Homogeneous Equations of Algebraic Petri Nets[*]

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Abstract

Algebraic Petri nets are a formalism for modeling distributed systems and algorithms, describing control and data flow by combining Petri nets and algebraic specification. One way to specify correctness of an algebraic Petri net model N is to specify a *linear equation* E over the places of N based on term substitution, and coefficients from an abelian group $\mathbb{G}$. Then, E is *valid* in N iff E is valid in each reachable marking of N. Due to the expressive power of Algebraic Petri nets, validity is generally undecidable. *Stable* linear equations form a class of linear equations for which validity is decidable. *Place invariants* yield a well-understood but incomplete characterization of all stable linear equations. In this paper, we provide a complete characterization of stability for the subclass of *homogeneous* linear equations, by restricting ourselves to the interpretation of terms over the Herbrand structure without considering further equality axioms. Based thereon, we show that stability is decidable for homogeneous linear equations if $\mathbb{G}$ is a cyclic group.

1998 ACM Subject Classification F.3.1 Specifying and Verifying and Reasoning about Programs, D.3.1 Formal Definitions and Theory

Keywords and phrases Algebraic Petri Nets, Invariants, Linear Equations, Validity, Stability

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2016.14

1 Introduction

The formalism of *algebraic Petri nets* (APNs) permits to formally model both control flow and data flow of distributed systems and algorithms, extending Petri nets with concepts from algebraic specification, namely a signature together with equality axioms. Thus, APNs combine the benefits of Petri nets, such as explicit modeling of concurrency and options for structural analysis, with the ability to describe data objects on a freely chosen level of abstraction. The price to pay for this expressive power is that many important behavioral properties, such as reachability of a certain marking, are undecidable. However, there are behavioral properties that can be proven based on *structural* properties, such as invariants.

In this paper, we study a particular class of behavioral properties, namely *linear equations*. Intuitively, a linear equation E formalizes a linear correlation between the tokens on different places, requiring that each reachable marking satisfies E. More formally, an APN N is defined over a signature Σ, and the tokens are ground terms over Σ. A linear equation E has the form $\sum_{p \in P} \gamma_p \kappa_p = b_1\mu_1 + \ldots + b_n\mu_n$, where P is the set of places, each γ_p and b_i are coefficients stemming from an abelian group, each κ_p is a term over Σ, and each μ_i is a ground term over Σ. A marking satisfies E if substituting each variable in each κ_p with the tokens on p yields an equality. *Validity* of E in N requires that each reachable marking of N satisfies E. Case

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[*] This work was partially funded by the DFG Graduiertenkolleg 1651 (SOAMED).

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27th International Conference on Concurrency Theory (CONCUR 2016).
Editors: Josée Desharnais and Radha Jagadeesan; Article No. 14; pp. 14:1–14:14
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

studies have shown that this class of properties permits to formalize important behavioral properties of distributed systems and algorithms. Unfortunately, verifying the validity of E in N is generally infeasible. However, if E is *stable* then validity of E becomes decidable. Stability requires the preservation of E along all—not necessarily reachable—steps, that is, if a marking satisfies E, then firing a transition yields a marking satisfying E. Now, if E is stable, validity of E coincides with the initial marking satisfying E.

Place invariants yield a subclass of stable linear equations. Intuitively, a place invariant is a solution of a homogeneous system of linear equations given by the structure of N, providing the coefficients γ_p and terms κ_p—the right hand side can be chosen arbitrarily. This characterization is known to be decidable but incomplete, that is, there are stable linear equations, such that the left hand side is not given by a place invariant. A decidable, complete characterization of stability—or an undecidability proof—is still an open problem.

In this paper, we contribute to this field of study as follows:

1. We show the undecidability of validity of homogeneous equations.
2. We provide a complete characterization of stability, restricting ourselves to
    - *homogeneous* linear equations, that is, $n = 1$ and $b_1 = 0$, and
    - the interpretation of terms in the Herbrand structure, that is, assuming coincidence of syntax and semantics of a term, without considering further equality axioms for terms.
3. We show that our characterization is decidable if the coefficients stem from a cyclic group.

Section 2 recalls required notions for equations of algebraic Petri nets. We summarize our main theorems in Section 3, and prove these theorems in Section 4 and Section 5. We discuss related work in Section 6, and conclude in Section 7. Due to lack of space, the reader is referred to the companion technical report [16] for the missing proofs.

2 Formalization

2.1 Preliminaries

We write $\mathbb{Z}$ for the set of all integers, and $\mathbb{N}$ denotes the set $\{0, 1, 2, \ldots\}$ of natural numbers including 0. Let $z \in \mathbb{Z}$. Then, $|z|$ denote the absolute value.

2.1.1 Polynomials over Abelian Groups

Polynomials over abelian groups serve as a common algebraic base to formalize APNs and linear equations of APNs.

▸ **Definition 1** (Abelian Group, Scalar Product)**.** An *abelian group* $(\mathbb{G}, \oplus)$ consists of a set $\mathbb{G}$, and an associative, commutative, binary operation $\oplus$ on $\mathbb{G}$ with an identity $0_\mathbb{G}$, and inverses $\ominus g$ for each $g \in \mathbb{G}$. Let $z \in \mathbb{Z}$ and $a \in \mathbb{G}$. We define the *scalar product* $za \in \mathbb{G}$ by

$$za := \begin{cases} \bigoplus_{i=0}^{z} a & \text{if } z \geq 0 \\ \ominus(-za) & \text{otherwise.} \end{cases}$$

$(\mathbb{G}, \oplus)$ is *cyclic* iff there exists $a \in \mathbb{G}$, such that $\mathbb{G} = \{za \mid z \in \mathbb{Z}\}$.

Whenever clear from context, we simply write $\mathbb{G}$ for $(\mathbb{G}, \oplus)$. Examples for abelian groups are the real numbers, rational numbers, integers, and the additive group $\mathbb{Z}/n\mathbb{Z}$ of integers modulo some $n \in \mathbb{N}$. The group $\mathbb{Z}$ is infinite and cyclic, the group $\mathbb{Z}/n\mathbb{Z}$ is finite and cyclic. In contrast to that, the group of rational numbers is not cyclic.

▸ **Definition 2** (Series, Polynomial, Monomial, Empty Polynomial)**.** Let M be a set, $\mathbb{G}$ be an abelian group, and $f : M \rightarrow \mathbb{G}$ be a function. Then, f is a (linear) *series* over M and $\mathbb{G}$ with *support* $\operatorname{supp}(f) \coloneqq \{m \in M \mid f(m) \neq 0_{\mathbb{G}}\}$. If $\operatorname{supp}(f)$ is finite, then f is a *polynomial*. We write $\mathbb{G}\langle M\rangle$ for the set of all polynomials over M and $\mathbb{G}$. If $\operatorname{supp}(f)$ is singleton, f is a *monomial*, and we denote f by (m, a) where $\operatorname{supp}(f) = \{m\}$ and $f(m) = a$. If $\operatorname{supp}(f) = \varnothing$, then f is *empty*, and we denote f by $0_{\mathbb{G}}$.

