Graph Minors for Preserving Terminal Distances Approximately – Lower and Upper Bounds*†

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Abstract

Given a graph where vertices are partitioned into k terminals and non-terminals, the goal is to compress the graph (i.e., reduce the number of non-terminals) using minor operations while preserving terminal distances approximately. The distortion of a compressed graph is the maximum multiplicative blow-up of distances between all pairs of terminals. We study the trade-off between the number of non-terminals and the distortion. This problem generalizes the Steiner Point Removal (SPR) problem, in which all non-terminals must be removed.

We introduce a novel black-box reduction to convert any lower bound on distortion for the SPR problem into a super-linear lower bound on the number of non-terminals, with the same distortion, for our problem. This allows us to show that there exist graphs such that every minor with distortion less than 2 / 2.5 / 3 must have $\Omega(k^2)$ / $\Omega(k^{5/4})$ / $\Omega(k^{6/5})$ non-terminals, plus more trade-offs in between. The black-box reduction has an interesting consequence: if the tight lower bound on distortion for the SPR problem is super-constant, then allowing any $\mathcal{O}(k)$ non-terminals will *not* help improving the lower bound to a constant.

We also build on the existing results on spanners, distance oracles and connected 0-extensions to show a number of upper bounds for general graphs, planar graphs, graphs that exclude a fixed minor and bounded treewidth graphs. Among others, we show that any graph admits a minor with $\mathcal{O}(\log k)$ distortion and $\mathcal{O}(k^2)$ non-terminals, and any planar graph admits a minor with $1+\varepsilon$ distortion and $\widetilde{\mathcal{O}}((k/\varepsilon)^2)$ non-terminals.

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1 Introduction

Graph compression generally describes a transformation of a *large* graph G into a *smaller* graph G' that preserves, either exactly or approximately, certain features (e.g., distance, cut, flow) of G. Its algorithmic value is apparent, since the compressed graph can be computed in

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a preprocessing step of an algorithm, so as to reduce subsequent running time and memory. Some notable examples are graph spanners, distance oracles and cut/flow sparsifiers.

In this paper, we study compression using minor operations, which has attracted increasing attention in recent years. Minor operations include vertex/edge deletions and edge contractions. It is naturally motivated since it preserves certain structural properties of the original graph, e.g., any minor of a planar graph remains planar, while reducing the size of the graph. We are interested in *vertex sparsification*, where G has a designated subset T of k vertices called the *terminals*, and the goal is to reduce the number of non-terminals in G' while preserving some feature among the terminals. Recent work in this field studied preserving cuts and flows. Our focus here is on preserving terminal distances approximately in a multiplicative sense, i.e., we want that for any terminals t, t', $d_G(t,t') \leq d_{G'}(t,t') \leq \alpha \cdot d_G(t,t')$, for a small *distortion* α. This problem, called *Approximate Terminal Distance Preservation (ATDP) problem*, has natural applications in multicast routing [6] and network traffic optimization [22]. It was also suggested in [15] that to solve the *subset travelling salesman problem*, one can compute a compressed minor with a small distortion as a preprocessing step for algorithms that solve the travelling salesman problem for planar graphs.

ATDP was initiated by Gupta [11], who introduced the related *Steiner Point Removal (SPR) problem*: Given a tree G with both terminals and non-terminals, output a weighted tree G' *with terminals only* which minimizes the distortion. Gupta gave an algorithm that achieves a distortion of 8. Chan et al. [4] observed that Gupta's algorithm returned always a minor of G. For general graphs, Kamma et al. [13] gave an algorithm to construct a minor with distortion $\mathcal{O}(\log^5 k)$. Krauthgamer et al. [15] studied ATDP and showed that every graph has a minor with $\mathcal{O}(k^4)$ non-terminals and distortion 1. It is then natural to ask, for different classes of graphs, what the trade-off between the distortion and the number of non-terminals is. In this paper, for different classes of graphs, and w.r.t. different allowed distortions, we provide lower and upper bounds on the number of non-terminals needed.

Further Related Work. Basu and Gupta [3] showed that for outer-planar graphs, SPR can be solved with distortion $\mathcal{O}(1)$. When randomization is allowed, Englert et al. [9] showed that for graphs that exclude a fixed minor, one can construct a randomized minor for SPR with $\mathcal{O}(1)$ expected distortion. Krauthgamer et al. [15] proved that solving ATDP with distortion 1 for planar graphs needs $\Omega(k^2)$ non-terminals.

