On Computing the Fréchet Distance Between Surfaces

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Abstract

We describe two $(1+\varepsilon)$-approximation algorithms for computing the Fréchet distance between two homeomorphic piecewise linear surfaces $\mathcal{R}$ and $\mathcal{S}$ of genus zero and total complexity n, with Fréchet distance δ.

1. A $2^{O\left(\left(n+\frac{\mathrm{Area}(\mathcal{R})+\mathrm{Area}(\mathcal{S})}{(\varepsilon\delta)^2}\right)^2\right)}$ time algorithm if $\mathcal{R}$ and $\mathcal{S}$ are composed of fat triangles (triangles with angles larger than a constant).
2. An $O(D/(\varepsilon\delta)^2)\cdot n + 2^{O(D^4/(\varepsilon^4\delta^2))}$ time algorithm if $\mathcal{R}$ and $\mathcal{S}$ are polyhedral terrains over $[0,1]^2$ with slope at most D.

Although, the Fréchet distance between curves has been studied extensively, very little is known for surfaces. Our results are the first algorithms (both for surfaces and terrains) that are guaranteed to terminate in finite time. Our latter result, in particular, implies a linear time algorithm for terrains of constant maximum slope and constant Fréchet distance.

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1 Introduction

Measuring similarity between geometric objects is a fundamental problem that appears in several applications in computer science (see, for example, references [27, 16]). A natural way of estimating such a similarity is formalized via the notion of the Fréchet distance. The Fréchet distance takes into account the endowed topology of the objects, which makes it a more desirable choice compared to other classical measures like the Hausdorff distance for many applications.

The Fréchet distance between curves can be computed efficiently [2], and it has been successfully used in many applications [20, 25, 21, 5, 28, 6, 7]. Surprisingly though, computing the Fréchet distance becomes much harder for higher dimensional objects, even surfaces. It is NP-hard to compute the Fréchet distance between a triangle and a (self-crossing) surface [17], between two terrains [8], and between two polygons with (the same number of) holes [8]. Furthermore, to the best of our knowledge, no exact or approximation algorithm exists for computing the Fréchet distance between two surfaces, even two terrains, that is guaranteed to terminate in finite time.

In this paper, we study the problem of computing the Fréchet distance between surfaces of genus zero (punctured spheres). Our input is composed of two homeomorphic piecewise

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linear orientable surfaces, $\mathcal{R}$ and $\mathcal{S}$, each constructed from a set of Euclidean triangles by identifying pairs of equal-length edges. The input also includes immersions $\varphi_{\mathcal{R}} : \mathcal{R} \to \mathbb{R}^3$ and $\varphi_{\mathcal{S}} : \mathcal{S} \to \mathbb{R}^3$ that are isometric on each triangle of $\mathcal{R}$ and $\mathcal{S}$, respectively. The Fréchet length of a homeomorphism $\sigma : \mathcal{R} \to \mathcal{S}$, is defined to be $\delta_F(\sigma) = \max_{x \in \mathcal{R}} ||\varphi_{\mathcal{R}}(x) - \varphi_{\mathcal{S}}(\sigma(x))||_2$. The Fréchet distance between (immersed) $\mathcal{R}$ and $\mathcal{S}$ is defined to be $\delta_F(\mathcal{R}, \mathcal{S}) = \inf_\sigma \delta_F(\sigma)$, where σ ranges over all homeomorphisms between $\mathcal{R}$ and $\mathcal{S}$. Note that $\delta_F(\mathcal{R}, \mathcal{S})$ depends on the immersions $\varphi_{\mathcal{R}}$ and $\varphi_{\mathcal{S}}$, which are not explicitly mentioned here and throughout the paper to simplify the notation.

1.1 Previous work

The Fréchet distance and its variants between curves have been extensively studied [3, 13, 12, 19, 4, 11], resulting in very efficient exact or approximation algorithms. In contrast, very little is known about computing the Fréchet distance between surfaces. Alt and Buchin [1] show that the Fréchet distance between triangulated surfaces is uppersemicomputable: there is an algorithm that generates an infinite sequence of real numbers that converges to the Fréchet distance. Buchin et al. [9] describe a polynomial time exact algorithm for computing the Fréchet distance between simple polygons. Their work is generalized to exact polynomial time algorithms for folded polygons and polygons with constant numbers of holes by Cook et al. [10], and Nayyeri and Sidiropoulos [22], respectively. To our knowledge, there is no exact or approximation algorithm that is guaranteed to terminate in finite time for surfaces or even polyhedral terrains.

1.2 Our contribution

In this paper, we describe $(1+\varepsilon)$-approximation algorithms for computing the Fréchet distance between two triangulated surfaces $\mathcal{R}$ and $\mathcal{S}$ (composed of fat triangles) immersed in $\mathbb{R}^3$.

▶ **Theorem 1.** *Let $\mathcal{R} = (\mathcal{R}_V, \mathcal{R}_E, \mathcal{R}_T)$ and $\mathcal{S} = (\mathcal{S}_V, \mathcal{S}_E, \mathcal{S}_T)$ be two triangulated surfaces composed of fat triangles, let $n = |\mathcal{R}_V| + |\mathcal{S}_V|$, and let $\varepsilon > 0$. There exists a $(1+\varepsilon)$-approximation algorithm for computing the Fréchet distance between $\mathcal{R}$ and $\mathcal{S}$ with running time*

$$2^{O\left(\left(|\mathcal{R}_V|+|\mathcal{S}_V|+\frac{Area(\mathcal{R})+Area(\mathcal{S})}{(\varepsilon\delta)^2}\right)^2\right)}.$$

Note that $\frac{\mathrm{Area}(\mathcal{R})+\mathrm{Area}(\mathcal{S})}{\delta^2}$ is invariant up to scaling, therefore, scaling the surfaces does not change the running time of our algorithm. We view this result as a first step towards designing efficient approximation algorithms for computing the Fréchet distance between surfaces.

We describe a significantly more efficient algorithm if $\mathcal{R}$ and $\mathcal{S}$ are polyhedral terrains over $[0,1]^2$. This algorithm works in linear time for surfaces of constant maximum slope with constant Fréchet distance.

▶ **Theorem 2.** *Let $\mathcal{R}$ and $\mathcal{S}$ be polyhedral terrains over $[0,1]^2$ of maximum slope D, and let $n = |\mathcal{R}_V| + |\mathcal{S}_V|$. There exists a $(1+\varepsilon)$-approximation algorithm for computing the Fréchet distance between $\mathcal{R}$ and $\mathcal{S}$ with running time*

$$O(D/(\varepsilon\delta)^2) \cdot n + 2^{O(D^4/(\varepsilon^4\delta^2))}.$$

1.3 Overview

First, we consider the problem for piecewise linear triangulations $\mathcal{R} = (\mathcal{R}_V, \mathcal{R}_E, \mathcal{R}_T)$ and $\mathcal{S} = (\mathcal{S}_V, \mathcal{S}_E, \mathcal{S}_T)$, composed of triangles of diameter at most r. We design an algorithm that, for each $\delta > 0$, returns an $(\mathcal{R}, \mathcal{S})$-homeomorphism of Fréchet length $\delta + O(r)$ if $\delta \geq \delta_F(\mathcal{R}, \mathcal{S})$. If $\delta < \delta_F(\mathcal{R}, \mathcal{S})$, our algorithm either returns an $(\mathcal{R}, \mathcal{S})$-homeomorphism of Fréchet length $\delta + O(r)$, or it correctly decides that the $\delta_F(\mathcal{R}, \mathcal{S}) > \delta$. We use binary search with this algorithm to approximate $\delta_F(\mathcal{R}, \mathcal{S})$.