We lift $\oplus$ and the scalar product to $\mathbb{G}\langle M\rangle$ by pointwise application:

▸ **Definition 3** (Addition of Polynomials)**.** Let M be a set and $\mathbb{G}$ be an abelian group. For $p_1, p_2 \in \mathbb{G}\langle M\rangle$, $m \in M$, and $z \in \mathbb{Z}$, we define the polynomials $p_1 \oplus p_2$ and zp_1 over M and $\mathbb{G}$ by

$$\begin{aligned}(p_1 \oplus p_2)(m) &\coloneqq p_1(m) \oplus p_2(m) \quad ,\\ (zp_1)(m) &\coloneqq zp_1(m) \quad .\end{aligned}$$

We lift associative binary operations from M to $\mathbb{G}\langle M\rangle \times \mathbb{Z}\langle M\rangle$ by applying the Cauchy product:

▸ **Definition 4** (Cauchy Product)**.** Let $\odot$ be an associative binary operation on a set M, $\mathbb{G}$ be an abelian group, $p_1 \in \mathbb{G}\langle M\rangle$, and $p_2 \in \mathbb{Z}\langle M\rangle$. We define the series $p_1 \odot p_2$ over M and G by

$$(p_1 \odot p_2)(m) \coloneqq \bigoplus_{m = m_1 \odot m_2} \underbrace{p_2(m_2)}_{\in \mathbb{Z}} \underbrace{p_1(m_1)}_{\in \mathbb{G}} .$$

Because p_1 and p_2 are polynomials, the set $\operatorname{supp}(p_1 \odot p_2) = \{m_1 \odot m_2 \mid m_1, m_2 \in \mathbb{G}, p_1(m_1) \neq 0_{\mathbb{G}}, p_2(m_2) \neq 0\}$ is finite, and thus $p_1 \odot p_2$ is again a polynomial over M and $\mathbb{G}$.

2.1.2 Terms

For this paper, we fix a set of variables VAR, a non-empty, finite index set $\mathbb{I}$, and a signature $\Sigma = (\dot{f}_i/\mathrm{a}_i)_{i\in\mathbb{I}}$ consisting of $|\mathbb{I}|$ distinct function symbols $\dot{f}_i$ with respective arity a_i.

▸ **Definition 5** (Term)**.** For a set $V \subseteq \mathrm{VAR}$, the set Θ_V of *terms* over variables V is the smallest set satisfying the following conditions:
1. $V \subset \Theta_V$.
2. Let $i \in \mathbb{I}$, and $\theta_1, \dots, \theta_{\mathrm{a}_i} \in \Theta_V$. Then, $\dot{f}_i(\theta_1, \dots, \theta_{\mathrm{a}_i}) \in \Theta_V$.

The elements of $\Theta_\varnothing$ are called *ground terms*.

As usual, if $\mathrm{a}_i = 0$, we abbreviate $\dot{f}_i()$ as $\dot{f}_i$. We abbreviate the set Θ_{VAR} of all terms as Θ.

A *substitution* maps each variable to a term. A substitution is an *assignment* if it maps each variable to a ground term.

▸ **Definition 6** (Substitution, Assignment)**.** Every function $\sigma : \mathrm{VAR} \rightarrow \Theta$ is a substitution. Let $\theta \in \Theta$. The term $\theta\sigma$ is defined by:

$$\theta\sigma \coloneqq \begin{cases} \sigma(\theta) & \text{if } \theta \in \mathrm{VAR} \\ \dot{f}_i(\theta_1\sigma, \dots, \theta_{\mathrm{a}_i}\sigma) & \text{if } \theta = \dot{f}_i(\theta_1, \dots, \theta_{\mathrm{a}_i}), i \in \mathbb{I}. \end{cases}$$

If $\sigma(x) \in \Theta_\varnothing$ for each $x \in \mathrm{VAR}$, then σ is an *assignment*, and we also write $[\![\theta]\!]_\sigma$ instead of $\theta\sigma$.

Obviously, if σ is an assignment, then $[\![\theta]\!]_\sigma \in \Theta_\varnothing$ for all $\theta \in \Theta$.

Unification is the problem of applying a substitution to terms, such that the resulting terms become identical.

▸ **Definition 7** (Unification problem, unifier, solvable)**.** A *unification problem* U is a finite subset $\{(\theta_1, \theta_1'), \ldots, (\theta_n, \theta_n')\}$ of $\Theta \times \Theta$, also denoted by $\{\theta_1 \doteq \theta_1', \ldots, \theta_n \doteq \theta_n'\}$. A substitution σ is a *unifier* for U iff for all $1 \leq i \leq n$: $\theta_i\sigma = \theta_i'\sigma$. If there exists a unifier for U, then U is *solvable.*

It is known that every solvable unification problem has a *most general unifier* (up to variants) that subsumes all other unifiers:

▸ **Lemma 8.** *Let U be a solvable unification problem. Then, there exists a unifier $\hat{\sigma}$ for U, such that: For each unifier σ for U, there exists a substitution σ' with $\sigma(x) = \hat{\sigma}(x)\sigma'$ for all $x \in \mathrm{VAR}$.*

We define a *product* on terms by means of term substitution: The product of ϱ and θ is defined by substituting every occurrence of any variable in ϱ by θ.

▸ **Definition 9** (Term Product)**.** Let $\varrho, \theta \in \Theta$ be terms, and σ be the substitution with $\sigma(x) = \theta$ for all $x \in \mathrm{VAR}$. Then, $\varrho \odot \theta := \varrho\sigma$ is the *product* of ϱ and θ.

We observe that $\odot$ is associative. If $\varrho \in \Theta_\varnothing$, then $\varrho \odot \theta = \varrho$.

We lift substitutions from terms to polynomials over terms and abelian groups by pointwise substitution and subsequent "simplification" of the polynomial:

▸ **Definition 10** (Substitutions in Polynomials over Terms)**.** Let $\mathbb{G}$ be an abelian group, and $p \in \mathbb{G}\langle\Theta\rangle$. Let σ be a substitution. We define $p\sigma \in \mathbb{G}\langle\Theta\rangle$ by

$$p\sigma(\theta) := \bigoplus_{\theta = [\![\theta']\!]_\sigma} p(\theta').$$

If σ is an assignment, we also write $[\![p]\!]_\sigma$ instead of $p\sigma$.

We observe $(\varrho \odot \theta)\sigma = \varrho \odot \theta\sigma$ for all $\varrho, \theta \in \Theta$, and $(p_1 \odot p_2)\sigma = p_1 \odot p_2\sigma$ for all $p_1, p_2 \in \mathbb{G}\langle\Theta\rangle$. Moreover, if σ is an assignment then $\mathrm{supp}([\![p]\!]_\sigma) \subseteq \Theta_\varnothing$.

2.1.3 Vectors

In this paper, a *P-vector* is a mapping from a set P into polynomials over terms and an abelian group.

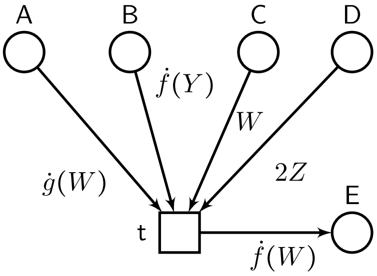
▸ **Definition 11** (P-vector)**.** Let P be a set, $(\mathbb{G}, \oplus)$ be an abelian group, and $\vec{k} : P \to \mathbb{G}\langle\Theta\rangle$. Then, $\vec{k}$ is a *P-vector* over $\mathbb{G}$. We write $\mathbb{G}\langle\Theta\rangle^P$ for the set of all P-vectors over $\mathbb{G}$. If $\vec{k}(p)$ is a monomial for each $p \in P$, then $\vec{k}$ is *simple.* If $\mathbb{G} = \mathbb{Z}$, and $\vec{k} \geq 0$ ($\vec{k} \leq 0$), then $\vec{k}$ is *semi-positive* (*semi-negative*).

In order to simplify notation, we lift the basis notions from polynomials to P-vectors:

▸ **Definition 12** (P-vectors: Support, emptiness, addition, Cauchy product, and assignments)**.** Let P be a set, $(\mathbb{G}, \oplus)$ be an abelian group, $\vec{k}, \vec{k}_1, \vec{k}_2 \in \mathbb{G}\langle\Theta\rangle^P$, and $\vec{k}' \in \mathbb{Z}\langle\Theta\rangle^P$.

