometimes omitted if  $T$  is clear from the context. A *subtree of (or below)  $v$*  is a tree  $t_T(c)$ , for a child  $c$  of  $v$  in  $T$ . For a tree  $T$ , denote by  $|T|$  the *size* (number of vertices) of  $T$ . We denote by  $\deg_T(v)$  the degree (indegree plus outdegree) of  $v$  in  $T$ . For a graph  $G$  we denote by  $E(G)$  the edge set of  $G$ . A *star* is a tree on  $n$  vertices that contains at least one vertex of degree  $n - 1$ . Such a vertex is a *center* of the star. A star on  $n \neq 2$  vertices has a unique center. For a star on two vertices, both vertices act as a center. When considered as a rooted tree, there are two different rooted stars on  $n \geq 3$  vertices. A star rooted at a center is called *central-star*, whereas a star rooted at a leaf that is not a center is called a *dangling star*. In particular, every star on one or two vertices is a central-star. A *nonstar* is a graph that is not a star. A *one-page book embedding* of a graph  $G$  is an embedding of  $G$  into a closed halfplane such that all vertices are placed on the bounding line. This line is called the *spine* of the book embedding.

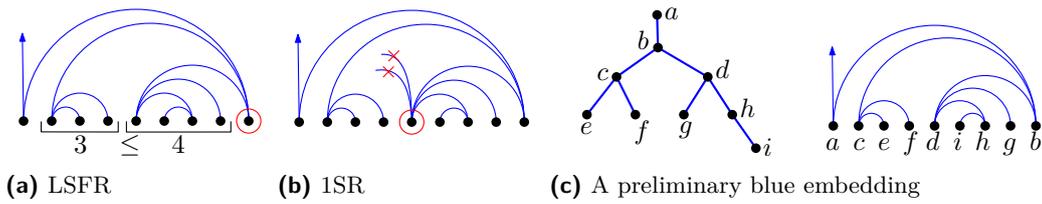
We embed vertices equidistantly along the positive  $x$ -axis and refer to them by their  $x$ -coordinate, that is,  $P = \{1, \dots, n\}$ . An *interval*  $[i, j]$  in  $P$  is a sequence of the form  $i, i + 1, \dots, j$ , for  $1 \leq i \leq j \leq n$ , or  $i, i - 1, \dots, j$ , for  $1 \leq j \leq i \leq n$ . Observe that we consider an interval  $[i, j]$  as oriented and so we can have  $i > j$ . Denote the *length* of an interval  $[i, j]$  by  $|[i, j]| = |i - j| + 1$ . To avoid notational clutter we often identify points from  $P$  with vertices embedded at them.

**Overview.** We construct a plane drawing of two  $n$ -vertex trees  $T_1$  and  $T_2$  on the point set  $P = [1, n]$ . We call  $T_1$  the *blue tree*; its edges are shown as solid blue arcs in figures. The tree  $T_2$  is called the *red tree*; its edges are shown as dotted red arcs. The algorithm first computes a preliminary one-page book embedding of  $T_1$  onto  $P$  (the *blue embedding*) in Section 3. In the second step we recursively construct an embedding for the red tree to pair up with the blue embedding. In principle we follow a similar strategy as in the first step, but we take the constraints imposed by the blue embedding into account. During this process we may reconsider and change the blue embedding locally. For instance, we may *flip* the embedding of some subtree of  $T_1$  on an interval  $[i, j]$ , that is, reflect the embedded tree at the vertical line  $x = \frac{i+j}{2}$  through the midpoint of  $[i, j]$ . In some cases we also perform more drastic changes to the blue embedding. In particular, the blue embedding may not be a one-page book embedding in the final packing. Although neither of the two trees  $T_1$  and  $T_2$  we start with is a star, it is possible – in fact, unavoidable – that stars appear as subtrees during the recursion. We have to deal with stars explicitly whenever they arise, because the general recursive step works for nonstars only. We introduce the necessary concepts and techniques in Section 4 and give the actual proof in Section 5.

### 3 A preliminary blue embedding

We begin by defining a preliminary one-page book embedding  $\pi : V_1 \rightarrow [1, n]$  for a tree  $T_1 = (V_1, E_1)$  rooted at  $r_1 \in V$ . In every recursive step, we are given a tree  $T$  rooted at a vertex  $r$  and an interval  $[i, j]$  of length  $|T|$ . Recall that we may have  $i < j$  or  $i > j$ . We place  $r$  at position  $i$  and recursively embed the subtrees of  $r$  on pairwise disjoint subintervals of  $[i, j] \setminus \{i\}$ . The embedding is guided by two rules illustrated in Figure 1.

- The *larger-subtree-first rule* (LSFR) dictates that for any two subtrees of  $r$ , the larger of the subtrees must be embedded on an interval closer to  $r$ . Ties are broken arbitrarily.
- The *one-side rule* (1SR) dictates that for every vertex all neighbors are mapped to the



■ **Figure 1** Illustrations for the two rules and an example embedding.

same side. That is, if  $N_T(v)$  denotes the set of neighbors of  $v$  in  $T$  (including its parent), then either  $\pi(u) < \pi(v)$  for all  $u \in N_T(v)$  or  $\pi(u) > \pi(v)$  for all  $u \in N_T(v)$ . These rules imply that every subtree  $T \subseteq T_1$  is embedded onto an interval  $[i, j] \subseteq [1, n]$  so that  $\{i, j\}$  is an edge of  $T$  and either  $i$  or  $j$  is the root of  $T$ . Together with  $\pi(r_1) = 1$ , these rules define the embedding (up to tiebreaking). See Figure 1c for an example.

#### 4 A red tree and a blue forest

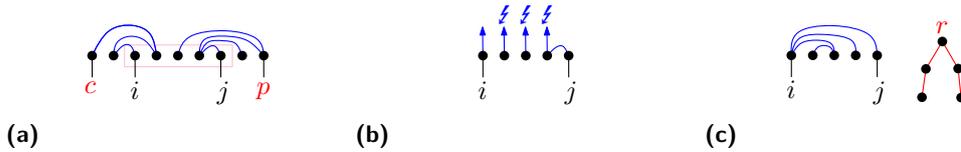
As common with inductive proofs, we prove a stronger statement than necessary. This stronger statement does not hold unconditionally but we need to impose some restrictions on the input. The goal of this section is to derive this more general statement – formulated as Theorem 2 – from which Theorem 1 follows easily.

Our algorithm receives as input a nonstar subtree  $R$  of the red tree and an interval  $I = [i, j]$  of size  $|R|$  along with a blue graph  $B$  embedded on  $I$ . Without loss of generality we assume  $i < j$ . In the initial call  $B$  is a tree, but in a general recursive call  $B$  is a *blue forest* that may consist of several components. For  $k \in [i, j]$  let  $B\langle k \rangle$  denote the component of  $B$  that contains  $k$ . For  $[x, y] \subseteq [i, j]$  let  $B[x, y]$  denote the subgraph of  $B$  induced by the vertices in  $[x, y]$ , and for  $k \in [x, y]$  let  $B[x, y]\langle k \rangle$  denote the component of  $B[x, y]$  that contains  $k$ .

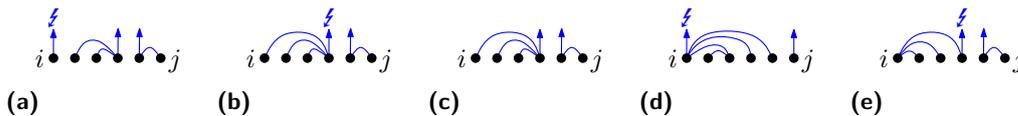
In general the algorithm sees only a small part of the overall picture because it has access to the vertices in  $I$  only. However, blue vertices in  $I$  may have edges to vertices outside of  $I$  and also vertices of  $R$  may have neighbors outside of  $I$ . We have to ensure that such *outside edges* are used by one tree only and can be routed without crossings. In order to control the effect of outside edges, we allow only one vertex in each component – that is, the root of  $R$  and the root of each component of  $B$  – to have neighbors outside of  $I$ . Whenever we change the blue embedding we need to maintain the relative order of these roots so as to avoid crossings among outside edges.