Consider a homeomorphism $g : \mathcal{R} \to \mathcal{S}$ of Fréchet length at most $\delta + r$. As the first step, our algorithm computes f_0, an approximation of $g|_{\mathcal{R}_V \cup g^{-1}(\mathcal{S}_V)}$ (the restriction of g into $\mathcal{R}_V \cup g^{-1}(\mathcal{S}_V)$). Precisely, for each vertex $u \in \mathcal{R}_V$ (resp. each vertex $v \in \mathcal{S}_V$), $f_0(u)$ and $g(u)$ (resp. $f_0^{-1}(v)$ and $g^{-1}(v)$) are in the same triangle of $\mathcal{S}_T$ (resp. $\mathcal{R}_T$). To compute $f_0(u)$, for each $u \in \mathcal{R}_V$ (resp. for each $v \in \mathcal{S}_V$), our algorithm enumerates over all possible triangles that can contain $g(u)$ (resp. $g^{-1}(v)$): triangles of $\mathcal{S}_T$ (resp. $\mathcal{R}_T$) whose distance from u (resp. v) is at most δ. Our algorithm refines $\mathcal{R}$ and $\mathcal{S}$ with regards to f_0. Let $\widetilde{\mathcal{R}}_V = \mathcal{R}_V \cup f_0^{-1}(\mathcal{S}_V)$, and $\widetilde{\mathcal{S}}_V = \mathcal{S}_V \cup f_0(\mathcal{R}_V)$. Then, let $\widetilde{\mathcal{R}} = (\widetilde{\mathcal{R}}_V, \widetilde{\mathcal{R}}_E, \widetilde{\mathcal{R}}_T)$ and $\widetilde{\mathcal{S}} = (\widetilde{\mathcal{S}}_V, \widetilde{\mathcal{S}}_E, \widetilde{\mathcal{S}}_T)$ be arbitrary refinements of $\mathcal{R}$ and $\mathcal{S}$, respectively.

Our algorithm seeks to extend f_0 to a homeomorphim $f : \widetilde{\mathcal{R}} \to \widetilde{\mathcal{S}}$, with Fréchet length at most $\delta + O(r)$. Let the homeomorphism $h : \widetilde{\mathcal{R}} \to \widetilde{\mathcal{S}}$ be an extension of f_0 of Fréchet length $\delta + O(r)$ (we show such a homeomorphism exists). So, $h_1 = h|_{\widetilde{\mathcal{R}}_E}$ is a one-to-one continuous map with the following properties: (1) $f_0 = h_1|_{\widetilde{\mathcal{R}}_V}$, (2) h_1 maps the boundary vertices to boundary vertices, and boundary edges to boundary edges, (3) h_1 preserves the cyclic order of edges around each vertex (the combinatorial embedding of $(\widetilde{\mathcal{R}}_V, \widetilde{\mathcal{R}}_E)$), and (4) $\delta_F(h_1) = \delta + O(r)$.

Moreover, we observe that any map $f_1 : \widetilde{\mathcal{R}}_E \to \widetilde{\mathcal{S}}$ with properties (1) to (4) can be extended to a homeomorphism between $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{S}}$ of Fréchet length $\delta + O(r)$. As a result, our problem of extending f_0 to $(\widetilde{\mathcal{R}}, \widetilde{\mathcal{S}})$-homeomorphism f reduces to finding an f_1 with properties (1) to (4), or equivalently finding the image of each edge $e \in \widetilde{\mathcal{R}}_E$ under f_1.

Let $e \in \widetilde{\mathcal{R}}_E$, and let $\gamma = h_1(e)$, so, $\delta_F(e, \gamma) \leq \delta_F(h_1) \leq \delta + O(r)$. The intersection of γ with any triangle $t \in \widetilde{\mathcal{S}}_T$ is a collection of paths $\{\gamma_1, \ldots, \gamma_k\}$. As the diameter of t is at most r, modifying γ_i's inside t can only increase the Fréchet length by $O(r)$. Since we allow an $O(r)$ additive approximation factor, we can overlook the exact positions of γ_i's in t, and focus only on their intersection pattern. When the intersection patterns in all triangles of $\widetilde{\mathcal{S}}_T$ are viewed as a whole, they specify the sequence of edges in $\widetilde{\mathcal{S}}_E$ that γ crosses, that is the homotopy class of γ in $\widetilde{\mathcal{S}} \backslash \widetilde{\mathcal{S}}_V$.

Potentially, the image of an edge $e \in \widetilde{\mathcal{R}}_E$ may belong to infinitely many homotopy classes (in $\widetilde{\mathcal{S}} \backslash \widetilde{\mathcal{S}}_V$). We bound the number of possible homotopy classes, by considering the interaction of $h_1(\widetilde{\mathcal{R}}_E)$ with $\widetilde{\mathcal{S}}_E$. We view $h_1(\widetilde{\mathcal{R}}_E)$ as an embedding of $(\widetilde{\mathcal{R}}_V, \widetilde{\mathcal{R}}_E)$ on $\widetilde{\mathcal{S}}$ that is combinatorially equivalent with its triangulated embedding on $\widetilde{\mathcal{R}}$. We show that if an edge $s \in \widetilde{\mathcal{S}}_E$ is crossed a sufficiently large number of times by $h_1(\widetilde{\mathcal{R}}_E)$, then we can shortcut the curves in $h_1(\widetilde{\mathcal{R}}_E)$ along s to obtain a different embedding of $(\widetilde{\mathcal{R}}_V, \widetilde{\mathcal{R}}_E)$ on $\widetilde{\mathcal{S}}$. Following Cook et al. [10], we observe that such shortcutting operations decrease the crossing number on s, but do not increase the Fréchet distance. We conclude the existence of an $f_1 : \widetilde{\mathcal{R}}_E \to \widetilde{\mathcal{S}}$ of Fréchet length $\delta + O(r)$ such that each edge $s \in \widetilde{\mathcal{S}}_E$ is crossed by $f_1(\widetilde{\mathcal{R}}_E)$ a bounded number of times.

Our algorithm uses normal coordinates to enumerate the set of homotopy classes for the curves of $f_1(\widetilde{\mathcal{R}}_E)$ in $\widetilde{\mathcal{S}} \backslash \widetilde{\mathcal{S}}_V$. Normal coordinates record, for each edge $s \in \widetilde{\mathcal{S}}_T$, the number of times it is crossed by $f_1(\widetilde{\mathcal{R}}_E)$. Our crossing bound implies a bound on the maximum coordinate of any set of normal coordinates that must be considered, thus, a bound on all possible normal coordinates.

Provided f_0 and a set of normal coordinates, our algorithm constructs an $f_1 : \widetilde{\mathcal{R}}_E \to \widetilde{\mathcal{S}}$ of Fréchet length $\delta + O(r)$ via constructing the images of all edges. Finally, our algorithm extends f_1 to include the interior of triangles, $\widetilde{\mathcal{R}}_T$, to obtain $f : \widetilde{\mathcal{R}} \to \widetilde{\mathcal{S}}$ of Fréchet length at most $\delta + O(r)$. The running time of our algorithm depends on the complexity of $\mathcal{R}$ and $\mathcal{S}$, as well as their vertex densities: the maximum number of their vertices in any ball of radius $\max(\delta, r)$.

To obtain our $(1+\varepsilon)$-approximation algorithm for two general triangulations $\mathcal{R}$ and $\mathcal{S}$, our algorithm refines them into triangulations composed of triangles of diameter $O(\varepsilon\delta)$. Then, the above algorithm can be applied to find a homeomorphism of Fréchet length $\delta + O(\varepsilon\delta)$. If $\mathcal{R}$ and $\mathcal{S}$ are terrains with slope at most D, our algorithms can find triangulations $\mathcal{R}'$ and $\mathcal{S}'$ within the Fréchet distance $\varepsilon\delta$ of $\mathcal{R}$ and $\mathcal{S}$, respectively. The complexity of $\mathcal{R}'$ and $\mathcal{S}'$, and their vertex densities are bounded by functions of ε, δ and D, therefore, we obtain a linear time $(1+\varepsilon)$-approximation algorithm if all these parameters are constant.

2 Preliminaries

Maps

Let $f : A \to B$ be a function, and let $U \subseteq A$. We define $f(U)$ to be $\{f(u) | u \in A\}$. The function $f|_U : U \to B$, the **restriction** of f to the subset U, is defined as for all $u \in U$, $f|_U(u) = f(u)$. In this case, we also say, that f is an **extension** of $f|_U$ to A. For topological spaces A and B, f is a **homeomorphism** if (1) it is a bijection, (2) it is continuous, and (3) its inverse is continuous.