- $\mathrm{supp}(\vec{k}) := \bigcup_{p \in P} \mathrm{supp}(\vec{k}(p))$ is the *support* of $\vec{k}$.
- If $\vec{k}(p) = 0_\mathbb{G}$ for all $p \in P$, then $\vec{k}$ is the empty P-vector, also denoted by $0_\mathbb{G}$.
- We define $(\vec{k}_1 \oplus \vec{k}_2)(p) := \vec{k}_1(p) \oplus \vec{k}_2(p)$ for all $p \in P$,
- We extend $\odot$ from $\mathbb{G}\langle\Theta\rangle \times \mathbb{Z}\langle\Theta\rangle \to \mathbb{G}\langle\Theta\rangle$ to $\mathbb{G}\langle\Theta\rangle^P \times \mathbb{Z}\langle\Theta\rangle^P \to \mathbb{G}\langle\Theta\rangle$ by defining $(\vec{k} \odot \vec{k}')(\theta) := \bigoplus_{p \in P} \vec{k}(p) \odot \vec{k}'(p)$ for all $\theta \in \Theta$,
- If σ is an assignment, we define $[\![\vec{k}]\!]_\sigma \in \mathbb{G}\langle\Theta\rangle^P$ by $[\![\vec{k}]\!]_\sigma(p) := [\![\vec{k}(p)]\!]_\sigma$ for all $p \in P$.

Let $\vec{k}_1 \in \mathbb{G}\langle\Theta\rangle^P$, $\vec{k}_2 \in \mathbb{Z}\langle\Theta\rangle^P$ and δ be a substitution. We observe: $\vec{k}_1 \odot (\vec{k}_2\delta) = \sum_{p \in P} \vec{k}_1(p) \odot (\vec{k}_2\delta)(p) = \sum_{p \in P} \vec{k}_1(p) \odot (\vec{k}_2(p)\delta) = \sum_{p \in P} (\vec{k}_1(p) \odot \vec{k}_2(p))\delta = (\vec{k}_1 \odot \vec{k}_2)\delta$.



(a) An APNS S_1.

$\Sigma = \{\dot{f}/1, \dot{g}/1, \dot{c}/0\}$

(b) The Signature of S_1.

	A	B	C	D	E	$\mathbb{G}$
E_1	$4\dot{f}(\mathsf{A})$	$3\dot{g}(\mathsf{B})$	$-5\dot{f}(\dot{g}(\mathsf{C}))$	$-\mathsf{D}$	0	$\mathbb{Z}$
E_2	$3\dot{c}$	0	0	$2\mathsf{D}$	0	$\mathbb{Z}/7\mathbb{Z}$

(c) Homogeneous equations over $\mathbb{Z}$ and $\mathbb{Z}/7\mathbb{Z}$.

Figure 1 An APNS S_1 with equations E_1 and E_2.

2.1.4 Algebraic Petri Nets

An *algebraic Petri net structure* consists of *places* P and *transitions* T. A place $p \in P$ describes a token store, and a transition t is given by two semi-positive P-vectors $\vec{t}^-$ and $\vec{t}^+$, describing token consumption and production, respectively.

▸ **Definition 13** (Transition, algebraic Petri net structure)**.** Let $P \neq \emptyset$ be a set. A *transition* $t = (\vec{t}^-, \vec{t}^+)$ over P consists of two semi-positive simple P-vectors $\vec{t}^-, \vec{t}^+$ over $\mathbb{Z}$. We define the *effect* $\vec{t}^\Delta \in \mathbb{Z}\langle\Theta\rangle^P$ of t by $\vec{t}^\Delta := -\vec{t}^- + \vec{t}^+$. Let T be a set of transitions over P. Then, (P, T) is an *algebraic Petri net structure* (APNS). We write $\mathrm{pre}(t)$ for $\{p \in P \mid \vec{t}^-(p) > 0\}$.

Figure 1 shows an example of an APNS S_1 with transition t, places A, B, C, D and E and signature Σ using two unary function symbols $\dot{f}$ and $\dot{g}$ and the constant $\dot{c}$. Transition t consists of $\mathsf{t}^- = (\dot{g}(W) \quad \dot{f}(Y) \quad W \quad 2Z \quad 0)$ and $\mathsf{t}^+ = (0 \quad 0 \quad 0 \quad 0 \quad \dot{f}(W))$.

A *token* is a ground term, a *marking* maps each place to a multiset of tokens:

▸ **Definition 14** (Marking)**.** Let (P, T) be an APNS. Let $\vec{m} \in \mathbb{Z}\langle\Theta\rangle^P$ be a semi-positive P-vector over $\mathbb{Z}$ with $\mathrm{supp}(\vec{m}) \subseteq \Theta_\emptyset$. Then, $\vec{m}$ is a *marking* of (P, T). We write $\mathbb{Z}\langle\Theta_\emptyset\rangle^P_{\geq 0}$ for the set of all markings of (P, T).

Algebraic Petri net semantics are defined by the notion of a *step* based on the effect of a transition, and the notion of a *firing mode*:

▸ **Definition 15** (Step)**.** Let (P, T) be an APNS, $\vec{m}, \vec{m}' \in \mathbb{Z}\langle\Theta_\emptyset\rangle^P_{\geq 0}$, $t \in T$, and σ be an assignment, such that $\vec{m} \geq [\![\vec{t}^-]\!]_\sigma$ and $\vec{m}' = \vec{m} + [\![\vec{t}^\Delta]\!]_\sigma$. Then, $\vec{m}$ *enables* transition t in *firing mode* σ, denoted by $\vec{m}\,[t\sigma\rangle$, and $(\vec{m}, t, \sigma, \vec{m}')$ is a step of (P, T), denoted by $\vec{m}\,[t\sigma\rangle\,\vec{m}'$.

We remark that our definition of *enabling* does not consider additional equality axioms; permitting such axioms is left for future work.

An *algebraic Petri net* APN is an APNS together with an *initial marking*. Subsequent steps from the initial marking are *runs*, the resulting markings are *reachable*:

▸ **Definition 16** (Algebraic Petri net, run, reachable)**.** Let (P, T) be an APNS, and $\vec{m}_0 \in \mathbb{Z}\langle\Theta_\emptyset\rangle^P_{\geq 0}$. Then, $(P, T, \vec{m}_0)$ is an *algebraic Petri net* (APN). Let $\vec{m}_0\,[t_1\sigma_1\rangle\,\vec{m}_1 \ldots \vec{m}_{n-1}\,[t_n\sigma_n\rangle\,\vec{m}_n$ be a sequence of steps. Then, $(t_1, \sigma_1)\ldots(t_n, \sigma_n)$ is a *run* of $(P, T, \vec{m}_0)$ and $\vec{m}_n$ is *reachable* in $(P, T, \vec{m}_0)$.

2.2 Homogeneous Linear Equations of APNs

A homogeneous *(linear) P-equation* over a set P of places has the form $\sum_{p \in P} \gamma_p \kappa_p = 0_{\mathbb{G}}$, where $\gamma_p \in \mathbb{G}$ $(p \in P)$ are elements of an abelian group $\mathbb{G}$ with $0_{\mathbb{G}}$ as neutral element and each κ_p $(p \in P)$ is a term. Formally, a homogeneous P-equation is given by a simple P-vector.

▸ **Definition 17** (Homogeneous P-equation)**.** Let P be a set, $\mathbb{G}$ be an abelian group and $\vec{k} \in \mathbb{G}\langle\Theta\rangle^P$ be simple. Then, $\vec{k}$ induces a *homogeneous P-equation* over $\mathbb{G}$.

Figure 1 shows two equations E_1 and E_2. E_1 is over the group of integer $\mathbb{Z}$ and E_2 is over the group of integers modulo 7, $\mathbb{Z}/7\mathbb{Z}$. The table shows the simple P-vectors. For instance, $\vec{k}_1(\mathsf{A}) \odot \mathcal{X}_\mathsf{A}$ is the monomial $(\dot{f}(\mathsf{A}), 4)$.