Recently, there has been a growing interest on cut/flow vertex sparsifiers [19, 16, 5, 18, 9, 7, 2, 21]; given a capacitated graph G with terminals $T \subset V$, the goal is to find a sparsifier H with $V(H) = T$ preserving all terminal cuts up to a factor $q \geq 1$, i.e. for all $S \subset T$, $\text{mincut}_G(S, T \setminus S) \leq \text{mincut}_H(S, T \setminus S) \leq q \cdot \text{mincut}_G(S, T \setminus S)$. In this setting, there is an equivalence between the construction of vertex cut/flow and distance sparsifiers [20, 9].

A related graph compression is spanners, where the objective is to reduce the number of edges by edge deletions only. We will use a spanner algorithm (e.g., [1]) to derive our upper bound results for general graphs. Although spanner operation enjoys much less freedom than minor operation, proving a lower bound result for it is notably difficult. Assuming the Erdös girth conjecture [10], there are lower bounds that match the best known upper bounds, but the conjecture seems far from being settled [25]. Woodruff [27] showed a lower bound result bypassing the conjecture, but only for *additive* spanners.

Graph	Upper Bound	Lower Bound
General	$\forall q \in \mathbb{N}$ $(2q-1, \mathcal{O}(k^{2+2/q}))$	$(2-\varepsilon, \Omega(k^2))$
General	–	$(2.5-\varepsilon, \Omega(k^{5/4}))$, $(3-\varepsilon, \Omega(k^{6/5}))$
Bounded-treewidth p	$\forall q \in \mathbb{N}$ $(2q-1, \mathcal{O}(p^{1+2/q}k))$	$(1, \Omega(pk))$ †
Excluded-Fixed-Minor	$(\mathcal{O}(1), \widetilde{\mathcal{O}}(k^2)$	–
Planar	$(3, \widetilde{\mathcal{O}}(k^2))$, $(1+\varepsilon, \widetilde{\mathcal{O}}((k/\varepsilon)^2)$	$(1+o(1), \Omega(k^2))$ †

Our Contributions. For various classes of graphs, we show lower and upper bounds on the number of non-terminals needed in the minor for low distortion. The table above summarizes our results (results with † are from [15]); see the full version for more details.

For our lower bound results, we use a novel black-box reduction to convert any lower bound on distortion for the SPR problem into a super-linear lower bound on the number of non-terminals for ATDP with the same distortion. Precisely, we show that given any graph G^* such that solving its SPR problem leads to a minimum distortion of α, we use G^* to construct a new graph G such that every minor of G with distortion less than α must have at least $\Omega(k^{1+\delta(G^*)})$ non-terminals, for some constant $\delta(G^*) > 0$. The lower bound results in the above table are obtained by using for G^* a complete ternary tree of height 2, which was shown that solving its SPR problem leads to minimum distortion 3 [11]. More trade-offs are shown by using for G^* a complete ternary tree of larger heights.

The black-box reduction has an interesting consequence. For the SPR problem on general graphs, there is a huge gap between the best known lower and upper bounds, which are 8 [4] and $\mathcal{O}(\log^5 k)$ [13]; it is unclear what the asymptotically tight bound would be. Our black-box reduction allows us to prove the following result concerning the tight bound: for general graphs, if the tight bound on distortion for the SPR problem is super-constant, then for any constant $C > 0$, even if Ck non-terminals are allowed in the minor, the lower bound will remain super-constant. See Theorem 15 for a formal statement of this result.

We also build on the existing results on spanners, distance oracles and connected 0-extensions to show a number of upper bound results for general graphs, planar graphs and graphs that exclude a fixed minor. Our techniques, combined with an algorithm in Krauthgamer et al. [15], yield an upper bound result for graphs with bounded treewidth. In particular, our upper bound on planar graphs implies that allowing quadratic number of non-terminals, we can construct a deterministic minor with arbitrarily small distortion.

2 Preliminaries

Let $G = (V, E, \ell)$ denote an undirected graph with terminal set $T \subset V$ of cardinality k, where $\ell : E \to \mathbb{R}^+$ is the length function over edges E. A graph H is a *minor* of G if H can be obtained from G by performing a sequence of vertex/edge deletions and edge contractions, but no terminal can be deleted, and no two terminals can be contracted together. In other words, all terminals in G must be *preserved* in H.