Surfaces

A **surface** $\mathcal{Q}$ (or a 2-manifold) is a space, in which every point has a neighborhood that is homeomorphic to the plane or half-plane. An **embedding** $\Phi : \mathcal{Q} \to \mathbb{R}^3$ is a continuous one-to-one map. An **immersion** $\varphi : \mathcal{Q} \to \mathbb{R}^3$ is a continuous map, such that for any $x \in \mathcal{Q}$ there is a neighborhood N_x of x, on which f is an embedding.

A **piecewise linear surface** is a surface $\mathcal{Q}$ that is constructed from a set of Euclidean polygons by identifying pairs of equal-length edges. In this paper, we assume that all the constituent polygons are triangles. We denote the constituent vertices, edges, and triangles of $\mathcal{Q}$ by $\mathcal{Q}_V$, $\mathcal{Q}_E$, and $\mathcal{Q}_T$, in order, thus, we write $\mathcal{Q} = (\mathcal{Q}_V, \mathcal{Q}_E, \mathcal{Q}_T)$. A **locally isometric immersion** is an immersion $\varphi : \mathcal{Q} \to \mathbb{R}^3$ such that for each $t \in \mathcal{Q}_T$, $\varphi|_t$ is an isometric map. In particular, t and $\varphi(t)$ are congruent triangles.

Embedded curves and graphs

Let $\mathcal{Q}$ be a surface and let $\alpha, \beta : [0,1] \to \mathcal{Q}$ be curves embedded on $\mathcal{Q}$ with the same endpoints, $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$. A **homotopy** $h : [0,1] \times [0,1] \to \mathcal{Q}$ is a continuous map such that $h[x,0] = \alpha$, $h[x,1] = \beta$, $h[0,t] = \alpha(0) = \beta(0)$, and $h[1,t] = \alpha(1) = \beta(1)$.

An **embedding** of a graph $G = (V, E)$ into a surface $\mathcal{Q}$ is a continuous function that maps vertices in V into distinct points in $\mathcal{Q}$ and edges in E into disjoint paths except for their endpoints. The **faces** of the embedding are maximal subsets of $\mathcal{Q}$ that are disjoint from the image of the graph. An embedding is **cellular** if all its faces are topological disks. In particular, each boundary component in a cellular embedding is covered by the image of the graph. A cellular embedding on an orientable surface can be described by a **rotation system**. A rotation system is composed of a cyclic (clockwise) order of edges around vertices. A rotation system is a combinatorial description of the embedding of a graph. Equivalent rotation systems of G on two different surfaces $\mathcal{Q}$ and $\mathcal{Q}'$ induce a one-to-one correspondence between the vertices, edges, and faces of the different embeddings. Therefore, they can be extended to a homeomorphism between $\mathcal{Q}$ and $\mathcal{Q}'$. Note that the homotopy class of an edge might be completely different in two combinatorially equivalent embeddings. For example, one can apply Dehn Twists on a cycle that avoids vertices of the embedding to change the homotopy class of edges without affecting the rotation system.

Fréchet distance

Let $\mathcal{R}$ and $\mathcal{S}$ be homeomorphic piecewise linear triangulations, and let $\varphi_{\mathcal{R}} : \mathcal{R} \to \mathbb{R}^3$ and $\varphi_{\mathcal{S}} : \mathcal{S} \to \mathbb{R}^3$ be locally isometric immersions. The Fréchet length of a homeomorphism $\sigma : \mathcal{R} \to \mathcal{S}$, is defined to be $\delta_F(\sigma) = \max_{x\in\mathcal{R}} ||\varphi_{\mathcal{R}}(x) - \varphi_{\mathcal{S}}(\sigma(x))||_2$. The Fréchet distance between (immersed) $\mathcal{R}$ and $\mathcal{S}$ is defined to be $\delta_F(\mathcal{R}, \mathcal{S}) = \inf_\sigma \delta_F(\sigma)$, where σ ranges over all homeomorphisms between $\mathcal{R}$ and $\mathcal{S}$. Note that $\delta_F(\mathcal{R}, \mathcal{S})$ depends on the immersions $\varphi_{\mathcal{R}}$ and $\varphi_{\mathcal{S}}$, which are not explicitly mentioned here and throughout the paper to simplify the notation. Also, note that, a homeomorphism that realizes $\delta_F(\mathcal{R}, \mathcal{S})$ does not necessarily exist as the Fréchet distance is defined as an infimum.

Neighborhoods

Given $U \subset \mathbb{R}^3$, and $r \in \mathbb{R}_{\geq 0}$, we define the set $B_r(U)$ to be composed of all the points in $\mathbb{R}^3$ that are not farther than r from U. In particular, for a $p \in \mathbb{R}^3$, $B_r(p)$ is a closed ball of radius r, and for a curve $\gamma \subseteq \mathbb{R}^3$, $B_r(\gamma)$ is the union of balls of radius r centered on all points of γ. Let $\mathcal{Q} = (\mathcal{Q}_V, \mathcal{Q}_E, \mathcal{Q}_T)$ be a triangulated surface. We use $U \cap \mathcal{Q}_V$ to denote the set of vertices inside U, $U \cap \mathcal{Q}_E$ to denote the set of edges of $\mathcal{Q}_E$ that intersect U, and $U \cap \mathcal{Q}_T$ to denote the set of triangles of $\mathcal{Q}_T$ that intersect U.

3 Fréchet distance between fine triangulaions

In this section, let $\mathcal{R}$ and $\mathcal{S}$ be two triangulated surfaces (of genus zero) composed of triangles with diameter at most r. Also, let $\varphi_{\mathcal{R}} : \mathcal{R} \to \mathbb{R}^3$ and $\varphi_{\mathcal{S}} : \mathcal{S} \to \mathbb{R}^3$ be immersions. We consider the Fréchet length and Fréchet distance with respect to $\varphi_{\mathcal{R}}$ and $\varphi_{\mathcal{S}}$, but to simplify the exposition we do not explicitly mention them when it is clear from the context. We describe an algorithm that, given $\delta > 0$, returns an $(\mathcal{R}, \mathcal{S})$-homeomorphism of Fréchet length $\delta + O(r)$ if $\delta \geq \delta_F(\mathcal{R}, \mathcal{S})$. If $\delta < \delta_F(\mathcal{R}, \mathcal{S})$, our algorithm either returns a homeomorphism of Fréchet length $\delta + O(r)$ or it correctly decides that $\delta < \delta_F(\mathcal{R}, \mathcal{S})$.

3.1 Mapping vertices

A bijection $f_0 : \widetilde{\mathcal{R}}_V \to \widetilde{\mathcal{S}}_V$ is an **approximate vertex map** if and only if it has the following properties.

1. $\widetilde{\mathcal{R}}_V = \mathcal{R}_V \cup \mathcal{R}'_V$, $\widetilde{\mathcal{S}}_V = \mathcal{S}_V \cup \mathcal{S}'_V$, $f_0(\mathcal{R}_V) = \mathcal{S}'_V$, and $f_0(\mathcal{R}'_V) = \mathcal{S}_V$.
2. f_0 maps boundary vertices to boundary vertices, and it preserves the cyclic order of boundary vertices on each boundary component.
3. There exists an $f : \mathcal{R} \to \mathcal{S}$ of Fréchet length at most $\delta + r$ such that
   (a) for each $u \in \mathcal{R}_V$, $f(u)$ and $f_0(u)$ are in the same triangle of $\mathcal{S}_T$, and
   (b) for each $v \in \mathcal{S}_V$, $f^{-1}(v)$ and $f_0^{-1}(v)$ are in the same triangle of $\mathcal{R}_T$.

   We say that f_0 **agrees** with f.

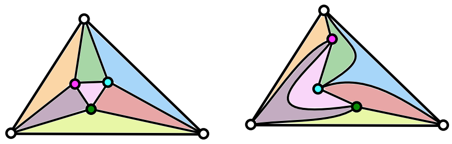


Figure 1 H and H'; corresponding faces and vertices have the same colors.

We show that an approximate vertex map can be extended to an $(\mathcal{R}, \mathcal{S})$-homeomorphism with Fréchet length $\delta + O(r)$. To that end, we find the following lemma helpful.