**Conflicts.** Typically  $r := \uparrow(R)$  has at least one neighbor outside of  $I$ : its parent  $p_{T_2}(r)$ . But  $r$  may also have children in  $T_2 \setminus R$ . We assume that all neighbors – parent and children – of  $r$  in  $T_2 \setminus R$  are already embedded outside of  $I$  when the algorithm is called for  $R$ . There are two principal obstructions for mapping  $r$  to a point  $v \in I$ :

- A vertex  $v \in I$  is in *edge-conflict* with  $r$ , if  $\{v, r'\} \in E(T_1)$  for some neighbor  $r'$  of  $r$  in  $T_2 \setminus R$ . Mapping  $r$  to  $v$  would make  $\{v, r'\}$  an edge of both  $T_1$  and  $T_2$  (Figure 2a–2b). In figures we mark vertices in edge-conflict with  $r$  by a lightning symbol ⚡.
- A vertex  $v \in I$  is in *degree-conflict* with  $r$  on  $I$  if  $\deg_R(r) + \deg_B(v) \geq |I|$ . If we map  $r$  to  $v$ , then no child of  $r$  in  $R$  can be mapped to the same vertex as a child of  $v$  in  $B$ . With only  $|I| - 1$  vertices available there is not enough room for both groups (Figure 2c).



■ **Figure 2** An interval  $[i, j]$  on which a tree  $R = t(r)$  is to be embedded. Two neighbors  $p$  and  $c$  of  $r$  in  $T_2 \setminus R$  are already embedded (a). Then the situation on  $[i, j]$  presents itself as in (b), where the three central vertices are in edge-conflict with  $r$  due to blue outside edges to  $p$  or  $c$ . In (c) the vertex  $i$  is in degree-conflict with  $r$  because  $\deg_R(r) + \deg_B(i) = 2 + 3 = 5 \geq |[i, j]|$ . We cannot map  $r$  to the blue vertex at  $i$  because there is not enough room for the neighbors of both in  $[i, j]$ .



■ **Figure 3** An interval  $[i, j]$  in edge-conflict (a)–(b), and examples where  $[i, j]$  is not in edge-conflict (c)–(e). In (c) the center of  $B^*$  is not in edge-conflict; it may be in degree-conflict, though, if  $\deg_R(r) \geq 3$ . In both (d) and (e) the tree  $B\langle i \rangle$  is not a central-star.

We cannot hope to avoid conflicts entirely and we do not need to. It turns out that is sufficient to avoid a very specific type of conflict involving stars.

- An interval  $[i, j]$  is in *edge-conflict* (*degree-conflict*) with  $R = t(r)$  if  $B^* := B\langle i \rangle$  is a central-star and the root of  $B^*$  is in edge-conflict (*degree-conflict*) with  $r$  (Figure 3).
- An interval  $I$  is in *conflict* with  $R$  if  $I$  is in edge-conflict or degree-conflict with  $R$  (or both).

We claim that  $R$  can be packed with  $B$  onto  $I$  unless  $I$  is in conflict with  $R$ . The following theorem presents a precise formulation of this claim. Only  $R$  and the graph  $B\langle i \rangle$  determine whether or not an interval  $[i, j]$  is in conflict with  $R$ . Therefore we can phrase the statement without referring to an embedding of  $B$  but just regarding it as a sequence of trees. The set  $C$  represents the set of roots from  $B$  that are in edge-conflict with  $r$ .

► **Theorem 2.** *Let  $R$  be a nonstar tree with  $r = \uparrow(R)$  and let  $B$  be a nonstar forest with  $|R| = |B| = n$ , together with an ordering  $b_1, \dots, b_k$  of the  $k \in \{1, \dots, n\}$  roots of  $B$  and a set  $C \subseteq \{b_1, \dots, b_k\}$ . Suppose (i)  $t_B(b_1)$  is not a central-star or (ii)  $b_1 \notin C$  and  $\deg_R(r) + \deg_B(b_1) < n$ . Then there is a plane packing  $\pi$  of  $B$  and  $R$  onto any interval  $I$  with  $|I| = n$  such that*

- $\pi(r) \notin \pi(C)$  and
- *we can access  $b_1, \dots, b_k, r$  in this order from the outer face of  $\pi$ , that is, we can add a new vertex  $v$  in the outer face of  $\pi$  and route an edge to each of  $b_1, \dots, b_k, r$  such that the resulting multigraph is plane and the circular order of neighbors around  $v$  is  $b_1, \dots, b_k, r$ . (If  $r = b_i$ , for some  $i \in \{1, \dots, k\}$ , then two distinct edges must be routed from  $v$  to  $r$  so that the result is a non-simple plane multigraph.)*

Such a packing  $\pi$  we call an *ordered plane packing* of  $B$  and  $R$  onto  $I$ .

Theorem 2 is a strengthening of Theorem 1 and so we obtain Theorem 1 as an easy corollary.

**Proof of Theorem 1 from Theorem 2.** Select roots arbitrarily so that  $T_1 = t(r_1)$  and  $T_2 = t(r_2)$ . Then use Theorem 2 with  $R = T_2$ ,  $B = T_1$ ,  $k = 1$ ,  $b_1 = r_1$ , and  $C = \emptyset$ . By assumption  $T_1$  is not a star and so (i) holds. Therefore we can apply Theorem 2 and obtain the desired plane packing of  $T_1$  and  $T_2$ . ◀



(a)  $b_1 \in C$  (b)  $\deg_R(r) + \deg_{t(b_1)}(b_1) \geq n$

Figure 4 The statement of Theorem 2 does not hold without (i) or (ii). In the examples the trees of  $B$  are ordered from left to right so that  $t(b_1)$  is a central-star. Vertices in  $C$  are labeled with  $\ell$ .

It is not hard to see that forbidding conflicts in Theorem 2 is necessary: The example families depicted in Figure 4 do not admit an ordered plane packing.

### 5 Embedding the red tree

In this section we outline our recursive embedding algorithm to prove Theorem 2. We are given a red tree  $R = t(r)$ , a blue forest  $B$  with roots  $b_1, \dots, b_k$ , an interval  $I = [i, j] \subseteq [1, n]$  with  $|I| = |R| = |B|$ , and a set  $C$  that we consider to be the vertices from  $B$  in edge-conflict with  $r$ . As a first step, we embed  $B$  onto  $I$  by embedding  $t(b_1), \dots, t(b_k)$  in this order from left to right, each time using the algorithm from Section 3.

► **Observation 3.** We may assume that  $R, B$  and  $I = [i, j]$  satisfy the following invariants:

- I1.  $I$  is not in conflict with  $R$ . (peace invariant)
- I2. Every component of  $B$  satisfies LFSR and 1SR. All edges of  $B$  are drawn in the upper halfplane (above the  $x$ -axis). All roots of  $B$  are visible from above (that is, a vertical ray going up from  $b_x$  does not intersect any edge of  $B$ ). (blue-local invariant)
- I3.  $i$  is not in edge-conflict with  $r$ . (placement invariant)

**Proof.** 1 follows from the assumption (i) or (ii) in Theorem 2. 2 is achieved by using the embedding from Section 3. If  $i$  is in conflict with  $r$ , then 1 implies that  $B\langle i \rangle$  is not a singleton (which would be a central-star). Therefore flipping  $B\langle i \rangle$  establishes 3 without affecting 1 or 2. ◀

Theorem 2 ensures that all roots of  $B$  along with  $r$  appear on the outer face in the specified order. We cannot assume that we can draw an edge to any other vertex of  $B$  or  $R$  without crossing edges of the embedding given by Theorem 2. Therefore it is important that whenever the algorithm is called recursively,

- I4. only the roots  $b_1, \dots, b_k$  and  $r$  have edges to the outside of  $I$ .