Besides the above standard description of minor operations, there is another equivalent way to construct a minor H from G [13], which will be more convenient for presenting some of our results. A partial partition of $V(G)$ is a collection of pairwise disjoint subsets of $V(G)$ (but their union can be a proper subset of $V(G)$). Let $S_1, \cdots, S_m$ be a partial partition of $V(G)$ such that (1) each induced graph $G[S_i]$ is connected, (2) each terminal belongs to exactly one of these partial partitions, and (3) no two terminals belong to the same partial partition. Contract the vertices in each S_i into one single "super-node" in H. For any vertex $u \in V(G)$, let $S(u)$ denote the partial partition that contains u; for any super-node $u \in V(H)$, let $S(u)$ denote the partial partition that is contracted into u. In H, super-nodes u_1, u_2 are adjacent *only if* there exists an edge in G with one of its endpoints in $S(u_1)$ and the other in $S(u_2)$. We denote the super-node that contains terminal t by t as well.

▶ **Definition 1.** The graph $H = (V', E', \ell')$ is an α-distance approximating minor (abbr. α-DAM) of $G = (V, E, \ell)$ if H is a minor of G and for any $t, t' \in T$, $d_G(t, t') \leq d_H(t, t') \leq \alpha \cdot d_G(t, t')$. H is an (α, y)-DAM of G if H is an α-DAM of G with at most y non-terminals.

We note that the SPR problem is equivalent to finding an $(\alpha, 0)$-DAM. One can also define a randomized version of distance approximating minor:

▶ **Definition 2.** Let π be a probability distribution over minors of $G = (V, E, \ell)$. We call π an α-randomized distance approximating minor (abbr. α-rDAM) of G if for any $t, t' \in T$, $\mathbb{E}_{H\sim\pi}[d_H(t,t')] \leq \alpha \cdot d_G(t,t')$, and for every minor H in the support of π, $d_H(t,t') \geq d_G(t,t')$. Furthermore, we call π an (α, y)-rDAM if π is an α-rDAM of G, and every minor in the support of π has at most y non-terminals.

3 Deterministic and Randomized Lower Bounds

For all the lower bound results, we use a tool in combinatorial design called *Steiner system* (or alternatively, *balanced incomplete block design*). Let $[k]$ denote the set $\{1, 2, \cdots, k\}$.

▶ **Definition 3.** Given a ground set $T = [k]$, an $(s, 2)$-Steiner system (abbr. $(s, 2)$-SS) of T is a collection of s-subsets of T, denoted by $\mathcal{T} = \{T_1, \cdots, T_r\}$, where $r = \binom{k}{2} / \binom{s}{2}$, such that every 2-subset of T is contained in *exactly* one of the s-subsets.

▶ **Lemma 4** ([26]). *For any integer $s \geq 2$, there exists an integer M_s such that for every $q \in \mathbb{N}$, the set $[M_s + qs(s-1)]$ admits an $(s, 2)$-SS.*

Our general strategy is to use the following black-box reduction, which proceeds by taking a *small* connected graph G^* as input, and it outputs a *large* graph G which contains many disjoint embeddings of G^*. Here is how it exactly proceeds:

- Let G^* be a graph with $s \geq 2$ terminals and $q \geq 1$ non-terminals. Let k be an integer, as given in Lemma 4, such that the terminal set $T = [k]$ admits an $(s, 2)$-SS $\mathcal{T}$.
- We construct $\mathcal{T}' \subseteq \mathcal{T}$ that satisfies *certain* property depending on the specific problem. For each s-set in $\mathcal{T}'$, we add q non-terminals to the s-set, which altogether form a *group*. The union of vertices in all groups is the vertex set of our graph G. We note that each terminal may appear in many groups, but each non-terminal appears in one group only.
- *Within* each of the groups, we embed G^* in the natural way.

The following two lemmas describe some basic properties of all minors of G output by the black-box above. Their proofs are deferred to the full version.

▶ **Lemma 5.** *Let H be a minor of G. Then for each edge (u_1, u_2) in H, there exists exactly one group R in G such that $S(u_1) \cap R$ and $S(u_2) \cap R$ are both non-empty.*

The above lemma permits us to legitimately define the notion R-edge: an edge (u_1, u_2) in H is an R-edge if R is the unique group that intersects both $S(u_1)$ and $S(u_2)$.