A marking $\vec{m}$ satisfies E if replacing P by $\vec{m}$ yields an identity. A homogeneous P-equation is *valid* in an APN if it is satisfied by each reachable marking.

▸ **Definition 18** (Satisfaction, validity)**.** Let (P, T) be an APNS, $\vec{m}$ be a marking, $\mathbb{G}$ be an abelian group, and E be a homogeneous P-equation over $\mathbb{G}$ given by the simple P-vector $\vec{k} \in \mathbb{G}\langle\Theta\rangle^P$. If $\vec{k} \odot \vec{m} = 0_\mathbb{G}$, then $\vec{m}$ *satisfies* E. If each reachable marking of $(P, T, \vec{m})$ satisfies E, then E is *valid* in $(P, T, \vec{m})$.

A homogeneous P-equation is *stable* if satisfaction is preserved by all steps:

▸ **Definition 19** (Stability)**.** Let (P, T) be an APNS, $t \in T$, $\mathbb{G}$ be an abelian group, and E be a homogeneous P-equation over $\mathbb{G}$. Then, E is *t-stable* in (P, T) iff for each step $\vec{m}\,[t\sigma\rangle\,\vec{m}'$ of (P, T): If $\vec{m}$ satisfies E, then $\vec{m}'$ satisfies E.

Stability together with satisfaction in the initial marking yields validity:

▸ **Lemma 20.** *Let $(P, T, \vec{m})$ be an APN, $\mathbb{G}$ be an abelian group, and E be a homogeneous P-equation over $\mathbb{G}$ given by a simple P-vector $\vec{k} \in \mathbb{G}\langle\Theta\rangle^P$. If E is t-stable for each $t \in T$, and $\vec{m}$ satisfies E, then E is valid in $(P, T, \vec{m})$.*

A *place invariant* $\vec{k}$ is a simple P-vector such that for each $t \in T$, we have $\vec{k} \odot \vec{t}^\Delta = 0_\mathbb{G}$. Then, the homogeneous equation induced by $\vec{k}$ is stable:

▸ **Lemma 21.** *Let (P, T) be an APN, $\mathbb{G}$ be an abelian group, and E be a homogeneous P-equation over $\mathbb{G}$ given by a simple P-vector $\vec{k} \in \mathbb{G}\langle\Theta\rangle^P$. Let $t \in T$ and $\vec{k} \odot \vec{t}^\Delta = 0_\mathbb{G}$. Then, E is t-stable in (P, T).*

3 Contributions

We summarize our contributions in the form of two main theorems which we prove in the subsequent sections. Our first contribution is a proof that validity of a given P-equation in an APN is undecidable. The proof can be found in Section 4 and bases on a reduction of the halting problem of Minsky machines.

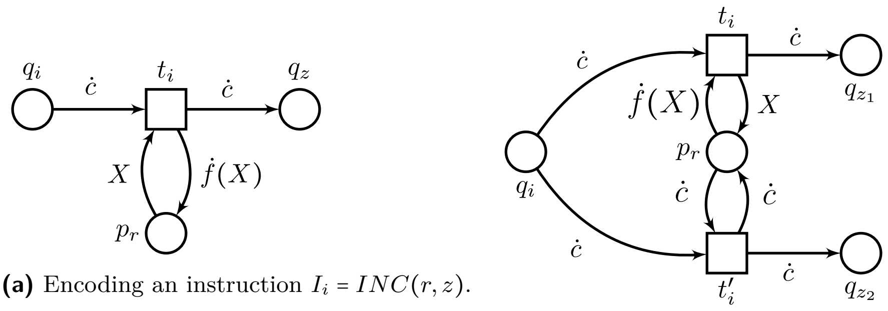
▸ **Theorem 22.** *Let $(P, T, \vec{m})$ be an APN and E a homogeneous P-equation. Then, validity of E in $(P, T, \vec{m})$ is undecidable.*

Proof. Follows from Lemma 25 and Lemma 29. ◂

Our second contribution is a decidability proof for the stability of a homogeneous P-equation in an APNS under the assumption that the coefficients stem from a cyclic group. Here, we develop a decidable, necessary and sufficient criterion, generalizing the invariant theorem (cf. Lemma 21), in Section 5.

▸ **Theorem 23.** *Let (P, T) be an APNS and E be a homogeneous P-equation over a cyclic group, then stability of E in (P, T) is decidable.*

Proof. Follows from Lemma 44 and Lemma 46. ◂



(a) Encoding an instruction $I_i = INC(r, z)$.

(b) Encoding an instruction $I_i = JZ(r, z_1, z_2)$.

Figure 2 Encoding Minsky Machines into APNs.

4 Undecidability of Validity of Homogeneous Equations

In this section, we give short description how to encode a *Minsky Machine* [10] M into an APN N_M using the Herbrand structure. Then, the halting problem in the Minsky Machine reduces to validity of an equation. This proof technique has been used before for Petri Nets, for example in [12]. First, we recall the required notions of a Minsky machine, its states and its steps:

▸ **Definition 24** (Minsky machine). A *Minsky Machine* $M = (\mathcal{I}, \mathcal{R})$ consists of number of registers $\mathcal{R} \in \mathbb{N}$ and a sequence $\mathcal{I} = I_1, \dots, I_n$ of instructions, where each instruction $I_i \in \{INC(r, z) \mid 1 \le r \le \mathcal{R}, 1 \le z \le n\} \cup \{JZ(r, z_1, z_2) \mid 1 \le r \le \mathcal{R}, 1 \le z_1 \le n-1, 1 \le z_2 \le n-1\}$ and $I_n = HALT$.

Every tuple $(\rho, \ell) \in \mathbb{N}^{\mathcal{R}} \times \{1, \dots, n\}$ is a *state* of M. If $I_\ell = INC(r, z)$, then $(\rho, \ell) \to (\rho', z)$ is a step in M with $\rho'(r) = \rho(r) + 1$ and $\rho'(q) = \rho(q)$ for all $q \neq r$. If $I_\ell = JZ(r, z_1, z_2)$ and $\rho(r) > 0$, then $(\rho, \ell) \to (\rho', z_1)$ is a step in M with $\rho'(r) = \rho(r) - 1$ and $\rho'(q) = \rho(q)$ for all $q \neq r$. If $I_\ell = JZ(r, z_1, z_2)$ $\rho(r) = 0$, then $(\rho, \ell) \to (\rho, z_2)$ is a step. We denote the reflexive transitive closure of $\to$ with $\to^*$.

We recall that the halting problem for Minsky machines is undecidable:

▸ **Lemma 25** ([10]). *Let M be a Minsky Machine. It is undecidable, whether M halts, i.e. the following problem is undecidable: $\exists \rho \in \mathbb{N}^{\mathcal{R}}$ such that $(0, 1) \to^* (\rho, n)$.*

To reduce the halting problem, we encode a Minsky Machine into an APNS.

▸ **Definition 26** (Encoding of Minsky Machine). Let M be a Minsky Machine M, then the APNS N_M encodes M, if:

- The signature is $\Sigma_M = \{\dot{f}/1, \dot{c}/0\}$,
- the set of places is $P = \{p_r \mid 1 \le r \le \mathcal{R}\} \cup \{q_i \mid 1 \le i \le n\}$,
- for every INC-instruction I_i, let t_i be the transition with the pattern shown in Figure 2a,
- and for every JZ-instruction I_i let t_i and t'_i be the transitions following the pattern shown in Figure 2b.