► **Observation 4.** If  $B$  satisfies 2 and 4 on an interval  $I$ , then both invariants also hold for  $B[x, y]$  on  $[x, y]$ , for every subinterval  $[x, y] \subseteq I$ .

In the remainder of the proof we will ensure and assume that invariants 1–4 hold for every call of the algorithm.

**The algorithm.** Let  $s$  denote a child of  $r$  that minimizes  $|t_R(c)|$  among all children  $c$  of  $r$  in  $R$ . Denote  $S = t_R(s)$  and  $R^- = R \setminus S$ . If  $|R^-| \geq 2$ , then  $R^-$  cannot be a central-star: if it were, then  $|S| = 1$  and  $R$  would be a star. Ideally, we can recursively embed  $S$  onto  $[j, j - |S| + 1]$  and  $R^-$  onto  $[i, j - |S|]$  (Figure 5a). But in general the invariants may not hold for the recursive subproblems. For instance, some of the subgraphs could be stars, or if

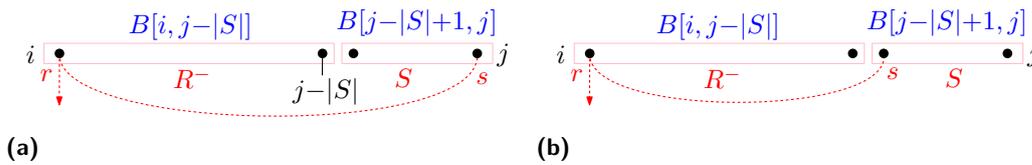


Figure 5 Our recursive strategy in an ideal world.

$\{i, j\} \in E(B)$ , then placing  $r$  at  $i$  may put  $[j, j - |S| + 1]$  in edge-conflict with  $S$ . Therefore, we explore a number of alternative strategies, depending on which – if any – of the four forests  $R^-$ ,  $S$ ,  $B[i, j - |S|]$  and  $B[j - |S| + 1, j]$  in our decomposition is a star.

Due to space constraints we can only discuss a small part of the proof in the main text. We chose to present the general case, in which none of  $R^-$ ,  $S$ ,  $B[i, j - |S|]$  and  $B[j - |S| + 1, j]$  is a star. The analysis for the remaining cases can be found in the full paper.

► **Lemma 5.** *If none of  $S$ ,  $R^-$ ,  $B[i, j - |S|]$ , and  $B[j - |S| + 1, j]$  is a star, then there is an ordered plane packing of  $B$  and  $R$  onto  $I$ .*

**Proof.** As  $S$  is a minimum size subtree of  $r$  in  $R$ , and neither  $S$  nor  $R^-$  is a star, we know that  $r$  has at least one more subtree other than  $S$  and every subtree of  $r$  in  $R$  has size at least four. (All trees on three or less vertices are stars.) It follows that

$$\deg_{R^-}(r) \leq (|R^-| - 1)/4. \tag{1}$$

The general plan is to use one of the following two options. In both cases we first embed  $R^-$  recursively onto  $[i, j - |S|]$ . Then we conclude as follows.

**Option 1:** Embed  $S$  recursively onto  $[j, j - |S| + 1]$  (Figure 5a).

**Option 2:** Embed  $S$  recursively onto  $[j - |S| + 1, j]$  (Figure 5b).

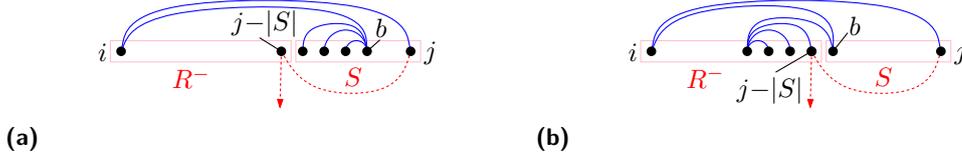
In some cases neither of these two options works and so we have to use a different embedding.

As we embed  $S$  after  $R^-$ , the (final) mapping for  $s$  is not known when embedding  $R^-$ . However, we need to know the position of  $s$  in order to determine the conflicts for embedding  $R^-$ . Therefore, before embedding  $R^-$  we *provisionally* embed  $s$  at  $\alpha := \uparrow(B[j - |S| + 1, j]\langle j \rangle)$  (Option 1) or  $\alpha := \uparrow(B[j - |S| + 1, j]\langle j - |S| + 1 \rangle)$  (Option 2). That is, for embedding  $R^-$  we pretend that some neighbor of  $r$  is embedded at  $\alpha$ . In this way we ensure that  $S$  is not in edge-conflict with the interval in its recursive embedding. The final placement for  $s$  is then determined by the recursive embedding of  $S$ , knowing the definite position of its parent  $r$ .

For the recursive embeddings to work, we need to show that the invariants 1, 2 and 4 hold (3 then follows as in Observation 3). For 2 and 4 this is obvious by construction and Observation 4, as long as we do not change the embedding of  $B$ . As we do not change the embedding in Option 1 and 2, it remains to ensure 1. So suppose that for both options, 1 does not hold for at least one of the two recursive embeddings. There are two possible obstructions for 1: edge-conflicts and degree-conflicts. We discuss both types of conflicts, starting with edge-conflicts.

**Case 1**  $[i, j - |S|]$  is not in degree-conflict with  $R^-$  and  $[j, j - |S| + 1]$  is not in degree-conflict with  $S$ . Then Option 1 works, unless  $[i, j - |S|]$  is in edge-conflict with  $R^-$ . Recall that  $[j, j - |S| + 1]$  is not in edge-conflict with  $S$  after embedding  $R^-$  onto  $[i, j - |S|]$ .

We claim that an edge-conflict between  $R^-$  and  $[i, j - |S|]$  implies  $\{i, j\} \in E(B)$ . To prove this claim, suppose that  $[i, j - |S|]$  is in edge-conflict with  $R^-$ . Then  $B[i, j - |S|]\langle i \rangle$  is a central-star whose root  $c$  is in edge-conflict with  $r$ . If  $c = i$ , then by 3 there was no such conflict initially. So, as claimed, the conflict can only come from a blue edge to  $s$



■ **Figure 6** A third embedding when the first two options fail.

(provisionally placed) at  $j$ . Otherwise,  $c > i$  and by 1SR there is no edge in  $B$  from  $c$  to any point in  $[c + 1, j]$ . It follows that  $B[i, j - |S|] \langle i \rangle = B \langle i \rangle$ . The conflict between  $c$  and  $r$  does not come from the edge to  $s$  but from an edge to a vertex outside of  $[i, j]$ . This contradicts 1 for  $R^-$  and  $[i, j]$ , which proves the claim.

The presence of the edge  $\{i, j\}$  implies that  $B$  is a tree and by 4 only (the root)  $i$  or  $j$  may have edges out of  $[i, j]$ . Consider Option 2, which embeds  $S$  onto  $[j - |S| + 1, j]$ , provisionally placing  $s$  at  $\uparrow(B[j - |S| + 1, j] \langle j - |S| + 1 \rangle)$ . There are two possible obstructions: an edge-conflict for  $R^-$  or a degree-conflict for  $S$ . In both cases we face a central-star  $B^* = B[j - |S| + 1, b]$  with center  $b \in [j - |S| + 1, j - 1]$ . Due to 1SR and  $\{i, j\} \in E(B)$ , we know that  $b = \uparrow(B[j - |S| + 1, j] \langle j - |S| + 1 \rangle)$ . We distinguish three cases.