▶ **Lemma 3.** *Let t be a triangle with diameter r, and let $P, P' \subseteq int(t)$ be finite point sets with the same cardinality. Also, let $g : P \to P'$ be a bijection. There exists a homeomorphism $h : t \to t$ such that*

1. *$h|_{\partial(t)}$ is the identity map.*
2. *$h|_P = g$.*
3. *$\delta_F(h) \leq r$.*

Proof. Let (x, y, z) be vertices of t. Let $g' : \{x, y, z\} \cup P \to \{x, y, z\} \cup P'$ be a bijection that is the identity map for $\{x, y, z\}$ and g for P. Let H be a triangulation (a plane graph) with vertex set $\{x, y, z\} \cup P$. Let H' be a graph with vertex set $V' = \{x, y, z\} \cup P'$, where $v, v' \in V'$ are adjacent if any only if there corresponding vertices via g' are adjacent in H. The isomorphism between H and H' naturally gives rise to a combinatorial embedding of H' that is equivalent to the embedding of H. The isomorphism between H and H' and their equivalent embedding provides bijections between vertex sets, edge sets, and face sets of H and H'. Let h be any homeomorphism that respects these bijections and that is identity on the boundary. By the construction, h has properties (1) and (2). Additionally, $\delta_F(h) \leq r$, as h maps points within t that has diameter r. ◀

▶ **Lemma 4.** *Let $\mathcal{R} = (\mathcal{R}_V, \mathcal{R}_E, \mathcal{R}_T)$ and $\mathcal{S} = (\mathcal{S}_V, \mathcal{S}_E, \mathcal{S}_T)$ be triangulated surfaces composed of triangles of diameter at most r. Any approximate vertex map $f_0 : \mathcal{R} \to \mathcal{S}$ can be extended to a $(\mathcal{R}, \mathcal{S})$-homeomorphism of Fréchet length $\delta + 3r$.*

Proof. Let $g : \mathcal{R} \to \mathcal{S}$ be a homeomorphism of Fréchet length $\delta + r$ that agrees with f_0. We construct homeomorphisms $g' : \mathcal{R} \to \mathcal{R}$ and $g'' : \mathcal{S} \to \mathcal{S}$, each of Fréchet length at most r, such that $g'' \circ g \circ g'$ is an extension of f_0.

We construct g' in three steps as follows. (1) For each $v \in \mathcal{R}_V$, we define $g'(v) = v$. (2) For each $e \in \mathcal{R}_E$, if e is not a boundary edge we define $g'|_e$ to be the identity map, otherwise $g'|_e$ is any homeomorphism, in which for any $p \in f_0^{-1}(\mathcal{S}_V) \cap e$ we have $g'|_e(p) = g^{-1}(f_0(p))$. Such a homeomorphism exists because both f_0 and g preserve the order of boundary edges along boundary cycles. (3) For each $t \in \mathcal{R}_T$, we define $g'|_t$ to be the extension of $g'|_{\partial(t)}$ to t such that for any $p \in f_0^{-1}(\mathcal{S}_V) \cap t$ we have $g'|_t(p) = g^{-1}(f_0(p))$. Lemma 3 implies that such an extension exists.

The construction of g'' is very similar. (1) For each $v \in \mathcal{S}_V$, we define $g''(v) = v$. (2) For each $e \in \mathcal{S}_E$, if e is not a boundary edge we define $g''|_e$ to be the identity map, otherwise $g''|_e$ is any homeomorphism, in which for any $p \in f_0(\mathcal{R}_V) \cap e$ we have $g''|_e(p) = g(f_0^{-1}(p))$. (3) For each $t \in \mathcal{S}_T$, we define $g''|_t$ to be the extension of $g''|_{\partial(t)}$ to t such that for any $p \in f_0(\mathcal{R}_V) \cap t$ we have $g''|_t(p) = g(f_0^{-1}(p))$. Lemma 3 implies that such an extension exists. ◀

Our algorithm computes a set of candidates for f_0 via guessing the triangles that contain the images and preimages of $\mathcal{R}_V$ and $\mathcal{S}_V$, respectively, under a homeomorphism of Fréchet length at most $\delta + r$.

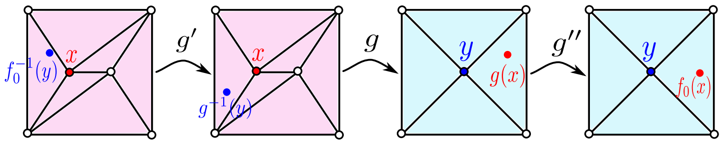


Figure 2 Proof of Lemma 4; $f_0 = g'' \circ g \circ g'$ on vertices $x \in \mathcal{R}_V$ (red) and $y \in \mathcal{S}_V$ (blue).

▶ **Lemma 5.** *Let $\mathcal{R}$ and $\mathcal{S}$ be triangulated surfaces composed of triangles of diameter at most r. There exists a set F_0 of size*

$$\prod_{u \in \mathcal{R}_V} |B_{\delta+r}(u) \cap \mathcal{S}_T| \cdot \prod_{v \in \mathcal{S}_V} |B_{\delta+r}(v) \cap \mathcal{R}_T|$$

that contains an approximate vertex map. ($B_{\delta+r}(u) \cap \mathcal{S}_T$ denotes the set of triangles of $\mathcal{S}_T$ that intersect $B_{\delta+r}(u)$, and $B_{\delta+r}(v) \cap \mathcal{R}_T$ denotes the set of triangles of $\mathcal{R}_T$ that intersect $B_{\delta+r}(v)$.)

Proof. Let f_0 be an approximate vertex map, and let f be the homeomorphism of Fréchet length $\delta + r$ that agrees with f_0. For each $u \in \mathcal{R}_V$, we need to guess the triangle $t \in \mathcal{S}_T$ that contains $f_0(u)$ or equivalently $f(u)$. Since $||f(u) - u|| \leq \delta + r$, t should intersect $B_{\delta+r}(u)$, thus, there are $|B_{\delta+r}(u) \cap \mathcal{S}_T|$ choices for the triangle that contains $f_0(u)$. Similarly, for each $v \in \mathcal{S}_V$, there are $|B_{\delta+r}(v) \cap \mathcal{R}_T|$ choices for the triangle that contains $f_0^{-1}(v)$. ◀

3.2 Mapping edges

Let $f_0 : \widetilde{\mathcal{R}}_V \to \widetilde{\mathcal{S}}_V$ be an approximate vertex map. Let $\widetilde{\mathcal{R}} = (\widetilde{\mathcal{R}}_V, \widetilde{\mathcal{R}}_E, \widetilde{\mathcal{R}}_T)$ and $\widetilde{\mathcal{S}} = (\widetilde{\mathcal{S}}_V, \widetilde{\mathcal{S}}_E, \widetilde{\mathcal{S}}_T)$ be refinements of $\mathcal{R}$ and $\mathcal{S}$, respectively.

A **scaffold map** is a continuous one-to-one map $f_1 : \widetilde{\mathcal{R}}_E \to \widetilde{\mathcal{S}}$ with the following properties.

1. $f_1(\widetilde{\mathcal{R}}_V) = \widetilde{\mathcal{S}}_V$.
2. f_1 is a cellular embedding of $(\widetilde{\mathcal{R}}_V, \widetilde{\mathcal{R}}_E)$ on $\widetilde{\mathcal{S}}$.
3. f_1 maps boundary edges to boundary edges (so, it preserves the cyclic order of boundary edges around boundary components).
4. f_1 preserves the cyclic order of edges around each vertex: for any $u \in \widetilde{\mathcal{R}}_V$ with neighbors $\{w_1, \ldots, w_k\}$, the cyclic order of the edges $\{(u, w_1), (u, w_2), \ldots, (u, w_k)\}$ around u is identical to the cyclic order of curves $\{f_1(u, w_1), \ldots, f_1(u, w_k)\}$ around $f_1(u)$.
5. For each $e \in \widetilde{\mathcal{R}}_E$ and each $t \in \widetilde{\mathcal{S}}_T$, $f_1(e) \cap t$ is a collection of straight line segments that intersect $\partial(t)$ at their endpoints.