▶ **Lemma 6.** *Suppose that in a minor H of G, (u_1, u_2) is a R_1-edge and (u_2, u_3) is R_2-edge, where $R_1 \neq R_2$. Then R_1 and R_2 intersect, and $S(u_2)$ contains the terminal in $R_1 \cap R_2$.*

We will show that for any minor H with low distortion, at least one of the non-terminals in each group must be retained, and thus H must have at least $|\mathcal{T}'|$ non-terminals. We first present some of our main theorems on lower bounds and then prove them; two more theorems are given in Section 3.3.

▶ **Theorem 7.** *For infinitely many $k \in \mathbb{N}$, there exists a bipartite graph with k terminals which does not have a $(2 - \epsilon, k^2/7)$-DAM, for all $\epsilon > 0$.*

▶ **Theorem 8.** *There exists a constant $c_1 > 0$, such that for infinitely many $k \in \mathbb{N}$, there exists a quasi-bipartite graph with k terminals which does not have an $(\alpha - \epsilon, c_1 k^{\gamma})$-DAM, for all $\epsilon > 0$, where α, γ are given in the table below.*

α	2.5	3	10/3	11/3	4	4.2	4.4
γ	5/4	6/5	10/9	11/10	12/11	21/20	22/21

▶ **Theorem 9.** *For infinitely many $k \in \mathbb{N}$, there exists a bipartite graph with k terminals which does not have a $\left(2 - \epsilon, \epsilon^3 k^2/150\right)$-rDAM, for any $1 \geq \epsilon > 0$.*

3.1 Proof of Theorem 7

We start by reviewing the lower bound for SPR problem on stars due to Gupta [11].

▶ **Lemma 10.** *Let $G^* = (T \cup \{v\}, E)$ be an unweighted star with $k \geq 3$ terminals, in which v is the center of the star. Then, every edge-weighted graph only on the terminals T with fewer than $\binom{k}{2}$ edges has distortion at least 2.*

We construct G using the black-box reduction above. Let $k \in \mathbb{N}$ be such that the terminals $T = [k]$ admits a $(3, 2)$-SS, denoted by $\mathcal{T}$. Here, we set $\mathcal{T}' = \mathcal{T}$ and G^* to be the star with 3 terminals, as described in Lemma 10.

By the definition of Steiner system, the shortest path between every pair of terminal t, t' in G is unique, which is the 2-hop path within the group that contains both terminals, i.e., $d_G(t, t') = 2$ for all $t, t' \in T$. Every other simple path between t, t' must pass through an extra terminal, so the length of such simple path is at least 4.

Let H be a minor of G. Suppose that the number of non-terminals in H is less than r, then there exists a group R in which its non-terminal is not retained (which means that it is either deleted, or contracted into a terminal in that group). By Lemma 10, there exists a pair of terminals in that group such that every simple path within R (which means a path comprising of R-edges only) between the two terminals has length at least 4. And every other simple path must pass through an extra terminal (just as in G), so again it has length at least 4. Thus, the distortion of the two terminals is at least 2.

Therefore, every $(2 - \epsilon)$-DAM of G must have $r > k^2/7$ non-terminals.

3.2 Proof of Theorem 8

We will give the proof for the case $\alpha = 2.5$ here, and discuss how to generalize this proof for other distortions. We will first define the notions of *detouring graph* and *detouring cycle*, and then use them to construct the graph G that allows us to show the lower bound.

Detouring Graph and Detouring Cycle. For any $s \geq 3$, let $k \in \mathbb{N}$ be such that the terminal set $T = [k]$ admits an $(s, 2)$-SS. Let $\mathcal{T} = \{T_1, \cdots, T_r\}$ be such an $(s, 2)$-SS. A *detouring graph* has the vertex set $\mathcal{T}$. By the definition of Steiner system, $|T_i \cap T_j|$ is either zero or one. In the detouring graph, T_i is adjacent to T_j if and only if $|T_i \cap T_j| = 1$. Thus, in the detouring graph, it is legitimate to give each edge (T_i, T_j) a *terminal label*, which is the terminal in $T_i \cap T_j$. A *detouring cycle* is a cycle in the detouring graph such that no two neighbouring edges of the cycle have the same terminal label.