**Case 1.1**  $\{i, b\} \in E(B)$ . Then we consider a third option: provisionally place  $s$  at  $j$ , embed  $R^-$  recursively onto  $[j - |S|, i]$  and then  $S$  onto  $[j, j - |S| + 1]$  (Figure 6a). The edge  $\{i, b\}$  of  $B$  prevents any edge-conflict between  $[j - |S|, i]$  and  $R^-$  (and, as before, for  $S$ ). Given that we assume in Case 1 that  $[j, j - |S| + 1]$  is not in degree-conflict with  $S$ , we are left with  $[j - |S|, i]$  being in degree-conflict with  $R^-$  as a last possible obstruction.

Then the tree  $B[i, j - |S|] \langle j - |S| \rangle$  is a central-star  $A^*$  with root  $a$  such that

$$\deg_{A^*}(a) + \deg_{R^-}(r) \geq |R^-|. \quad (2)$$

The following lemma holds in general. Its proof is omitted due to space constraints.

► **Lemma 6.** *If  $\deg_R(r) \geq 2$ , then  $|R^-| \geq |S| + \deg_{R^-}(r)$ .*

Combining Lemma 6 with (2) we get  $|A^*| = \deg_{A^*}(a) + 1 \geq |S| + 1 \geq 5$ . Note that  $A^*$  can be huge, but we know that it does not include  $i$  (because  $B[i, j - |S|]$  is not a star). We also know that  $a \neq j - |S|$ : If  $a = j - |S|$ , then by 1SR we have  $p_B(a) \in [i, j - |S| - 1]$ , in contradiction to  $a = \uparrow(B[i, j - |S|] \langle j - |S| \rangle)$ . Therefore  $a = j - |S| - |A^*| + 1$  and by 1SR its parent is to the right. Due to  $\{i, b\} \in E(B)$  and since  $B[j - |S| + 1, b]$  is a tree rooted at  $b$ , we have  $p_B(a) = b$ . As  $A^*$  is a subtree of  $b$  in  $B$  on at least five vertices, by LSFRR  $b$  cannot have a leaf at  $b - 1$ . Therefore, the star  $B[j - |S| + 1, j] \langle j - |S| + 1 \rangle$  consists of a single vertex only, that is,  $b = j - |S| + 1$  (Figure 6b). We consider two subcases. In both the packing is eventually completed by recursively embedding  $S$  onto  $[j, j - |S| + 1]$ .

**Case 1.1.1**  $\{x, b\} \in E(B)$ , for some  $x \in [i + 1, a - 1]$  (Figure 7a). Select  $x$  to be maximal with this property. Then we exchange the order of the two subtrees  $t(x)$  and  $A^*$  of  $b$  (Figure 7b). This may violate LSFRR for  $B$  at  $b$ , but 2 holds for both  $B[i, j - |S|]$  and  $B[j - |S| + 1, j]$ . Clearly there is still no edge-conflict for  $[j - |S|, i]$  with  $R^-$  after this change. We claim that there is no degree-conflict anymore, either.

To prove the claim, note that by LSFRR at  $b$  we have  $|t(x)| \leq |A^*|$ . As the size of both subtrees combined is at most  $|R^-|$ , we have  $|t(x)| \leq |R^-|/2$ . Then, using (1),  $|t(x)| - 1 + \deg_{R^-}(r) < |R^-|/2 + \deg_{R^-}(r) < 3|R^-|/4 < |R^-|$ . Therefore after the exchange  $[j - |S|, i]$  is not in degree-conflict with  $R^-$ , which proves the claim and concludes this case.

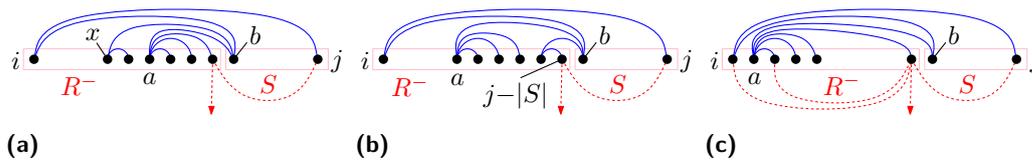


Figure 7 Swapping two subtrees of  $b$  in Case 1.1.1 and an explicit embedding for Case 1.1.2.

**Case 1.1.2**  $i$  and  $a = j - |S| - |A^*| + 1$  are the only neighbors of  $b$  in  $B$ . We claim that in this case  $A^*$  extends all the way up to  $i + 1$ , that is,  $A^* = B[i + 1, j - |S|]$ . To prove this claim, suppose to the contrary that  $a \geq i + 2$ . Then there is another subtree of  $i$  to the left of  $a$  and, in particular,  $\{i, a - 1\} \in E(B)$ . By LSFR this closer subtree is at least as large as  $A^*$ . Using (1) and (2) we get  $|[i + 1, a - 1]| + |A^*| \geq 2|A^*| > 2(|R^-| - \deg_{R^-}(r)) > 3|R^-|/2 > |R^-|$ , in contradiction to  $|[i + 1, a - 1]| + |A^*| < |R^-|$ . Therefore  $a = i + 1$ , as claimed (Figure 7c).

The vertex  $a$  has high degree in  $B$  but it is not adjacent to  $i$ . Therefore, we can embed  $R^-$  as follows: put  $r$  at  $j - |S|$  and embed an arbitrary subtree  $Y$  of  $r$  onto  $[i, i + |Y| - 1]$  recursively or, if it is a star, explicitly, using the locally isolated vertex at  $i$  for the center (and  $i + |Y| - 1$  for the root in case of a dangling star). As  $i$  is isolated on  $[i, i + |Y| - 1]$  there is no conflict between  $[i, i + |Y| - 1]$  and  $Y$ . As  $|Y| \geq |S| \geq 4$ , the remaining graph  $B[i + |Y|, j - |S| - 1]$  consists of isolated vertices only, on which we can explicitly embed any remaining subtrees of  $r$  using the algorithm from Section 3.

**Case 1.2**  $\{i, b\} \notin E(B)$  and  $b = p_B(j - |S|)$ . Then  $j - |S|$  is a locally isolated vertex in  $B[i, j - |S|]$ , whose only neighbor in  $B$  is at  $b \notin B[j - |S| + 1, j](j)$ . Therefore, we can provisionally place  $s$  at  $j$  so that  $[j - |S|, i]$  is not in conflict with  $R^-$ . By the assumption of Case 1  $[j, j - |S| + 1]$  is not in degree-conflict with  $S$ . Therefore, we obtain the claimed packing by first embedding  $R^-$  onto  $[j - |S|, i]$  recursively and then  $S$  onto  $[j, j - |S| + 1]$ .

**Case 1.3**  $\{i, b\} \notin E(B)$  and  $b \neq p_B(j - |S|)$ . As  $\{i, b\} \notin E(B)$  and  $s$  is provisionally placed at  $b$ , the interval  $[i, j - |S|]$  is not in edge-conflict with  $R^-$ . Thus, Option 2 (Figure 5b) succeeds unless  $[j - |S| + 1, j]$  is in degree-conflict with  $S$ . Hence suppose

$$\deg_S(s) + \deg_{B^*}(b) \geq |S|. \tag{3}$$

The following lemma holds in general. Its easy proof is omitted due to space constraints.