▸ **Definition 27.** Let $(\rho, \ell) \in \mathbb{N}^{\mathcal{R}}$ be a state of M. For $x \in \mathbb{N}$, we define $\theta_x \in \Theta$ by

$$\theta_x := \begin{cases} \dot{c} & \text{if } x = 0 \\ \dot{f}(\theta_{x-1}) & \text{otherwise.} \end{cases}$$

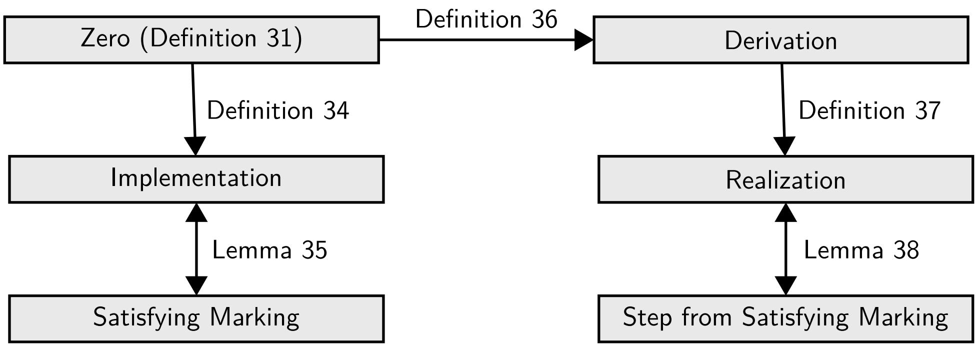


Figure 3 Overview of the proof of Theorem 23.

Then, we define the marking $\vec{m}_\ell^\rho \in \mathbb{Z}\langle\Theta_\varnothing\rangle_{\geq 0}^P$ of N_M as follows for $p \in P$ and $\theta \in \Theta$:

$$\vec{m}(p) := \begin{cases} (\dot{c}, 1) & \text{if } p = q_\ell \\ (\theta_{\rho(r)}, 1) & \text{if } p = p_r \\ 0 & \text{otherwise} \end{cases}$$

Now, we can relate the steps of a Minsky Machine M to the steps of the encoding N_M.

▸ **Lemma 28.** *Let $(\rho, \ell), (\rho', \ell')$ be states of M with $(\rho, \ell) \to (\rho', \ell')$. Then:*

1. *There exists a step $\vec{m}_\ell^\rho\, [t\sigma\rangle\, \vec{m}'$ of N_M.*
2. *If $\vec{m}_\ell^\rho\, [t\sigma\rangle\, \vec{m}'$ is a step of N_M, then $\vec{m}' = \vec{m}_{\ell'}^{\rho'}$.*

Finally, we reduce the halting problem for M to the validity of the homogeneous P-equation $q_n = 0$ in $(N_M, \vec{m}_0^1)$. The P-vector over $\mathbb{Z}$ that induces the P-equation is zero for all places $p \in P \smallsetminus \{q_n\}$ and 1 for q_n. Inductively applying Lemma 28 reduces reachability of the $HALT$ state in M to non-emptiness of the place q_n in $(N_M, \vec{m}_0^1)$ and thus to validity of $q_n = 0$.

▸ **Lemma 29.** *The equation $q_n = 0$ is valid in $(N_M, \vec{m}_0^1)$ if and only if the Minsky Machine M does not halt.*

5 Deciding Stability of Homogeneous Equations over Cyclic Groups

In this section, we show that stability of a homogeneous P-equation E given by a simple P-vector $\vec{k}$ in an APNS $N = (P, T)$ is decidable, if $\mathbb{G}$ is a cyclic group. To this end, we identify a decidable, necessary and sufficient condition for stability, which generalizes the necessary but not sufficient condition given by the classical invariant theorem (cf. Lemma 21). We develop our condition based on the following lemma, which directly follows from applying additivity arguments to the definition of stability:

▸ **Lemma 30.** *Let $t \in T$ be a transition. Then, the following statements are equivalent:*

1. *E is t-stable.*
2. *For all steps $\vec{m}\, [t\sigma\rangle\, \vec{m}'$: If $\vec{k} \odot \vec{m} = 0_{\mathbb{G}}$, then $\vec{k} \odot [\![\vec{t}^\Delta]\!]_\sigma = 0_{\mathbb{G}}$.*

Lemma 30 generalizes Lemma 21 in the sense that we can derive Lemma 21 from Lemma 30, but not vice versa. However, the condition stated in Lemma 30 does not directly infer a decision procedure, because the set of steps $\vec{m}\, [t\sigma\rangle\, \vec{m}'$ with $\vec{k} \odot [\![\vec{t}^\Delta]\!]_\sigma = 0_{\mathbb{G}}$ is infinite, that

	A	B	C	D	Zero of E_1?	Zero of E_2?	$\varrho(\nu_i)$
ν_1	0	1	0	3	yes	no	$\dot g(\mathsf{B})$
ν_2	5	0	4	0	yes	no	$\dot f(\dot g(\mathsf{C}))$
ν_3	0	2	0	6	yes	no	$\dot g(\mathsf{B})$
ν_4	1	1	1	2	no	yes	$\dot c$
ν_5	2	0	0	4	no	yes	$\dot c$

(a) Zeros $\nu_1, \dots, \nu_5 \in \mathbb{N}^P$.

	A	B	C	D	impl. ν_1 for E_1?	impl. ν_2 for E_1?	impl. ν_5 for E_2?
$\vec m_1$	0	$\dot c$	0	$3\dot g(\dot c)$	yes	no	no
$\vec m_2$	0	$2\dot f(\dot c)$	0	$6\dot g(\dot f(\dot c))$	yes	no	no
$\vec m_3$	$5\dot g(\dot c)$	0	$4\dot c$	0	no	yes	no
$\vec m_4$	$2\dot g(\dot c)$	0	0	$4\dot c$	no	no	yes

(b) Implementations of zeros ν_1 (w.r.t. E_1), ν_2 (w.r.t. E_1) and ν_5 (w.r.t. E_2).

Figure 4 Examples for zeros, realizations, and implementations.

is, one has to reason about infinitely many markings $\vec m$ and firing modes σ. Our approach copes with this challenge by applying symbolic techniques, that is, we finitely characterize the infinite set of all such $\vec m$ and σ conveniently for computation. Figure 3 summarizes the notions applied in our proof: We first symbolically describe the set of E-satisfying markings by means of *zeros* and their *implementations*. Then, we *derive* symbolically described firing modes from zeros, and characterize stability by means of *realizability*.

In order to simplify notation, we fix for this section an APNS (P, T), an abelian group $\mathbb{G}$, and a homogeneous P-equation E given by a simple vector $\vec k \in \mathbb{G}\langle\Theta\rangle^P$. Moreover, we assume that for each $p \in P$, $\vec k(p)$ is the monomial (κ_p, γ_p), that is, $\gamma_p = \vec k(p)(\kappa_p) \in \mathbb{G}$ is the coefficient of the only term κ_p in $\mathrm{supp}(\vec k(p))$.

Our first goal is to abstractly characterize infinite sets of E-satisfying markings by means of a *zero*. Intuitively, an E-satisfying marking assigns "right number" of a "right kind of tokens" to each place.

▸ **Definition 31** (Zero). Let $\nu : P \to \mathbb{N}$ such that $\sum_{p \in P} \nu(p)\gamma_p = 0$. If the unification problem $U = \{\kappa_p \doteq \kappa_{p'} \mid p, p' \in P, \gamma_p, \gamma_{p'}, \nu(p), \nu(p') \neq 0\}$ is solvable, ν is a *zero* of E, and we write $\underline{\nu}$ for the most general unification of U.

We observe that 0 is always a zero. Furthermore, the sum of two zeros ν_1, ν_2 yield again $\sum_{p \in P} (\nu_1(p) + \nu_2(p)) = 0$, but the unification problem is not necessarily solvable. However, a zero may be the sum of other zeros.