Suppose that two edges in the detouring graph have a common vertex, and their terminal labels are different, denoted by t, t'. Then the common vertex must be an s-set in $\mathcal{T}$ containing both t, t'. By the definition of Steiner system, the s-set is uniquely determined.

▶ **Claim 11.** *In the detouring graph, number of detouring cycles of size $\ell \geq 3$ is at most k^{ℓ}.*

Our key lemma is: for any $L \geq 3$, we can retain $\Omega_s(k^{L/(L-1)})$ vertices in the detouring graph, such that the induced graph on these vertices has *no* detouring cycle of size L or less.

▶ **Lemma 12.** *For any integer $L \geq 3$, given a detouring graph with vertex set $\mathcal{T} = \{T_1, T_2, \cdots, T_r\}$, there exists a subset $\mathcal{T}' \subset \mathcal{T}$ of cardinality $\Omega_s(k^{L/(L-1)})$ such that the induced graph on $\mathcal{T}'$ has no detouring cycle of size L or less.*

Proof. We choose the subset $\mathcal{T}'$ by the following randomized algorithm:

1. Each vertex is picked into $\mathcal{T}'$ with probability $\delta k^{-(L-2)/(L-1)}$, where $\delta = \delta(s) < 1$ is a positive constant which we will derive explicitly later.
2. While (there is a detouring cycle of size L or less in the induced graph of $\mathcal{T}'$)
   Remove a vertex in the detouring cycle from $\mathcal{T}'$

After Step 1, $\mathbb{E}[|\mathcal{T}'|] = r \cdot \delta k^{-(L-2)/(L-1)} \geq \frac{\delta}{2s(s-1)} k^{L/(L-1)}$. Using Claim 11, the expected number of detouring cycles of size L or less is at most $\sum_{\ell=3}^{L} k^{\ell} \cdot (\delta k^{-(L-2)/(L-1)})^{\ell} \leq 2\delta^3 k^{L/(L-1)}$. Thus, the expected number of vertices removed in Step 2 is at most $2\delta^3 k^{L/(L-1)}$. Now, choose $\delta = 1/\sqrt{8s(s-1)}$. By the end of the algorithm,

$$\mathbb{E}[|\mathcal{T}'|] \geq \frac{\delta}{2s(s-1)} k^{L/(L-1)} - 2\delta^3 k^{L/(L-1)} = \Omega(k^{L/(L-1)}).$$ ◀

Construction of G and the Proof. Recall the black-box reduction. Let k be an integer such that $T = [k]$ admits a $(9,2)$-SS $\mathcal{T}$. By Lemma 12, we choose $\mathcal{T}'$ to be a subset of $\mathcal{T}$ with $|\mathcal{T}'| = \Omega(k^{5/4})$, such that the induced graph on $\mathcal{T}'$ has no detouring cycle of size 5 or less. We choose G^* to be a complete ternary tree of height 2, in which the 9 leaves are the terminals. For each $T_i \in \mathcal{T}'$, we add four non-terminals to T_i, altogether forming a *group*.

The following lemma is a direct consequence that the induced graph on $\mathcal{T}'$ has no detouring cycle of size 5 or less.

▶ **Lemma 13.** *For any two terminals t, t' in the same group, let R denote the group. Then, in any minor H of G, every simple path from t to t' either comprises of R-edges only, or it comprises of edges from at least 5 groups other than R.*

Proof. Proof of Theorem 8 Let H be a $(2.5 - \epsilon)$-DAM of G, for some $\epsilon > 0$. Suppose that there exists a group such that all its non-terminals are not retained in H. By [11], there exists a pair of terminals t, t' in that group such that every simple path between t and t', which comprises of edges of that group only, has length at least $3 \cdot d_G(t, t')$.

By Lemma 13 and Lemma 6, any other simple path P between t and u passes through at least 4 other terminals, say they are t_a, t_b, t_c, t_d in the order of the direction from t to t'. We denote this path by $P := t \to t_a \to t_b \to t_c \to t_d \to t'$, by ignoring the non-terminals along the path. Between every pair of consecutive terminals in P, the length is at least 2. Thus, the length of P is at least 10. Since $d_G(t, t') \leq 4$, the length of P is at least $2.5 \cdot d_G(t, t')$.