► **Lemma 7.** *If an interval  $[i, j]$  is in degree-conflict with a nonstar subtree  $R$  of  $T_2$ , then  $B(i)$  is a central-star on at least three vertices.*

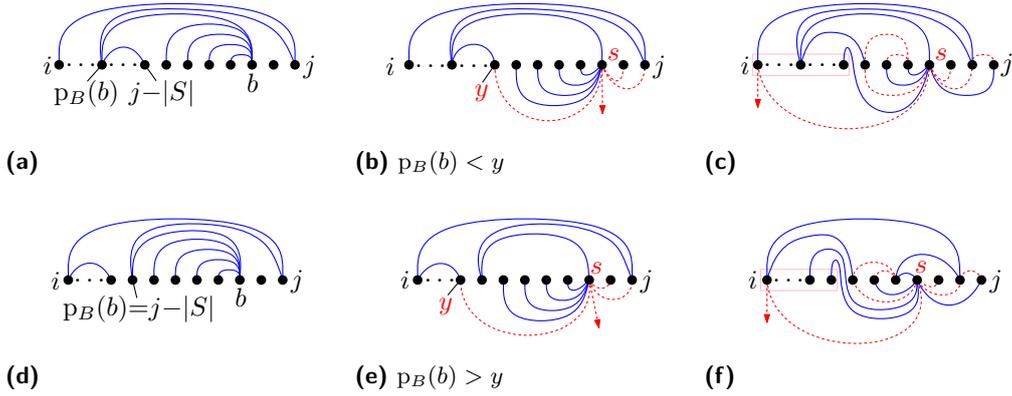
Hence,  $|B^*| \geq 3$  by Lemma 7. As  $b \neq p_B(j - |S|)$ , by LSFR  $b$  has exactly one neighbor in  $B$  outside of  $B^*$ : its parent  $p_B(b) \in [i + 1, j - |S|]$  (Figure 8a). Let  $B^+ = B^* \cup \{p_B(b)\}$ . The plan is to explicitly embed  $S$  such that the children of  $s$  that do not fit on  $[b + 1, j]$  – there is no room for all by (3) – are put on  $[i + 1, j] \setminus B^+$ , and the rest of  $S$  on  $B^*$ . For this to work, we need

$$|S| - \deg_S(s) \leq |B^*| \text{ and } \deg_S(s) \leq |I| - 1 - |B^+|. \tag{4}$$

The first inequality follows from (3) and the second from  $|B^*| + 1 + \deg_S(s) \leq (|S| - 1) + 1 + (|S| - 1) \leq |S| + (|R^-| - 1) - 1 = |I| - 2$ . We embed  $S$  as follows: put  $s$  at  $b$  and embed a child of  $s$  onto every vertex in  $[b + 1, j]$ . By (3), at least one child of  $s$  remains. Let  $y$  be such that the remaining children of  $s$  fit exactly in  $[y, j - |S|] \setminus \{p_B(b)\}$  and embed them there.

Suppose  $p_B(b) < y$ . Denote the  $d := \deg_S(s)$  children of  $s$  in  $S$  by  $c_1, \dots, c_d$  from right to left ( $c_1$  is embedded at  $j$  and  $c_d$  at  $y$ ). For each  $c_k$ ,  $1 \leq k \leq d$ , we need an additional

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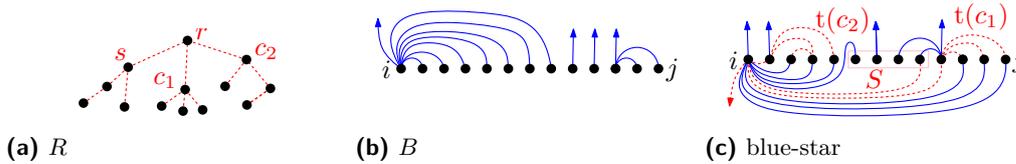
■ **Figure 8** Explicit embedding of  $S$  in Case 1.3. The edge  $\{p_B(b), j\}$  need not be present in  $B$ .

$|t_S(c_k)| - 1$  vertices to embed  $t_S(c_k)$ . For this, we use the leaves of  $B^*$ . Mirror  $B^*$  vertically into the bottom halfplane (Figure 8b). For each  $c_k$ , move  $|t_S(c_k)| - 1$  leaves of  $B^*$  immediately to the right of where  $c_k$  is embedded. This requires  $|S| - \deg_S(s) - 1$  leaves in  $B^*$ , which is ensured by the first inequality from (4). In this way, each  $c_k$  has  $|t_S(c_k)| - 1$  locally isolated vertices to its right, on which we embed  $t_S(c_k) \setminus \{c_k\}$  explicitly in the *upper* halfplane (Figure 8c), using the algorithm from Section 3. This explicit embedding works also if  $t_S(c_k)$  is a star. Any remaining leaves of  $B^*$  are placed immediately to the left of  $c_d$ . For every such leaf, the blue edge to  $s$  is routed so that it starts upwards and crosses the spine immediately to the left of  $y = c_d$ . In this way the remaining blue vertices in  $I$  form an interval.

Otherwise  $p_B(b) > y$ , which in this case by LSFR for  $p_B(b)$  and (4) implies  $p_B(b) = j - |S|$  (Figure 8d–8e). If we proceed as above, then the remaining blue vertices do not form an interval because  $p_B(b)$  is separated from the rest. We address this problem as follows. Since  $p_B(b) > y$ , the procedure above embeds a child of  $s$  at  $p_B(b) - 1$ . As  $S$  is not a star,  $s$  has a non-leaf child  $v$ . We modify the procedure to embed  $v$  at  $p_B(b) - 1$  and let  $p_B(b)$  take the role of a leaf of  $B^*$  for the purpose of embedding  $t_S(v)$  (Figure 8f).

After embedding  $S$  we complete the packing by recursively embedding  $R^-$  onto the interval  $[i, j - |S|]$ . This interval differs from the interval  $[i, j - |S|]$  before embedding  $S$  only in that some (possibly empty) suffix of vertices has been replaced by locally isolated vertices. These vertices correspond to leaves of the star  $B^*$ , and since we embedded  $s$  at the center of  $B^*$ , they are all in edge-conflict with  $r$ . The “new” interval  $[i, j - |S|]$  is not in edge-conflict with  $R^-$  by 3,  $\{i, j\} \in E(B)$ , and  $\{i, b\} \notin E(B)$ . Suppose towards a contradiction that  $[i, j - |S|]$  is in degree-conflict with  $R^-$ . Then due to  $\{i, j\} \in E(B)$  there is a central-star  $D^* = B[i, j - |S|](i)$  rooted at  $i$ . By LSFR  $B[i, j - |S|](i)$  was a central-star of size at least  $|D^*|$  before embedding  $S$ . But then  $[i, j - |S|]$  is in degree-conflict with  $R^-$  before embedding  $S$ , contrary to the assumption of Case 1. This completes the proof of Case 1.

**Blue-star embedding.** The procedure we used to embed  $S$  explicitly in Case 1.3 is useful in general. Hence we define this procedure, which we call *blue-star embedding*, in a more general form. Let  $A = t_R(a)$  be a subtree rooted at a vertex  $a$  of  $R$ , and let  $\sigma \in [i, j]$  be the center of a star  $B^* = t_B(\sigma)$ . Either  $\sigma$  is the root of  $B(\sigma)$  or  $\tau := p_B(\sigma) \in [i, j]$ . Denote by  $B^+$  the subgraph of  $B$  induced by  $\sigma$  and all its neighbors (parent and children). Note that either  $B^+ = B^*$  or  $B^+ = B^* \cup \{\tau\}$ . Put  $d = \deg_A(a)$  and let  $\varphi = (v_1, \dots, v_d)$  be a sequence of elements from  $B \setminus B^+$ . Furthermore, suppose the following four conditions hold:



■ **Figure 9** Handling a degree-conflict for  $R^-$  in Case 2.1.