3.2.1 Sufficiency of scaffold maps

We show that a scaffold map $f_1 : \widetilde{\mathcal{R}}_E \to \widetilde{\mathcal{S}}$ can be extended to a $(\widetilde{\mathcal{R}}, \widetilde{\mathcal{S}})$-homeomorphism of Fréchet length arbitrarily close to $\delta_F(f_1)$. As Properties (2) to (4) of a scaffold map f_1 imply, $f_1(\widetilde{\mathcal{R}}_E)$ is an embedding of $(\widetilde{\mathcal{R}}_V, \widetilde{\mathcal{R}}_E)$ on the surface $\widetilde{\mathcal{S}}$ that is combinatorially equivalent to the embedding of $(\widetilde{\mathcal{R}}_V, \widetilde{\mathcal{R}}_E)$ on $\widetilde{\mathcal{R}}$. In particular, f_1 gives a one-to-one correspondence between the triangles in $\widetilde{\mathcal{R}}_T$ and the faces of $f_1(\widetilde{\mathcal{R}}_E)$. Property (5) of a scaffold map implies that each triangle corresponds to a **folded polygon**: a piecewise linear triangulated (sub-)surface (of $\widetilde{\mathcal{S}}$) whose interior is disjoint from $\widetilde{\mathcal{S}}_V$.

To extend f_1 to a homeomorphism, for each triangle $t \in \widetilde{\mathcal{R}}_T$ and its corresponding folded polygon p_t with boundary $f_1(\partial(t))$, we extend $f_1|_{\partial(t)}$ to a (t, p_t)-homeomorphism. The proof of Lemma 1 of Cook et al. [10] actually proves the following stronger statement, which facilitates the extension of $f_1|_{\partial(t)}$.

▶ **Lemma 6** (Cook et al. [10]). *Let t be a triangle, p be a folded polygon with n triangles, and $g : \partial(t) \to \partial(p)$ a homeomorphism. For any $\epsilon > 0$, the map g can be extended to a homeomorphism, $h : t \to p$, for which $\delta_F(h) \leq \delta_F(g) + \epsilon$, in polynomial time in n.*

The one-to-one correspondence of triangles to folded polygons and Lemma 6 imply that a scaffold map can be extended to an $(\widetilde{\mathcal{R}}, \widetilde{\mathcal{S}})$-homeomorphism of arbitrary close Fréchet length.

▶ **Lemma 7.** *Let $f_1 : \widetilde{\mathcal{R}}_E \to \widetilde{\mathcal{S}}$ be a scaffold map that maps each triangle to a folded polygon composed of at most m triangles. For any $\epsilon > 0$, the map f_1 can be extended to a homeomorphism $f : \widetilde{\mathcal{R}} \to \widetilde{\mathcal{S}}$, for which $\delta_F(f) \leq \delta_F(f_1) + \epsilon$, in $O(m|\widetilde{\mathcal{R}}_V + \widetilde{\mathcal{S}}_V|)$ time.*

In light of Lemma 7 we focus on computing a scaffold map with Fréchet length $\delta + O(r)$.

3.2.2 Crossing bounds

We show that any approximate vertex map can be extended to a scaffold map of Fréchet length $\delta + O(r)$, whose image intersects each edge $s \in \widetilde{\mathcal{S}}$ a bounded number of times. (Note that, a priori, these intersection numbers can be arbitrarily large, giving rise to infinitely many scaffold maps.) To this end, we consider a homeomorphism $h : \widetilde{\mathcal{R}} \to \widetilde{\mathcal{S}}$ of Fréchet length $\delta + O(r)$ that is an extension of an approximate vertex map, which exists by Lemma 4. We modify $h_1 = h|_{\widetilde{\mathcal{R}}_E}$ via a sequence of shortcutting operations (defined below) to obtain a scaffold map of Fréchet length $\delta + O(r)$, whose image intersects each edge of $\widetilde{\mathcal{S}}$ a bounded number of times.

Let $\alpha : [0, 1] \to \mathbb{R}^d$ be an immersed curve, let $0 \leq t_1 < t_2 \leq 1$, and let $\ell : [t_1, t_2] \to \mathbb{R}^d$ be a line segment with endpoints $\alpha[t_1]$ and $\alpha[t_2]$. Finally, let $\alpha' : [0, 1] \to \mathbb{R}^d$ be $\alpha[0, t_1) \cup \ell[t_1, t_2] \cup \alpha(t_2, 1]$, that is α' coincides with α in $[0, t_1) \cup (t_2, 1]$, and coincides with the line segment ℓ on $[t_1, t_2]$. We say that α' is obtained from α via a **shortcutting operation**. The following lemma is key in our arguments.

▶ **Lemma 8** (Lemma 3 of Buchin et al. [9]). *Let $\alpha : [0, 1] \to \mathbb{R}^d$ and $\alpha' : [0, 1] \to \mathbb{R}^d$ be two curves, and let s be a line segment. If α' is obtained from α via a sequence of shortcutting operations then $\delta_F(\alpha', s) \leq \delta_F(\alpha, s)$.*

The following lemma follows from Lemma 4.1 and Lemma 4.2 of Erickson and Nayyeri [14]. Also, see Schaefer and Štefankovič [24] for similar shortcutting arguments.

▶ **Lemma 9** (Erickson and Nayyeri [14]). *Let $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ be a set of non-crossing curves on a triangulated surface (of genus zero) $\mathcal{Q} = (\mathcal{Q}_V, \mathcal{Q}_E, \mathcal{Q}_T)$. There exists a set of non-crossing curves $\Gamma' = \{\gamma'_1, \gamma'_2, \ldots, \gamma'_k\}$ with the following properties.*

1. *For each i, γ'_i is obtained from γ_i via a sequence of shortcutting operations along the edges in $\mathcal{Q}_E$.*
2. *For each $\gamma' \in \Gamma'$ and $t \in \mathcal{Q}_T$, each connected component of $\gamma' \cap t$ is*
   - **(a)** *a path with endpoints on different sides of t, or*
   - **(b)** *a path with one point being a vertex of t and the other on its opposite side, or*
   - **(c)** *a side of t, in this case γ' coincide with the side of t.*
3. *For each $e \in \mathcal{Q}_E$, if e is crossed by m different curves of Γ' then it is crossed at most 2^m times.*

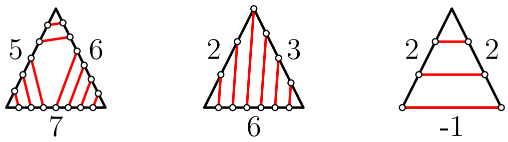


Figure 3 Examples for normal coordinates in triangles.

We remark that if Γ is composed of edges of a cellularly embedded graph on $\mathcal{Q}$, Γ' is the set of edges of a combinatorially equivalent cellular embedding on $\mathcal{Q}$. This is because the shortcutting operation does not change the boundary edges or the cyclic order of edges around vertices. Now, we are ready to show the existence of our desired scaffold map.

▶ **Lemma 10.** *Any approximate vertex map $f_0 : \widetilde{\mathcal{R}}_V \to \widetilde{\mathcal{S}}_V$ can be extended to a scaffold map $f_1 : \widetilde{\mathcal{R}}_E \to \widetilde{\mathcal{S}}$ of Fréchet length $\delta + 3r$ such that any edge $s \in \widetilde{\mathcal{S}}_E$ is crossed by $f_1(\widetilde{\mathcal{R}}_E)$ at most $2^{|B_{\delta+3r}(s)\cap\widetilde{\mathcal{R}}_E|}$ times. ($B_{\delta+3r}(s) \cap \widetilde{\mathcal{R}}_E$ denotes the subset of edges of $\widetilde{\mathcal{R}}_E$ that intersect $B_{\delta+3r}(s)$.)*

Proof. By Lemma 4, f_0 can be extended to a homeomorphism h of Fréchet length at most $\delta+3r$. Let $\Gamma = \{\gamma_e = h(e)|e \in \widetilde{\mathcal{R}}_E\}$, and note that $\delta_F(e, \gamma_e) \leq \delta+3r$. Because $\delta_F(h) \leq \delta+3r$, for any $s \in \widetilde{\mathcal{S}}_E$, if γ_e intersects s then e must intersect $B_{\delta+3r}(s)$. Thus, s is crossed by at most $|B_{\delta+3r}(s) \cap \widetilde{\mathcal{R}}_E|$ different γ_e's. Now, let $\Lambda = \{\lambda_e|e \in \widetilde{\mathcal{R}}_E\}$ be the set of paths obtained from Γ via Lemma 9 that has properties (1), (2), and (3). In particular, any segment $s \in \widetilde{\mathcal{S}}$ is crossed at most $2^{|B_{\delta+3r}(s)\cap\widetilde{\mathcal{R}}_E|}$ times.