Figure 4a shows some examples for zeros using the net structure and equations shown in Figure 1. In this section, we ignore the place E, as it is irrelevant for enabling t. ν_1 is a zero of E_1 as $3 - 3 = 0$, and $\dot g(\mathsf{B}) \doteq \mathsf{D}$ can be unified with $D \mapsto \dot g(\mathsf{B})$. ν_2 is a zero of E_1 as $20 - 20 = 0$ and $\mathsf{A} \mapsto \dot g(\mathsf{C})$ unifies $\dot f(\mathsf{A}) \doteq \dot f(\dot g(\mathsf{C}))$. For ν_4 and E_1 we have $4 + 3 - 5 - 2 = 0$, but it is not a zero of E_1 as $\dot f(\mathsf{A}) \doteq \dot g(\mathsf{B})$ cannot be unified. ν_5 is not a zero for E_1 as $8 - 4 \neq 0$. Regarding E_2, ν_1 and ν_2 aren't zeros as $6 \not\equiv_7 0$ and $15 \not\equiv_7 0$. ν_4 is a zero for E_2 as $3 + 4 \equiv_7 0$ and $\mathsf{D} \mapsto \dot c$ unifies $\dot c \doteq \mathsf{D}$. Finally, ν_5 is also a zero of E_2, as $6 + 8 \equiv_7 = 0$ and as for ν_4 the unification problem is solvable as for ν_4.

Because $\underline{\nu}$ is a unifier, applying $\underline{\nu}$ to κ_p yields the same result for every $p \in P$ satisfying $\gamma_p \neq 0_{\mathbb{G}}$ and $\nu(p) \neq 0$.

▸ **Lemma 32.** *Let ν be a zero. The set $\{\kappa_p \underline{\nu} \mid p \in P, \gamma_p \neq 0_{\mathbb{G}}, \nu(p) \neq 0\}$ is singleton.*

▸ **Definition 33** (Result of the unification)**.** We define $\varrho(\nu)$ by $\{\varrho(\nu)\} = \{\kappa_p \underline{\nu} \mid p \in P, \gamma_p \neq 0_{\mathbb{G}}, \nu(p) \neq 0\}$

Intuitively, an *implementation* of a zero ν is a marking which satisfies E "in the same way" as ν. Formally, we define this based on an assignment transforming the result of the unification to a marking.

▸ **Definition 34** (Implementation of a zero)**.** Let $\vec{m} \in \mathbb{Z}\langle\Theta_\varnothing\rangle^P_{\geq 0}$ be a marking and ν be a zero for E. Then, $\vec{m}$ *implements* ν, or: $\vec{m}$ is an *implementation* of ν, if for all $p \in P$ with $\nu(p) \neq 0$ and $\gamma_p \neq 0_{\mathbb{G}}$:

1. $\nu(p) = \sum_{\theta \in \mathrm{supp}(\vec{m}(p))} \vec{m}(p)(\theta)$, and
2. there exists an assignment σ, such that $\{[\![\varrho(\nu)]\!]_\sigma\} = \mathrm{supp}(\vec{k}(p) \odot \vec{m}(p))$.

As an example, in Figure 4b, the marking $\vec{m}_1$ implements ν_1 for E_1 as for assignment σ_1 with $\sigma_1(\mathsf{B}) = \dot{c}$ we have $[\![B]\!]_{\sigma_1} = \dot{c} = \mathsf{D} \odot \dot{g}(\dot{c})$. $\vec{m}_2$ implements ν_1 for E_1, because for assignment σ_2 with $\sigma_2(\mathsf{B}) = \dot{f}(\dot{c})$, we have $[\![\mathsf{B}]\!]_{\sigma_2} = \mathsf{D} \odot \dot{f}(\dot{g}(\dot{c}))$. $\vec{m}_3$ implements ν_2 for E_1, because for assignment σ_3 with $\sigma_3(\mathsf{C}) = \dot{c}$, we have $[\![\dot{f}(\dot{g}(\mathsf{C}))]\!]_{\sigma_2} = \dot{f}(\dot{g}(\mathsf{C})) = \dot{f}(A) \odot \dot{g}(\dot{c}) = \dot{f}(\dot{g}(\mathsf{C})) \odot \dot{c}$. Moreover, $\vec{m}_4$ implements ν_5 for E_2 as for assignment σ_4 with $\sigma_4(\mathsf{D}) = \dot{c}$ we have $[\![\dot{c}]\!]_{\sigma_4} = \dot{c} \odot \dot{g}(\dot{c}) = \mathsf{D} \odot \dot{c}$.

Next, we show that the set of all zeros exactly characterizes the set of all E-satisfying markings: For every term ω used by an E-satisfying marking $\vec{m}$ we can identify an implementation $\vec{m}_\omega$ of a zero. Because the set of E-satisfying markings is closed under addition, the converse also holds.

▸ **Lemma 35.** *Let $\vec{m}$ be a marking, the following are equivalent:*

1. $\vec{k} \odot \vec{m} = 0_{\mathbb{G}}$.
2. *There exist zeros $\nu_1, \ldots, \nu_n$ of E, and markings $\vec{m}_1, \ldots, \vec{m}_n$, such that: $\vec{m} = \sum_{1 \leq i \leq n} \vec{m}_i$ and $\vec{m}_i$ implements ν_i for all $i = 1, \ldots, n$.*

Our next goal is to abstractly describe sets of firing modes *derivable* from a set of zeros. Formally, we describe such a set of derived firing modes by a substitution, abstractly describing a way of enabling a transition.

▸ **Definition 36** (Derivable)**.** Let $t \in T$. Let $\mathcal{S}$ be a set of zeros. For every $q \in \mathrm{pre}(t)$ let $X_q \in \mathrm{VAR}$ be a fresh variable, such that X_q does not occur in E or t and $X_q = X_{q'}$ implies $q = q'$. Let $\nu_q \in S$ be a zero with $\nu_q(q) \geq 1$. Let $U = \{\varrho(\nu_q) \odot X_q \doteq \kappa_q \odot \theta_{q,t} \mid q \in \mathrm{pre}(t)\}$, where $\{\theta_{q,t}\} = \mathrm{supp}(\vec{t}^-(q))$. Let U be solvable by most general unification δ. Then, δ is *derivable* from $\mathcal{S}$.

In the example of Figure 5, we can derive δ_1 for E_1 with $\nu_{\mathsf{A}} = \nu_{\mathsf{C}} = \nu_4$ and $\nu_{\mathsf{B}} = \nu_{\mathsf{D}} = \nu_1$. For E_2, we can derive δ_2 with $\nu_{\mathsf{A}} = \nu_{\mathsf{B}} = \nu_{\mathsf{C}} = \nu_{\mathsf{D}} = \nu_5$.

A *realization* is an assignment which refines a derivable substitution:

▸ **Definition 37** (Realization)**.** Let $\mathcal{S}$ be a set of zeros and δ be derivable from $\mathcal{S}$. Then, σ is a *realization* of δ, if there exists an assignment σ' with $\sigma(X) = [\![\delta(X)]\!]_{\sigma'}$ for all $X \in \mathrm{VAR}$.

The assignment σ_1 shown in Figure 5 is a realization of δ_1. The assignment σ with $\sigma(X_{\mathsf{C}}) = \sigma(X_{\mathsf{B}}) = \dot{c}$ gives $\sigma_1(\mathsf{A}) = [\![X_{\mathsf{A}}]\!]_\sigma = \dot{c}$, $\sigma_1(\mathsf{B}) = [\![X_{\mathsf{B}}]\!]_\sigma = \dot{c}$ and $\sigma_1(\mathsf{C}) = [\![\dot{g}(X_{\mathsf{B}})]\!]_\sigma = \dot{g}(\dot{c})$.