- BS1.**  $a$  is not in edge-conflict with  $\sigma$ ,
- BS2.**  $|A| \leq |B^*| + \deg_A(a)$  and  $|B^+| + \deg_A(a) \leq |R| - 1$ ,
- BS3.** at least one of  $B \setminus (B^* \cup \varphi)$  or  $B \setminus (B^+ \cup \varphi)$  forms an interval, and
- BS4.** if  $B \setminus (B^* \cup \varphi)$  does not form an interval, then  $A$  is not a central-star,  $v_1 = \tau \pm 1$  and  $\{v_1, \tau\} \notin E(B)$ .

Then we can *blue-star embed*  $A$  starting from  $\sigma$  with  $\varphi$  such that  $a$  is embedded at  $\sigma$ , the children of  $a$  in  $A$  are embedded onto  $\varphi$ , the remaining vertices of  $A$  are embedded on  $B^* \setminus \{\sigma\}$ , and the remaining blue vertices form an interval  $[i', j']$ . A precise formulation with proof can be found in the full paper; here we provide some intuition only. 1 is necessary to put  $a$  at  $\sigma$ . 2 corresponds to (4). 3 ensures that the remaining blue vertices form an interval on which we can continue to work and 4 allows us to pull the trick of using  $p_B(\sigma)$  in place of a leaf of  $B^*$ , if applicable.

**Case 2**  $[i, j - |S|]$  is in degree-conflict with  $R^-$ . Then  $B[i, j - |S|]\langle i \rangle$  is a central-star  $B[i, x]$

$$\text{with } \deg_{R^-}(r) + (x - i) \geq |R^-| \tag{5}$$

and  $|B[i, x]| = x - i + 1 \geq 3$  by Lemma 7. We distinguish two cases.

**Case 2.1**  $B\langle i \rangle = B[i, x]$ . Then  $B\langle i \rangle \neq B\langle j \rangle$ . If necessary, flip  $B\langle i \rangle$  to put its center at  $i$ . If  $B\langle j \rangle$  is a central-star on  $\geq 3$  vertices, then – if necessary – flip  $B\langle j \rangle$  to put its root at  $j$ . We use a blue-star embedding for  $R^-$  starting from  $\sigma = i$  with  $\varphi = (x + 1, \dots)$ . As  $\varphi$  consists of  $d := \deg_{R^-}(r)$  vertices, we have  $[i, j] \setminus (B[i, x] \cup \varphi) = [x + d + 1, j]$ . If  $B[x + d + 1, j]\langle j \rangle$  is a central-star on  $\geq 3$  vertices, then use  $\varphi = (j, x + 1, \dots)$  instead (and note that  $\uparrow(B[x + d + 1, j]\langle j \rangle) = j$ ).

In the notation of the blue-star embedding we have  $B^* = B^+ = B[i, x]$ . We need to show that the conditions for this embedding hold. 1 holds by 1 (for embedding  $R$  onto  $[i, j]$ ). For 2 we have to show  $|R^-| \leq |B^*| + \deg_{R^-}(r) \leq |R| - 1$ . The first inequality holds by (5) and  $|B^*| \geq x - i$ . The second inequality holds due to 1 (for embedding  $R$  onto  $[i, j]$ ), which implies  $\deg_R(r) + (x - i) \leq |R| - 1$ . As  $|B[i, x]| = x - i + 1$  and  $\deg_R(r) = \deg_{R^-}(r) + 1$ , 2 follows. 3 is obvious by the choice of  $\varphi$  and 4 is trivial for  $B^* = B^+$  due to 3. That leaves us with an interval  $[i', j']$ , where  $j' \in \{j, j - 1\}$ . We claim that  $[j', i']$  is not in conflict with  $S$ .

To prove the claim we consider two cases. If  $j' = j - 1$ , then initially  $B[x + d + 1, j]\langle j \rangle$  was a central-star on  $\geq 3$  vertices rooted at  $j$ . By the choice of  $\varphi$  a leaf of this star is at  $j'$  whose only neighbor in  $B$  is at  $j \neq i = r$ . Therefore  $[j', i']$  is not in edge-conflict with  $S$ . As  $B[j', i']\langle j' \rangle$  is an isolated vertex, by Lemma 7 there is no degree-conflict between  $[j', i']$  and  $S$ , either, which proves the claim.

Otherwise,  $j' = j$  and  $B[j', i']\langle j' \rangle = B[x + d + 1, j]\langle j \rangle$  is not a central-star on  $\geq 3$  vertices. Therefore by Lemma 7 there is no degree-conflict between  $[j', i']$  and  $S$ . In order to show that there is no edge-conflict, either, it is enough to show that  $\uparrow(B[j', i']\langle j' \rangle)$  is not adjacent to  $i = r$  in  $B$ . If  $\uparrow(B[j', i']\langle j' \rangle) \neq j'$  this follows from 1SR. Otherwise  $\uparrow(B[j', i']\langle j' \rangle) = j' = j$ , and  $\{i, j\} \notin E(B)$  because  $B\langle i \rangle$  is a star but  $B$  is not. Therefore the claim holds and we can complete the packing by recursively embedding  $S$  onto  $[j', i']$ .

**Case 2.2**  $B\langle i \rangle \neq B[i, x]$ . By 1SR this means that  $i = p_B(x)$  and  $i$  has at least one more neighbor in  $[i, j] \setminus B[i, x]$ . Since by assumption  $B[i, j - |S|]$  is not a star, we have  $x \leq j - |S| - 1$ . Since  $B[i, x]$  is a central-star and  $x \leq j - |S| - 1$ , by LSFR for  $i$  the only neighbor of  $i$  in  $B$  outside of  $B[i, x]$  is its parent  $p_B(i) \in [j - |S| + 1, j]$ . We claim that such a configuration is impossible. To prove the claim, note that  $p_B(i)$  has at least two children in  $B[i, j - |S|]$  because  $x \leq j - |S| - 1$  and  $p_B(i) \geq j - |S| + 1$ . By LSFR, the corresponding subtrees have size at least  $|B[i, x]| = x - i + 1$ , and so  $|R| \geq |B[i, p_B(i)]| \geq 2|B[i, x]| + 1 \geq 2(|R^-| - \deg_{R^-}(r) + 1) + 1$ , where the last inequality uses (5). Rewriting and using (1) yields

$$|R^-| \leq \frac{|R| - 1}{2} + \deg_{R^-}(r) - 1 < \frac{|R| - 1}{2} + \frac{|R^-|}{4}.$$

It follows that  $|R^-| < \frac{2}{3}(|R| - 1)$  and hence that  $|S| > \frac{1}{3}(|R| - 1)$ . Since  $S$  is a smallest subtree of  $r$  in  $R$ , this means that  $r$  is binary in  $R$  and thus unary in  $R^-$ . This, finally, contradicts the degree-conflict for  $[i, j - |S|]$  with  $R^-$  because  $x < j - |S|$  and hence  $\deg_{R^-}(r) + \deg_{B[i, x]}(i) = 1 + (x - i) < 1 + (j - |S|) - i = |R^-|$ .

**Case 3**  $[j, j - |S| + 1]$  is in degree-conflict with  $S$  and  $[i, j - |S|]$  is not in degree-conflict with  $R^-$ . Then  $B[j - |S| + 1, j]\langle j \rangle$  is a central-star  $Z = t_B(z)$  with  $|Z| \geq 3$  by Lemma 7 and

$$\deg_S(s) + \deg_Z(z) \geq |S|. \quad (6)$$

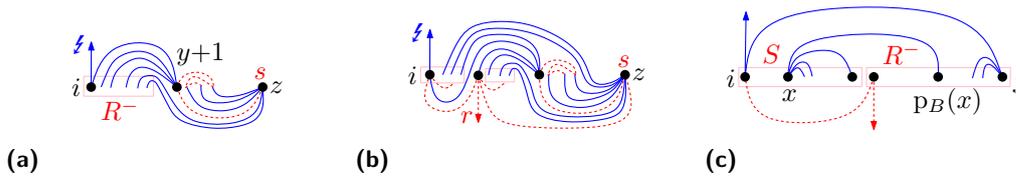
**Case 3.1**  $\{i, j\} \notin E(B)$ . Then we claim that we may assume  $z = j$  and  $Z = B\langle j \rangle$ .