We further modify Λ to obtain a set of piecewise linear paths $\Omega = \{\omega_e|e \in \widetilde{\mathcal{R}}_E\}$. For each $\lambda_e \in \Lambda$ and each $t \in \widetilde{\mathcal{S}}_T$, let $\{\lambda_e^1, \ldots, \lambda_e^k\}$ be the connected components of $\lambda_e \cap t$. For each $1 \leq i \leq k$, we define ω_e^i to be the straight line segment between the endpoints of λ_e^i. Property (2) of Lemma 9 implies that each ω_e^i is (a) a line segment between different sides of t, (b) a line segment between a vertex of t and its opposite side, or (c) a side of t (in this case $\omega_e = \omega_e^i$). Since each ω_e is obtained from λ_e via a sequence of shortcutting operations, which itself is obtained from γ_e via a sequence of shortcutting operations, we have $\delta_F(e, \omega_e) \leq \delta_F(e, \lambda_e) \leq \delta_F(e, \gamma_e) \leq \delta + 3r$ (Lemma 8). ◀

3.2.3 Enumeration via normal coordinates

Let $g_1 : \widetilde{\mathcal{R}}_E \to \widetilde{\mathcal{S}}$ be a scaffold map, and let $t \in \widetilde{\mathcal{S}}_T$. The intersection of $g_1(\widetilde{\mathcal{R}}_E)$ with t is a collection of **elementary segments**: straight line segments with endpoints on $\partial(t)$. The intersection pattern of $g_1(\widetilde{\mathcal{R}}_E) \cap t$ can be presented (up to continuous deformation) with three numbers, one per edge. For each edge $s \in \mathcal{S}_E$ we define its **(extended) normal coordinate**, denoted by $N(e)$, as follows: (1) $N(e) = -1$ if $e \in g_1(\widetilde{\mathcal{R}}_E)$, and (2) $N(e)$ is the number of elementary segments intersecting the interior of e, otherwise. See Figure 3 for examples of extended normal coordinates in triangles. (Our (extended) normal coordinates are straight forward extensions of normal coordinates defined for normal curves in surfaces, or normal surfaces in 3-manifolds. See references [23, 26, 15] for a detailed exposition of normal curves, the two dimensional variant of standard normal surfaces introduced by Haken [18].)

The set of normal coordinates of $g_1(\widetilde{\mathcal{R}}_E)$ is a vector of $|\widetilde{\mathcal{S}}_E|$ numbers, one per edge in $\widetilde{\mathcal{S}}_E$. Provided the normal coordinates, there is a unique way of locating the elementary segments inside each $t \in \widetilde{\mathcal{S}}_T$ (up to a continuous deformation) so that they do not cross. Hence, the normal coordinates specify, for every $e \in \widetilde{\mathcal{R}}_E$, the sequence of edges that $g_1(e)$ crosses (its homotopy class in $\widetilde{\mathcal{S}} \backslash \widetilde{\mathcal{S}}_E$). The following corollary is implied by Lemma 10.

▶ **Corollary 11.** *There is a collection $\mathcal{N}$ of sets of normal coordinates of size at most*

$$\prod_{s \in \widetilde{\mathcal{S}}_E} 2^{|B_{\delta+3r}(s) \cap \widetilde{\mathcal{R}}_E|},$$

such that for any approximate vertex map $f_0 : \widetilde{\mathcal{R}}_V \to \widetilde{\mathcal{S}}_V$, $\mathcal{N}$ contains the set of normal coordinates of a scaffold map $f_1 : \widetilde{\mathcal{R}}_E \to \widetilde{\mathcal{S}}$ with the following properties:

1. *f_1 is an extension of f_0.*
2. *$\delta_F(f_1) \leq \delta + 3r$.*

Next, we show that provided the normal coordinates of a scaffold map g_1, we can build another scaffold map f_1 with the same normal coordinates within the Fréchet distance $O(r)$ of g_1. In fact, we show the following stronger lemma.

▶ **Lemma 12.** *Let $g_1 : \widetilde{\mathcal{R}}_E \to \widetilde{\mathcal{S}}$ be a scaffold map of Fréchet length δ', and let $e \in \widetilde{\mathcal{R}}_E$. Let $T \subseteq \widetilde{\mathcal{S}}_T$ be the set of all triangles that intersect $g_1(e)$. For any point $x \in e$ and any point $y \in t \in T$, we have $||x - y|| \leq \delta' + 2r$.*

Proof. Let $z \in g(e) \cap t$, and let $x' = g^{-1}(z)$. Because x and x' are both on e, we have $||x - x'|| \leq r$. Since y and z are in the same triangle t, we have $||z - y|| \leq r$. Therefore,

$$||x - y|| \leq ||x - x'|| + ||x' - z|| + ||z - y|| \leq r + \delta' + r \leq \delta' + 2r\,. \qquad ◀$$

The following corollary is immediate observing that the normal coordinates uniquely specify the sequence of triangles each curves cross.

▶ **Corollary 13.** *Let $g_1 : \widetilde{\mathcal{R}}_E \to \widetilde{\mathcal{S}}$ and $f_1 : \widetilde{\mathcal{R}}_E \to \widetilde{\mathcal{S}}$ be two scaffold maps with identical sets of normal coordinates. For each $e \in \widetilde{\mathcal{R}}$, we have $\delta_F(e, f_1(e)) \leq \delta_F(e, g_1(e)) + 2r$.*

▶ **Lemma 14.** *Let N be the set of normal coordinates of a scaffold map $g_1 : \widetilde{\mathcal{R}}_E \to \widetilde{\mathcal{S}}$. Provided N, there is an algorithm to compute a scaffold map $f_1 : \widetilde{\mathcal{R}}_E \to \widetilde{\mathcal{S}}$ such that $\delta_F(f_1) \leq \delta_F(g_1) + 2r$.*

Proof. By Corollary 13, it suffices to find any scaffold map with normal coordinates N. For each $s \in \widetilde{\mathcal{S}}_E$, we arbitrarily select $N(s)$ points on s. N uniquely determines, for each $t \in \widetilde{\mathcal{S}}_T$, which points should be connected with elementary segments. For each edge $e \in \widetilde{\mathcal{R}}_E$, we obtain a path γ_e composed of elementary segments. Lemma 12 implies that any homeomorphism between e and γ_e has Fréchet length at most $\delta_F(g_1) + 2r$. Our f_1 will be the union of all such homeomorphisms, one per each $e \in \widetilde{\mathcal{R}}_E$. ◀

3.3 Summing up

Now, we are ready to prove the main lemmas of this section. The following lemma describes an algorithm that for a sufficiently large δ returns a homeomorphism of Fréchet length close to δ, and for a sufficiently small δ it decides that no homeomorphism with Fréchet length δ exists.

▶ **Lemma 15.** *Let $\mathcal{R}$ and $\mathcal{S}$ be two triangulated surfaces composed of triangles of diameter at most r, and let $\delta > 0$. Let ρ be the maximum number of (immersed) vertices of $\mathcal{R}_V \cup \mathcal{S}_V$ in any ball of radius $\max(\delta, r)$. There exists a $2^{O(\rho \cdot (|\mathcal{R}_V| + |\mathcal{S}_V|))}$ time algorithm with the following properties.*

1. *If $\delta \geq \delta_F(\mathcal{R}, \mathcal{S})$, it computes a homeomorphism, $f : \mathcal{R} \to \mathcal{S}$, of Fréchet length at most $\delta + 5r$,*

2. *If $\delta < \delta_F(\mathcal{R}, \mathcal{S}) - 5r$, it decides that $\delta < \delta_F(\mathcal{R}, \mathcal{S})$.*
3. *If $\delta_F(\mathcal{R}, \mathcal{S}) - 5r \leq \delta < \delta_F(\mathcal{R}, \mathcal{S})$, either it computes a homeomorphism, $f : \mathcal{R} \to \mathcal{S}$, of Fréchet length at most $\delta + 5r$, or it decides that $\delta \leq \delta_F(\mathcal{R}, \mathcal{S})$.*