	W	Y	Z	Derivable from some E_j?	$\vec{k}_j \odot (\vec{t}^{\Delta}\delta_i)$
δ_1	X_{C}	X_{B}	$\dot{g}(X_{\mathsf{B}})$	yes, for $j = 1$	$-\dot{f}(\dot{g}(X_{\mathsf{C}})) + \dot{g}(X_{\mathsf{B}})$ $(j = 1)$
δ_2	X_{C}	X_{B}	$\dot{c}$	yes, for $j = 2$	0 $(j = 2)$

	W	Y	Z	Realization of	$\vec{k}_1 \odot [\![\vec{t}^{\Delta}]\!]_{\sigma_1}$
σ_1	$\dot{c}$	$\dot{c}$	$\dot{g}(\dot{c})$	δ_1	$-\dot{f}(\dot{g}(\dot{c})) + \dot{g}(\dot{c})$

Figure 5 Derivable substitutions δ_1 and δ_2, and a realization σ_1 of δ_1.

Next, we show that the derived substitutions from the set of all zeros exactly characterize the set of E-satisfying, t-enabling markings: If an E-satisfying marking $\vec{m}$ enables t in firing mode σ, then σ is a realization of some derivable substitution, and vice versa:

▸ **Lemma 38.** *Let $\mathcal{S}$ be the set of all zeros and σ be an assignment. Then, the following two statements are equivalent:*

1. *There exists a marking $\vec{m}$ with: $\vec{m} \geq [\![\vec{t}^{-}]\!]_{\sigma}$ and $\vec{k} \odot \vec{m} = 0_{\mathbb{G}}$.*
2. *There exists a δ that is derivable from $\mathcal{S}$ and σ is a realization of δ.*

A derivable substitution δ generally has infinitely many realizations. We show that the choice of the realization does not matter for deciding stability.

▸ **Lemma 39.** *Let $\mathcal{S}$ be a set of zeros and δ be derivable from $\mathcal{S}$. Then, the following two statements are equivalent:*

1. *$\vec{k} \odot (\vec{t}^{\Delta}\delta) = 0$*
2. *$\vec{k} \odot [\![\vec{t}^{\Delta}]\!]_{\sigma} = 0$ for all σ that are realizations of δ.*

Our proof of "2.⇒1." utilizes the existence of a realization σ preserving the distinctness of terms in $\vec{k} \odot \vec{t}^{\Delta}$, that is, if two terms θ_1, θ_2 occur in $\vec{k} \odot \vec{t}^{\Delta}$ with $\theta_1\delta \neq \theta_2\delta$, then $[\![\theta_1]\!]_{\sigma} \not\equiv [\![\theta_2]\!]_{\sigma}$.

Now, we prove that t-stability can be characterized by the set of all derivable substitutions:

▸ **Lemma 40.** *Let $\mathcal{S}$ be the set of all zeros. The following are equivalent:*

1. *E is t-stable.*
2. *For all δ derivable from $\mathcal{S}$ holds: $\vec{k} \odot (\vec{t}^{\Delta}\delta) = 0$.*

In the example shown in Figure 1, E_1 is not stable. Consider the marking $\vec{m}_5 := \vec{m}_1 + \vec{m}_2 + \vec{m}_3$. There, t is enabled. But, for the firing mode σ_1, we have $\vec{k}_1 \odot \sigma_1 \not\equiv 0$. On the other hand, E_2 is stable, although we have $\vec{k}_2 \odot \vec{t}^{\Delta} \not\equiv 0$.

The following lemma proves a closure property for the derived substitutions: If one combines zeros from a set $\mathcal{S}$ to a new zero ν, then for every realizable substitution derivable from $\mathcal{S} \cup \{\nu\}$, there exists a realizable substitution derivable from $\mathcal{S}$.

▸ **Lemma 41.** *Let $\mathcal{S}$ be a set of zeros and $\nu \notin \mathcal{S}$ with $\nu = \sum_{i=1}^{n} \nu_i$ where $\nu_i \in \mathcal{S}$. Let δ be derivable from $\mathcal{S} \cup \{\nu\}$ and σ be assignments that realizes δ. Then, there exists δ' such that: δ' is derivable from $\mathcal{S}$ and σ realizes δ'.*

We observe that we can only derive finite sets of substitutions from finite sets of zeros.

▸ **Lemma 42.** *Let $\mathcal{S}$ be a finite set of zeros. The set $\{\delta : \mathrm{VAR} \to \Theta \mid \delta$ is derivable from $S\}$ is finite and computable.*

Our next goal is to combine Lemma 41 and Lemma 42. To this end, we first define the notion of a *spanning set* of zeros: A set capable of generating all zeros by means of addition.

▸ **Definition 43** (Spanning Set). Let S be a set of zeros of E, such that for each zero ν of E, there exist $\nu_1, \dots, \nu_n \in S$, with $\nu(p) = \sum_{i=1}^{n} \nu_i(p)$ for all $p \in P$. Then, S is a *spanning set* (of zeros) of E.

Now, we show that given a *finite* spanning set of zeros, we can decide t-stability.

▸ **Lemma 44.** *Given a finite spanning set $\mathcal{S}$ of zeros, t-stability of E is decidable.*

Proof. By Lemma 41, for every δ that is derivable from the set of zeros, there exists a δ' derivable from $\mathcal{S}$. By Lemma 42, the set of all these δ' is finite and computable. By Lemma 40, E is stable if and only if for every δ' we have $\vec{k} \odot \vec{t}^{\Delta} \delta' = 0$, which is computable. ◂

The last step in our proof of Theorem 23 is showing that a finite spanning set of zeros can be computed if $\mathbb{G}$ is cyclic. For infinite cyclic groups, we apply that there exists a computable isomorphism into the integers. As a prerequisite, we observe that every spanning set contains every *indecomposable zero*, i.e., a zero which cannot be written as a sum of other zeros. For example, consider the zeros ν_1, ν_2 and ν_3 from Figure 4a: ν_1 and ν_2 are indecomposable, but $\nu_3 = \nu_1 + \nu_1$ is not. Thus, we show that there exists an upper bound for the coefficients of indecomposable zeros. To this end, we first show an auxiliary lemma, based on the maximum coefficient $\overline{\gamma}$, and the absolute value $\underline{\gamma}$ of the minimal coefficient in γ. In the example equation E_1 from Figure 1, we have $\overline{\gamma} = 4$ and $\underline{\gamma} = 5$. Intuitively, if the maximum constituent in a zero ν over places with negative (resp. positive) coefficients is less than $\overline{\gamma}$ (resp. $\underline{\gamma}$), then the sum of the constituents in ν is bounded by $2|P|\overline{\gamma}\underline{\gamma}$. For E_1, the upper bound is $2 \cdot 5 \cdot 4 \cdot 5 = 200$.

▸ **Lemma 45.** *Let $\nu \in \mathbb{N}^P$. Let $\eta \in \mathbb{Z}^P$ be mixed with $\sum_{p \in P} \nu(p) \cdot \eta(p) = 0$. Let $\overline{\eta} := \max\{\eta(p) \mid p \in P\}$ and $\underline{\eta} := \max\{|\eta(p)| \mid \eta(p) < 0, p \in P\}$ with:*
1. $\max\{\nu(p) \mid \eta(p) < 0, p \in P\} < \overline{\eta}$
2. *or* $\max\{\nu(p) \mid \eta(p) > 0, p \in P\} < \underline{\eta}$.
Then, $\sum_{p \in P} \nu(p) < 2|P|\overline{\eta}\underline{\eta}$

Finally, we show the computability of a finite spanning set of zeros. To this end, we utilize Lemma 45 to show that the sum of constituents of each indecomposable zero is bounded by $2|P|\overline{\gamma}\underline{\gamma}$: We assume a zero ν with $\sum_{p \in P} \nu(p) \geq 2|P|\overline{\gamma}\underline{\gamma}$, and show that ν decomposes into two zeros $\hat{\nu}$ and $\nu - \hat{\nu}$. Thus, extracting all zeros from the finite set of all $\nu \in \mathbb{N}^P$ with $\sum_{p \in P} \nu(p) < 2|P|\overline{\gamma}\underline{\gamma}$ yields a set of zeros containing all indecomposable zeros, and hence a finite spanning set.