Let us prove this claim. If  $z = j - |Z| + 1$ , then by 1SR it does not have any neighbor in  $B \setminus Z$ . Flipping  $Z = B\langle j \rangle$  establishes the claim. Otherwise,  $z = j$ . Suppose that  $z$  has a neighbor  $y \in B \setminus Z$ . As  $z$  is the root of  $Z = B[j - |S| + 1, j]\langle j \rangle$ , it does not have a neighbor in  $[j - |S| + 1, j - |Z|]$  and therefore  $y \in [i + 1, j - |S|]$ . By LSFR and because  $B[j - |S| + 1, j]$  is not a star,  $y = p_B(z)$ . In particular, since  $|Z| \geq 3$ , LSFR for  $y$  implies  $\{y, y + 1\} \notin E(B)$ . It follows that after flipping  $B\langle j \rangle$  the resulting subtree  $B[j - |S| + 1, j]\langle j \rangle$  is not a central-star anymore and so there is no conflict for embedding  $S$  onto  $[j, j - |S| + 1]$  anymore. Therefore we can proceed as above in Case 1 (the conflict situation for  $R^-$  did not change because  $B\langle i \rangle$  remains unchanged). Hence we may suppose that there is no such neighbor  $y$  of  $z$ , which establishes the claim.

We blue-star embed  $S$  starting from  $\sigma = j = z$  with  $\varphi = (j - |Z|, j - |Z| - 1, \dots)$ . In the terminology of the blue-star embedding we have  $B^* = B^+ = Z$ . Let us argue that the conditions for the embedding hold. 1 is trivial because no neighbor of  $s$  is embedded yet. For 2 we have to show  $|S| \leq |Z| + \deg_S(s) \leq |R| - 1$ . The first inequality holds by (6) and the second by  $|S| \leq (|R| - 1) / \deg_R(r) \leq (|R| - 1) / 2$ , which implies  $|Z| + \deg_S(s) \leq 2(|S| - 1) \leq |R| - 3$ . 3 is obvious by the choice of  $\varphi$  and given  $B^* = B^+$ , 4 is trivial. That leaves us with an interval  $[i', j']$ , where  $i' = i$ .

The plan is to recursively embed  $R^-$  onto  $[i, j']$ . This works fine, unless  $[i, j']$  and  $R^-$  are in conflict. So suppose that they are in conflict. Then there is a central-star  $Y = B[i, j']\langle i \rangle$ . Considering how  $\varphi$  consumes the vertices in  $I$  from right to left,  $Y$  appears as a part of some component of  $B$ , that is,  $Y = B[i, y]$ , for some  $y \in [i, j - |S|]$ .

We claim that  $\uparrow(Y) = i$ . To prove the claim, suppose to the contrary that  $\uparrow Y = y = p_B(i)$ . Then by 1SR  $Y$  is a component of  $B$ . Thus, a degree-conflict contradicts the assumption of Case 3 that  $[i, j - |S|]$  is not in degree-conflict with  $R^-$ , and an edge-conflict contradicts 1 for embedding  $R$  onto  $[i, j]$  together with the fact that by 1SR  $y$  is not adjacent to any vertex outside of  $Y$  in  $B$  – in particular not to  $j$ , where  $s$  was placed. This proves the claim and, furthermore, that  $p_B(i) \in [y + 1, j - |S|]$  and  $p_B(i)$  appears in  $\varphi$ .



**Figure 10** (a)–(b): Relocating some leaves of the stars  $Y = t_B(y + 1)$  and  $Z = t_B(z)$  in Case 3.1. One subtree of  $t_R(c)$  of  $R^-$  is embedded at  $i$  and the leaves to the right of  $i$ ; all other subtrees of  $R^-$  are embedded to the right of  $r$ . Both from  $y + 1$  and from  $z$  we can route as many blue edges as desired to either of these “pockets”. (c): Evading a degree-conflict for  $S$  in Case 3.2.1.

By 3 for embedding  $R$  onto  $[i, j]$  and  $\{i, j\} \notin E(B)$  we know that  $[i, j']$  and  $R^-$  are not in edge-conflict and so they are in degree-conflict. In particular,  $\deg_Y(i) + \deg_{R^-}(r) \geq |R^-|$ .

Undo the blue-star embedding. We claim  $\{i, y + 1\} \in E(B)$ . To prove the claim, suppose to the contrary that  $\{i, y + 1\} \notin E(B)$ . Then  $\{y + 1, p_B(i)\} \in E(B)$  because in  $B$  the vertex  $y + 1$  lies below the edge  $\{i, p_B(i)\}$ . By LSFR the subtree of  $p_B(i)$  rooted at  $y + 1$  is at least as large as  $Y$ . Therefore,

$$|t_B(p_B(i))| \geq 2|Y| + 1 = 2 \deg_Y(i) + 3 \geq 2(|R^-| - \deg_{R^-}(r)) \geq \frac{3}{2}|R^-|,$$

where the last inequality uses (1). This is in contradiction to  $p_B(i) \leq j - |S|$ , which implies  $|t_B(p_B(i))| \leq |R^-|$ . Therefore, the claim holds and  $\{i, y + 1\} \in E(B)$ .

Flip  $B[i, y + 1]$  and perform the blue-star embedding again. Although 1SR may be violated at  $y + 1$ , this is of no consequence for the blue-star embedding. As  $Y = B[i, j'] \setminus \{i\} = B[i, y]$ , we know that  $y + 1$  appears in  $\varphi$  and so the offending vertex is not part of  $[i, j']$  after the blue-star embedding. Furthermore, in this way we also get rid of the high-degree vertex of  $B$  that was at  $i$  initially so that the vertices in  $[i, y] \subset [i, j']$  are isolated. In particular,  $i$  is isolated in  $[i, j']$  and its only neighbor in  $B$  is at  $y + 1 \neq j$ . Therefore,  $[i, j']$  and  $R^-$  are not in conflict, unless  $y + 1 = p_B(i)$  initially and  $p_B(i)$  is in edge-conflict with  $r$ .

In other words, it remains to consider the case  $B \setminus \{i\} = B[i, y + 1] = t_B(y + 1)$  is a dangling star whose root  $i$  (at  $y + 1$  before flipping) is in edge-conflict with  $r$  (Figure 10a). Then  $[i, j']$  is an independent set in  $B$  that consists of leaves of the two stars  $Y$  and  $Z$  plus the isolated vertex at  $i$ . Yet we cannot simply embed  $R^-$  using the algorithm from Section 3 because  $i$  is and  $j'$  may be in edge-conflict with  $r$ . Given that  $|Y| \geq 3$  and  $\varphi$  gets to  $y + 1$  only, at least two leaves of  $Y$  remain in  $[i, j']$  and so, in particular,  $i + 1$  is not in conflict with  $r$ . We explicitly embed  $R^-$  as follows (Figure 10b): place  $r$  at  $i + 1$  and a child  $c$  of  $r$  in  $R^-$  at  $i$ . Then collect  $|t_R(c)|$  leaves from  $Z$  and/or  $Y$  and put them right in between  $i$  and  $i + 1$ . First – from left to right – the leaves of  $Z$  whose blue edges leave them upwards to bend down and cross the spine immediately to the right of the vertices of the red subtree rooted at  $y + 1$  (the leftmost subtree of  $S$ ) and then reach  $z$  from below. Next come the leaves of  $Y$  whose blue edges to  $y + 1$  are drawn as arcs in the upper halfplane. In order to make room for those leaves, the blue edge  $\{i, y + 1\}$  is re-routed to leave  $i$  downwards to bend up and cross the spine immediately to the left of  $i + 1$  in order to reach  $y + 1$  from above. Using the algorithm from Section 3 we can now embed  $t_R(c)$  onto these leaves and any remaining subtrees of  $r$  can be embedded explicitly on the vertices  $i + 2, \dots$  (ignoring the change of numbering caused by the just discussed repositioning of leaves).