Proof. First, consider the case $\delta \geq \delta_F(\mathcal{R}, \mathcal{S})$. By Lemma 5, there is a set of size $O(\rho^{|\mathcal{R}_V|+|\mathcal{S}_V|})$ that contains an approximate vertex map. Let $f_0 : \widetilde{\mathcal{R}} \to \widetilde{\mathcal{S}}$ be an approximate vertex map. Let $\widetilde{\rho}$ be the maximum number of vertices of $\widetilde{\mathcal{R}}_V \cup \widetilde{\mathcal{S}}_V$ in any ball of radius $\max(\delta, r)$. Let B be any ball of radius $\max(\delta, r)$, and let $\widetilde{B}$ be the concentric ball of radius $\max(\delta, r) + (\delta + r)$. For any vertex $u \in \mathcal{R}_V \cap B$, we have $f_0(u) \in \widetilde{B}$, and for any vertex $v \in \mathcal{S}_V \cap B$, we have $f_0^{-1}(v) \in \widetilde{B}$. Since, $\widetilde{B}$ can be covered by a constant number of balls of radii $\max(\delta, r)$, it contains $O(\rho)$ vertices of $\mathcal{R}_V \cup \mathcal{S}_V$. That is, B contains $O(\rho)$ vertices of $\widetilde{\mathcal{R}}_V \cup \widetilde{\mathcal{S}}_V$. We conclude that $\widetilde{\rho} = O(\rho)$.

Let $e \in \mathcal{R}_E$, and consider $B_{\delta+3r}(e) \cap \widetilde{\mathcal{S}}_E$, the set of edges in $\widetilde{\mathcal{S}}_E$ that intersect $B_{\delta+3r}(e)$. Any such an edge must have both of its endpoints in $B_{\delta+4r}(e)$, for the length of each edge is at most r. Now, consider the graph H induced by the vertices in $B_{\delta+4r}(e)$, which includes all edges in $B_{\delta+3r}(e) \cap \widetilde{\mathcal{S}}_E$. H is a collection of planar graphs, thus, its number of vertices and edges are within a constant factor. That is, $|B_{\delta+3r}(e) \cap \widetilde{\mathcal{S}}_E| = O(|B_{\delta+4r}(e) \cap \widetilde{\mathcal{S}}_V|)$. On the other hand, $B_{\delta+4r}(e)$ can be covered by a constant number of balls of radius $\max(\delta, r)$, therefore, $|B_{\delta+4r}(e) \cap \widetilde{\mathcal{S}}_V| = O(\widetilde{\rho}) = O(\rho)$. Overall, $|B_{\delta+3r}(e) \cap \widetilde{\mathcal{S}}_E| = O(\rho)$.

Therefore, Corollary 11 implies that a collection $\mathcal{N}$ of normal coordinate sets exists such that (1) $|\mathcal{N}| = 2^{O(\rho \cdot |\widetilde{\mathcal{S}}_E|)} = 2^{O(\rho \cdot (|\mathcal{R}_V|+|\mathcal{S}_V|))}$, and (2) $\mathcal{N}$ includes the normal coordinates of a scaffold map of Fréchet length $\delta + 3r$ that is an extension of f_0. Provided these normal coordinates, the algorithm of Lemma 14 computes a scaffold map f_1 of Fréchet length $\delta + 5r$. The total running time is dominated by the size of $\mathcal{N}$.

For any value of δ, our algorithm either computes a correct homeomorphism via computing a scaffold map or it fails to compute such a homeomorphism. In the former case, our algorithm returns the homeomorphism only if its Fréchet length is at most $\delta + 5r$. Therefore, if $\delta < \delta_F(\mathcal{R}, \mathcal{S}) - 5r$ our algorithms recognizes that $\delta < \delta_F(\mathcal{R}, \mathcal{S})$. For intermediate δ values, our algorithm either returns a homeomorphism of Fréchet length $\delta + 5r$, or it recognizes that a homeomorphism of Fréchet length δ does not exist. ◀

Next, we use binary search with Lemma 15 to estimate the value of the Fréchet distance.

▶ **Lemma 16.** *Let $\mathcal{R}$ and $\mathcal{S}$ be two triangulated surfaces composed of triangles of diameter at most r. Let ρ be the maximum number of vertices of $\mathcal{R}_V \cup \mathcal{S}_V$ in any ball of radius $\max(\delta_F(\mathcal{R}, \mathcal{S}), r)$. There exists a $2^{O(\rho \cdot (|\mathcal{R}_V|+|\mathcal{S}_V|))} \log(\delta_F(\mathcal{R}, \mathcal{S})/r)$ time algorithm to compute an $(\mathcal{R}, \mathcal{S})$-homeomorphism of Fréchet length at most $\delta_F(\mathcal{R}, \mathcal{S}) + 6r$*

Proof. We use Lemma 15 as a black box in an exponential and a binary search for $\delta_F(\mathcal{R}, \mathcal{S})$. Initially, we set $\delta = r$, we perform an exponential search to find the smallest $2^k \cdot r$, for which Lemma 15 returns a homeomorphism. Hence, we know that $2^{k-1} \cdot r \leq \delta_F(\mathcal{R}, \mathcal{S}) < 2^k \cdot r$. Then, we perform a binary search to reduce the size of this gap from $O(2^k \cdot r)$ to r, to guarantee that our homeomorphism has Fréchet length at most $\delta_F(\mathcal{R}, \mathcal{S}) + 6r$. Both the exponential and the binary search can be done in $O(k)$ time. Since $2^k \cdot r > \delta_F(\mathcal{R}, \mathcal{S})$, we have $k = O\left(\log(\frac{\delta_F(\mathcal{R}, \mathcal{S})}{r})\right)$. Thus, the total running time is

$$2^{O(\rho \cdot (|\mathcal{R}_V|+|\mathcal{S}_V|))} \cdot \log(\frac{\delta_F(\mathcal{R}, \mathcal{S})}{r}).$$ ◀

4 General surfaces

In this section, we describe an algorithm to compute the Fréchet distance between two arbitrary surfaces (of genus zero) $\mathcal{R} = (\mathcal{R}_V, \mathcal{R}_E, \mathcal{R}_T)$ and $\mathcal{S} = (\mathcal{S}_V, \mathcal{S}_E, \mathcal{R}_T)$. To simplify our running time analysis, we assume that $\mathcal{R}$ and $\mathcal{S}$ are composed of fat triangles, that is all their angles are larger than a constant $\theta > 0$. In general, our running time would depend on the minimum angle of the constituent triangles of the surfaces.

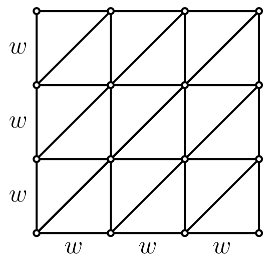
We define an r-**refinement** of a triangulation to be a refinement composed of triangles of diameter at most r. Before refining $\mathcal{R}$ and $\mathcal{S}$, we define triangulated grids, which we find helpful here and in the next section.

Triangulated grids. For any $w \in \mathbb{R}$, let $G_w = (V_w, E_w)$ be a triangulated grid of width w. That is

$$V_w = \{(iw, jw) \,|\, i, j \in \mathbb{Z}\},$$

and

$$E_w = \{((iw, jw), (i'w, j'w)) \,|\, i, j, i', j' \in \mathbb{Z} \wedge (i - i', j - j') \in \{(0, 1), (1, 0), (1, 1)\}\}.$$



▶ **Lemma 17.** *Let $\mathcal{Q} = (\mathcal{Q}_V, \mathcal{Q}_E, \mathcal{Q}_T)$ be a triangulated surface composed of fat triangles, and let $r \in \mathbb{R}^+$. There exists an $O(|\mathcal{Q}_V| + Area(\mathcal{Q})/r^2)$ time algorithm to compute an r-refinement of $\mathcal{Q}$ of size $O(|\mathcal{Q}_V| + Area(\mathcal{Q})/r^2)$.*

Proof. For each triangle $t \in \mathcal{Q}$ with diameter larger than r, we show how to refine it into a new triangulation composed of $O(\text{Area}(t)/r^2)$ triangles with diameter r. Note that when we put these triangulations together their vertices do not necessarily match on the border of different triangles. As a result, we may see flat polygons with more than three sides, but, we can triangulate them without introducing new vertices.