▸ **Lemma 46.** *If $\mathbb{G}$ is cyclic, a finite spanning set $\mathcal{S}$ of zeros is computable.*

Proof. Assume $\vec{k}$ is semi-positive or semi-negative, then 0 is the only zero. In the following, we assume $\vec{k}$ to have mixed coefficients. We distinguish the cases whether $\mathbb{G}$ is finite or infinite.

- First case: $\mathbb{G}$ is infinite. As $\mathbb{G}$ is cyclic, there exists a computable isomorphism to $\mathbb{Z}$ (see for instance [17]). Thus, we assume w.l.o.g that $\mathbb{G} = \mathbb{Z}$. Let $\overline{\gamma} := \max\{\gamma(p) \mid p \in P\}$ and $\underline{\gamma} := \max\{|\gamma(p)| \mid \gamma(p) < 0, p \in P\}$. Let ν be a zero with $\sum_{p \in P} \nu(p) > 2|P|\underline{\gamma}\overline{\gamma}$ (*). We show that that then, there exist $\underline{p}, \overline{p} \in P$ with: $\gamma_{\overline{p}} > 0 \wedge \gamma_{\underline{p}} < 0 \wedge \nu(\overline{p}) \geq |\gamma_{\underline{p}}| \wedge \nu(\underline{p}) \geq \gamma_{\overline{p}}$. Assume the opposite: Then, $\max\{\nu(p) \mid \gamma_p < 0, p \in P\} < \overline{\gamma}$ or $\max\{\nu(p) \mid \gamma_p > 0, p \in P\} < \underline{\gamma}$. By Lemma 45, then $\sum_{p \in P} \nu(p) < 2|P|\overline{\gamma}\underline{\gamma}$, which contradicts (*).
  Now, let $\hat{\nu} : P \to \mathbb{N}$ with:
  $$\hat{\nu}(p) = \begin{cases} |\gamma_{\underline{p}}| & \text{if } p = \overline{p} \\ \gamma_{\overline{p}} & \text{if } p = \underline{p} \\ 0 & \text{otherwise} \end{cases}$$

By definition, we have $\hat{\nu} \leq \nu$, moreover as $\sum_{p \in P} \hat{\nu}(p) \leq \underline{\gamma}\overline{\gamma} < \sum_{p \in P} \nu(p)$, we have $\hat{\nu} < \nu$. Let $\nu' = \nu - \hat{\nu}$. Then, $\nu' : P \to \mathbb{N}$ and $\nu' > 0$.

Now we show that $\hat{\nu}$ and ν' are zeros. For $\hat{\nu}$ we have $\sum_{p \in P} \nu(p) = |\gamma_{\underline{p}}|\gamma_{\overline{p}} + \gamma_{\underline{p}}\gamma_{\overline{p}} = -\gamma_{\underline{p}}\gamma_{\overline{p}} + \gamma_{\underline{p}}\gamma_{\overline{p}} = 0$ and accordingly $0 = \sum p \in P\nu(p) = \sum p_{\in}P\hat{\nu}(p) + \sum_{p \in P} \nu'(p) = 0 + \sum_{p \in P} \nu'(p)$. It remains to show that the unification problems of $\hat{\nu}$ and ν' are solvable. We observe $\hat{\nu} \leq \nu$ ($\nu' \leq \nu$) implies that unification problem of $\hat{\nu}$ (ν') is a subset of the unification problem of ν. Thus, ν is a sum of the zeros ν' and $\hat{\nu}$.

Now, we see that $\sum_{p \in P} \hat{\nu}(p) < 2|P|\overline{\gamma}\underline{\gamma}$. Assume additionally $\sum_{p \in P} \nu'(p) \leq 2|P|\overline{\gamma}\underline{\gamma}$, then we can continue. Otherwise, if $\sum_{p \in P} \nu'(p) > 2|P|\overline{\gamma}\underline{\gamma}$, we can apply induction, as $\nu' < \nu$, Hence, ν is the sum of other zeros $\nu_1, \ldots, \nu_n$, where for each $1 \leq i \leq n$: $\sum_{p \in P} \nu_i(p) \leq 2|P|\underline{\gamma}\overline{\gamma}$. Finally, $\{\nu \in \mathbb{N}^p \mid \sum_{p \in P} |\nu(p)| \leq |2P|\overline{\gamma}\underline{\gamma}$ and ν is zero$\}$ is finite, spanning and computable.

- Second Case: Let $\mathbb{G}$ be finite with order $o \in \mathbb{N} \setminus \{0\}$. As $\mathbb{G}$ is cyclic, there exists the generator $e \in \mathbb{G}$. Let $g \in \mathbb{G}$. Then, it holds that $g + oe = g$. Thus, for every $\nu : P \to \mathbb{N}$, and $p \in P$ with $\nu(p) > o$, we have $\nu(p)\gamma_p = (\nu(p) - o)\gamma_p$. Hence, for every zero ν we can find a zero ν' with $\nu'(p) \leq o$ and $\sum_{p \in P} \gamma_p \nu(p) = \sum p \in P\gamma_p\nu'(p)$. Therefore, $\{\nu \in \mathbb{N}^p \mid \nu(p) \leq o$ and ν is zero$\}$ is finite, spanning and computable. ◀

6 Related Work

APNs or similar "high level net"-formalisms are an established, expressive modeling language for distributed systems[11, 2]. Moreover, tools for Colored Petri Nets support simulation and (partial) verification [7, 8]. The idea to prove stable properties in Petri nets that use distinguishable tokens has been pursued at least since the early 80s [5]. Ever since, the class of invariants became a substantial part of Petri Net analysis [9, 2, 11]. Other stable properties for Algebraic Petri Nets have been studied in the context of Traps/Co-Traps [15]. In elementary Petri Nets (P/T-Nets), stable properties such as traps and co-traps have been studied [11] and been shown as useful for verification [11, 4]. Compared to this, the number of publications regarding stable properties in APNs is comparatively small. In the last years, Petri Net variants with distinguishable tokens gained more attention to model data in distributed systems and applying analytic methods such as [3, 6, 13].

The concept of stability has been used in other areas of research; the most similar maybe being abstract interpretation as a technique for verification of iterative programs [1]. In the context of data-aware business processes, stability has been used in a similar context, following a graph-oriented approach focusing on data modeling [14].

7 Concluding Remarks

Throughout this paper, we applied three restrictions: First, we only considered the interpretation of terms in the Herbrand structure, second, we only considered homogeneous P-equations, and third, we required for the decidability proof that the group of coefficients is cyclic.

If one chooses another structure for the interpretation of terms than the Herbrand structure, one can observe that validity and stability are preserved in one direction: If a P-equation is valid (stable) w.r.t. the Herbrand structure, then it is valid (stable) w.r.t. every generated structure. Because the Herbrand structure is a specific structure, the undecidability result (Theorem 22) could be generalized by allowing an arbitrary, but not fixed, structure. For the decidability result (Theorem 23), we observe that we can use our decision procedure as a sufficient but not necessary criterion for an arbitrary fixed structure.

The restriction to homogeneous P-equations yields that satisfying markings are closed under addition, which is not the case if one allowed arbitrary constants on the right hand side. Here, our approach of finding a finite spanning set symbolically describing all satisfying markings does not work. The main challenge for generalizing our approach is that markings have natural numbers as coefficients (in contrast to integers).

For our decidability result, we require that the coefficients stem from a cyclic group. Here, we explicitly exploit in the proofs that there exist a distinct generator element, and an isomorphism to the integers, or the integers modulo some natural number n.

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