**Case 3.2**  $\{i, j\} \in E(B)$ . Then  $z = j$  because  $j - |Z| + 1$  is enclosed by  $\{i, j\}$  and therefore cannot be the root of  $Z$ . Moreover,  $\uparrow(B) = i$  by LSFR and since  $B$  is not a star. By LSFR  $j$  does not have any child in  $B \setminus Z$  and as  $B[j - |S| + 1, j]$  is not a star,  $Z \subseteq B[j - |S| + 2, j]$ .

In particular,  $j$  is not adjacent to any vertex in  $B[i+1, j-|S|+1]$ . We provisionally place  $s$  at any vertex in  $[i+1, j-|R^-|]$ , say, at  $j-|R^-|$ . Then  $[j, j-|R^-|+1]$  is not in edge-conflict with  $R^-$ . We claim that it is not in degree-conflict, either. As  $Z$  is a star on  $|Z| \leq |S| - 1$  vertices, by Lemma 6 we have  $\deg_{R^-}(r) + \deg_Z(z) \leq \deg_{R^-}(r) + |S| - 2 \leq |R^-| - 2$  and the claim follows. We recursively embed  $R^-$  onto  $[j, j-|R^-|+1]$ , treating all local roots of  $B$  other than  $j$  as in conflict with  $r$ . It remains to recursively embed  $S$  onto  $[j-|R^-|, i]$ .

Suppose towards a contradiction that  $[j-|R^-|, i]$  is in conflict with  $S$ . Then there is a central-star  $X = B[x, j-|R^-|] = B[i, j-|R^-|]\langle j-|R^-| \rangle$ . Due to  $\{i, j\} \in E(B)$  and 1SR we have  $\uparrow(X) = x$  and  $p_B(x) > j-|R^-|$ . Together with LSFR for  $j$  it follows that  $p_B(x) \in [j-|R^-|+1, j-|Z|]$ . Due to the conflict setting for embedding  $R^-$ ,  $i$  is the only vertex in  $[j-|R^-|, i]$  that may be in edge-conflict with  $s$ . As  $X$  is a central-star and  $\uparrow(B) = i$ , we cannot have  $x = i$  because then  $B$  would be a star. It follows that  $x > i$  and so  $[j-|R^-|, i]$  is not in edge-conflict with  $S$ . Therefore  $[j-|R^-|, i]$  and  $S$  are in degree-conflict. Then  $|X| \geq 3$  by Lemma 7 and

$$\deg_S(s) + \deg_X(x) \geq |S|. \quad (7)$$

Depending on  $p_B(x)$  we consider two final subcases.

**Case 3.2.1**  $p_B(x) \in [j-|R^-|+2, j-|Z|]$  (Figure 10c). Then the edge  $\{x, p_B(x)\}$  encloses  $j-|R^-|+1$  so that, in particular,  $\{i, j-|R^-|+1\} \notin E(B)$ . We provisionally place  $s$  at  $i = \uparrow(B[i, j-|R^-|]\langle i \rangle)$  and claim that  $[j-|R^-|+1, j]$  and  $R^-$  are not in conflict.

To prove the claim, consider  $W^* := B[j-|R^-|+1, j]\langle j-|R^-|+1 \rangle$  and suppose it is a central-star. (If it is not, then we are done.) If  $\uparrow(W^*) > j-|R^-|+1$ , then by 1SR and  $\{x, p_B(x)\} \in E(B)$  we have  $p_B(\uparrow(W^*)) = x$ , in contradiction to LSFR for  $x$ . Therefore  $\uparrow(W^*) = j-|R^-|+1$ . In order for  $j-|R^-|+1$  to be the local root for  $W^*$  in the presence of  $\{x, p_B(x)\} \in E(B)$ , it follows that  $p_B(j-|R^-|+1) = x$  and so by 1SR  $|W^*| = 1$ . Therefore by Lemma 7 there is no degree-conflict between  $[j-|R^-|+1, j]$  and  $R^-$ . As  $\{x, p_B(x)\} \in E(B)$  prevents any connection in  $B$  from  $j-|R^-|+1$  to  $i$  and to vertices outside of  $[i, j]$ , there is no edge-conflict between  $[j-|R^-|+1, j]$  and  $R^-$ , either. This proves the claim. Recursively embed  $R^-$  onto  $[j-|R^-|+1, j]$ . Recall that  $\uparrow(B) = i$ . There is no conflict for embedding  $S$  onto  $[i, j-|R^-|]$  since  $\{i, r\} \notin E(B)$  and  $B[i, j-|R^-|]\langle i \rangle$  is not a central-star of size at least 2 by LSFR at  $i$ . Finish the packing by recursively embedding  $S$  onto  $[i, j-|R^-|]$ .

**Case 3.2.2**  $p_B(x) = j-|R^-|+1$ . Then by 1SR  $p_B(x)$  is the only neighbor of  $x$  outside of  $X$  in  $B$ . We provisionally place  $r$  at  $j$  and employ a blue-star embedding for  $S$ , starting from  $\sigma = x$  with  $\varphi = (i, \dots)$ , that is,  $\varphi$  takes vertices from left to right, skipping over  $[x, p_B(x)]$ . Let us argue that the conditions for the blue-star embedding hold.

In the terminology of the blue-star embedding we have  $B^* = X$  and  $B^+ = X \cup \{p_B(x)\}$ . 1 holds because  $\{x, j\} \notin E(B)$ . For the first inequality of 2 we have to show  $|S| \leq |X| + \deg_S(s)$ , which is immediate from (7). For the second inequality of 2 we have to show  $|X| + 1 + \deg_S(s) \leq |I| - 1$ . This follows from  $|X| + 1 + \deg_S(s) \leq |S| + (|S| - 1) \leq |R^-| + |S| - 1 = |I| - 1$ . Regarding 3 note that in  $\varphi$  we take the vertices of  $B \setminus B^+$  from left to right. As there are not enough vertices in  $[i, x-1]$  to embed the neighbors of  $s$  (which causes the degree-conflict),  $\varphi$  reaches beyond  $p_B(x)$  and so  $B \setminus (B^+ \cup \varphi)$  forms an interval. In particular,  $\varphi$  includes  $p_B(x) + 1$  and we may simply move  $p_B(x) + 1$  to the front of  $\varphi$ , establishing the second condition in 4. Regarding the remaining two conditions in 4 note that  $S$  is not a star by assumption and that  $p_B(x) + 1$  is not a neighbor of  $x$  in  $B$  because  $p_B(x)$  is the only neighbor of  $x$  outside of  $X$ .

Therefore, we can blue-star embed  $S$  as claimed, which leaves us with an interval  $[i', j']$ , where  $j = j'$ . As  $\{x, j\} \notin E(B)$  and  $j$  is not the local root of  $B$  ( $i$  is), there is no edge-conflict

between  $[j', i']$  and  $R^-$ . As there is no degree-conflict between  $[j, j - |R^-| + 1]$  and  $R^-$  and the number of neighbors of  $j$  in  $B[i', j']$  can only decrease compared to  $B[j - |R^-| + 1, j]$  (if they appear in  $\varphi$ ), there is no degree-conflict between  $[j', i']$  and  $R^-$ , either. Therefore, we can complete the packing by embedding  $R^-$  onto  $[j', i']$  recursively. ◀

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