Let $t \in \mathcal{Q}_T$ with side lengths ℓ, ℓ' and ℓ'', with $\max(\ell, \ell', \ell'') > r$. Place t on G_r, the triangulated grid of width r. Let $\bar{t}$ be the triangulation of t induced by G_r. The number of triangles in $\bar{t}$ is $|\bar{t}| = O((\ell + \ell' + \ell'')/r + \text{Area}(t)/r^2)$. Since all angles of t are larger than a constant, ℓ, ℓ', and ℓ'' are within a constant factor of each other, and $\text{Area}(t) = \Theta(\ell^2)$. Therefore, $|\bar{t}| = O(\ell/r + \ell^2/r^2) = O(\ell^2/r^2) = O(\text{Area}(t)/r^2)$. ◀

Our algorithm for general surfaces is implied by Lemma 16 and Lemma 17.

Proof of Theorem 1. Let $\delta = \delta_F(\mathcal{R}, \mathcal{S})$, and let $r = (\varepsilon\delta)/6$. Let $\overline{\mathcal{R}}$ and $\overline{\mathcal{S}}$ be r-refinements of $\mathcal{R}$ and $\mathcal{S}$, respectively, obtained by applying the algorithm of Lemma 17, thus, $|\overline{\mathcal{R}}_V| = O(|\mathcal{R}_V| + \text{Area}(\mathcal{R})/r^2)$, and $|\overline{\mathcal{S}}_V| = O(|\mathcal{S}_V| + \text{Area}(\mathcal{S})/r^2)$. Also, $\overline{\mathcal{R}}$ and $\overline{\mathcal{S}}$ can be computed in linear time with respect to their sizes. Trivially, the number of vertices of $\overline{\mathcal{R}}_V \cup \overline{\mathcal{S}}_V$ in each ball of radius $\max(\delta, r)$ is at most $\overline{n} = |\overline{\mathcal{R}}_V| + |\overline{\mathcal{S}}_V|$. Thus, by Lemma 16 for $\overline{\mathcal{R}}$ and $\overline{\mathcal{S}}$, there is an $2^{O(\overline{n}^2)}\log(1/\varepsilon)$ time algorithm to compute an $(\mathcal{R},\mathcal{S})$-homoemorphism of Fréchet length at most $\delta + 6r = (1+\varepsilon)\delta$. We have,

$$2^{O(\overline{n}^2)}\log(\frac{1}{\varepsilon}) = 2^{O(|\overline{\mathcal{R}}_V|+|\overline{\mathcal{S}}_V|)^2}\log(\frac{1}{\varepsilon}) = 2^{O\left(\left(|\mathcal{R}_V|+|\mathcal{S}_V|+\frac{\mathrm{Area}(\mathcal{R})+\mathrm{Area}(\mathcal{S})}{(\varepsilon\delta)^2}\right)^2\right)}. \blacktriangleleft$$

5 Terrains

In this section, we describe an algorithm to compute the Fréchet distance between two polyhedral terrains $\mathcal{R}$ and $\mathcal{S}$ over $[0,1]^2$ (i.e. the images of the immersions $\varphi_{\mathcal{R}} : \mathcal{R} \to \mathbb{R}^3$ and $\varphi_{\mathcal{S}} : \mathcal{S} \to \mathbb{R}^3$ are polyhedral terrains over $[0,1]^2$). Let $\delta = \delta_F(\mathcal{R},\mathcal{S})$, and let D be the maximum slope of $\mathcal{R}$ and $\mathcal{S}$ for any point in their domain, $[0,1]^2$.

5.1 Sampling

Let $\mathcal{Q} : [0,1]^2 \to \mathbb{R}$ be a polyhedral terrain, let $1/r \in \mathbb{Z}$, and let $G_r = (V_r, E_r)$ be a grid of width r. Here we use $\mathcal{Q}$ both to refer to the triangulated surface and to the function over $[0,1]^2$. The **r-coarse approximation** of $\mathcal{Q}$, is a polyhedral terrain $\overline{\mathcal{Q}}$, whose vertex set is,

$$\overline{\mathcal{Q}}_V = \{(x,y,\mathcal{Q}(x,y))\,|(x,y)\in V_r\}\,,$$

and whose edge set is

$$\overline{\mathcal{Q}}_E = \{((x,y,\mathcal{Q}(x,y)),(x',y',\mathcal{Q}(x',y')))\,|(x,y),(x',y')\in E_r\}\,.$$

Again, we view $\overline{\mathcal{Q}}$ as a triangulated surface as well as a function $\overline{\mathcal{Q}} : [0,1]^2 \to \mathbb{R}$, thus, we use $\overline{\mathcal{Q}}(x,y)$ for a point $(x,y)\in[0,1]^2$.

▶ **Lemma 18.** *Let $\mathcal{Q} : [0,1]^2 \to \mathbb{R}$ be a polyhedral terrain with maximum slope D, and let $\overline{\mathcal{Q}}$ be its r-coarse approximation, where $1/r \in \mathbb{Z}$. We have $\delta_F(\mathcal{Q},\overline{\mathcal{Q}}) \le 2\sqrt{2}\cdot rD$.*

Proof. Let $f : \mathcal{Q} \to \overline{\mathcal{Q}}$ be the projection map along the z-axis. That is, for any $(x,y,z)\in\mathcal{Q}$, $f(x,y,z) = (x,y,\overline{\mathcal{Q}}(x,y))$. Let t be the triangle in G_r that contains (x,y), and let (x',y') be any vertex of t. We have $||(x,y)-(x',y')|| \le \sqrt{2}\cdot r$, which implies,

$$||(x,y,\mathcal{Q}(x,y)) - (x',y',\mathcal{Q}(x',y'))|| \le \sqrt{2}\cdot rD,$$

and

$$||(x,y,\overline{\mathcal{Q}}(x,y)) - (x',y',\overline{\mathcal{Q}}(x',y'))|| \le \sqrt{2}\cdot rD,$$

for the maximum slope of $\overline{\mathcal{Q}}$ is bounded by D too.

On the other hand, since (x',y') is a grid point $\overline{\mathcal{Q}}(x',y') = \mathcal{Q}(x',y')$. Thus, by the triangle inequality,

$$||(x,y,\mathcal{Q}) - (x,y,\overline{\mathcal{Q}})|| \le 2\sqrt{2}\cdot rD. \blacktriangleleft$$

Proof of Theorem 2. Let $r' = \min(\varepsilon\delta/12, \varepsilon\delta/(8\sqrt{2}D))$, let $1/r$ be the smallest integer larger than $1/r'$, and let $\overline{\mathcal{R}}$ and $\overline{\mathcal{S}}$ be r-refinements of $\mathcal{R}$ and $\mathcal{S}$, respectively. Consider any point $p = (x,y,z)\in\mathbb{R}^3$. The number of vertices of $\overline{\mathcal{R}}_V \cup \overline{\mathcal{S}}_V$ in $B_{\max(\delta,r)}(p)$ is at most the number of grid points, vertices of V_r, in a disk of radius $\max(\delta,r)$ with center (x,y), which is $O(\delta^2/r^2)$. Thus, Lemma 16 implies that an $(\overline{\mathcal{R}},\overline{\mathcal{S}})$-homeomorphism of Fréchet length $\delta+6r$ can be computed in $2^{O(\delta^2/r^4)} = 2^{O(D^4/(\varepsilon^4\delta^2))}$ time. Composing this homeomorphism with the $(\mathcal{R}, \overline{\mathcal{R}})$-homeomorphism and the $(\overline{\mathcal{S}}, \mathcal{S})$-homeomorphism of Lemma 18, we obtain a homeomorphism of Fréchet length

$$\delta_F(S,T) + 6r + 4\sqrt{2} \cdot rD \leq \delta + \varepsilon\delta/2 + \varepsilon\delta/2 = (1+\varepsilon)\delta.$$

We need to sample $O(D/(\varepsilon\delta)^2)$ points from $\mathcal{R}$ and $\mathcal{S}$ to compute $\overline{\mathcal{R}}$ and $\overline{\mathcal{S}}$, which takes $O(D/(\varepsilon\delta)^2 n)$ time. Therefore, overall, we obtain the desired asymptotic time bound. ◀

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