Thus, the length of *every* simple path from t to t' in H is at least $2.5 \cdot d_G(t, t')$, a contradiction. Therefore, at least one non-terminal in each group is retained in H. As there are $\Omega(k^{5/4})$ groups, we are done. ◀

For the other results in Theorem 8, we follow the above proof almost exactly, with the following modifications. Set $s = 3^h$ for some $h \geq 2$, and set G^* to be a complete ternary tree with height h, in which the leaves are the terminals. Let α_h be a lower bound on the distortion for the SPR problem on G^*. Apply Lemma 12 with some integer $h < L \leq \lceil \alpha_h h \rceil$. Following the above proof, attaining a distortion of $\min \left\{ \frac{L}{h}, \alpha_h \right\} - \epsilon$ needs $\Omega(k^{L/(L-1)})$ non-terminals.

The last puzzle we need is the values of α_h. Chan et al. [4] proved that for complete binary trees of height h, $\lim_{h\to+\infty} \alpha_h = 8$, but they did not give explicit values of α_h. We apply their ideas to complete ternary tree of height h, to obtain explicit values for $h \le 5$, which are used to prove all the results in Theorem 8. The explicit values are $\alpha_2 = 3$, $\alpha_3 = \alpha_4 = 4$ and $\alpha_5 = 4.4$. We discuss the details for computing these values in the full version.

3.3 Full Generalization of Theorem 8, and its Interesting Consequence

Indeed, we can set G^* as *any* graph. In our above proofs we used a tree for G^* because the only known lower bounds on distortion for the SPR problem are for trees. If one can find a graph G^* (either by a mathematical proof, or by computer searches) such that its distortion for the SPR problem is at least α, applying the black-box reduction with this G^*, and reusing the above proof show that there exists a graph G with k terminals such that attaining a distortion of $\alpha - \epsilon$ needs $\Omega(k^{1+\delta(G^*)})$ non-terminals, for some $\delta(G^*) > 0$.

▶ **Theorem 14.** *Let G^* be a graph with s terminals, and the distance between any two terminals is between 1 and β. Suppose the distortion for the* SPR *problem on G^* is at least α. Then, for any positive integer $\max\{2, \lceil\beta\rceil\} \le L \le \lceil\alpha\beta\rceil$, there exists a constant $c_4 := c_4(s) > 0$, such that for infinitely many $k \in \mathbb{N}$, there exists a graph with k terminals which does not have a $\left(\min\{L/\beta, \alpha\} - \epsilon, c_4 k^{L/(L-1)}\right)$-DAM, for all $\epsilon > 0$.*

The above theorem has an interesting consequence. For the SPR problem on general graphs, the best known lower bound is 8, while the best known upper bound is $\mathcal{O}(\log^5 k)$ [13]. There is a huge gap between the two bounds, and it is not clear where the tight bound locates in between. Suppose that the tight lower bound on SPR is super-constant. Then for any positive constant α, there exists a graph G^*_α with $s(\alpha)$ terminals and some non-terminals, such that the distortion is larger than α. By Theorem 14, G^*_α can be used to construct a family of graphs with k terminals, such that to attain distortion α, the number of non-terminals needed is super-linear in k. Recall that in SPR, no non-terminal can be retained. In other words, Theorem 14 implies that: *if retaining no non-terminal will lead to a super-constant lower bound on distortion, then having the power of retaining any linear number of non-terminals will not improve the lower bound to a constant.*

Formally, we define the following generalization of SPR problem. Let LSPR_y denote the problem that for an input graph with k terminals, find a DAM with at most yk non-terminals so as to minimize the distortion; the SPR problem is equivalent to LSPR_0.

▶ **Theorem 15.** *For general graphs,* SPR *has super-constant lower bound on distortion if and only if for any constant $y \ge 0$, LSPR_y has super-constant lower bound on distortion.*

3.4 Proof of Theorem 9

In this subsection we give a lower bound for rDAM. The strategy we follow will be very similar to that of Theorem 7. In fact, one can view it as a randomized version of that proof. We start with the following lemma, which generalizes the deterministic SPR lower bound of Gupta in Lemma 10 to randomized minors.

▶ **Lemma 16.** *Let $G^* = (T \cup \{v\}, E)$ be an unweighted star with $k \ge 3$ terminals, in which v is the center of the star. Then, for every probability distribution over minors of G^* with vertex set T, there exists a terminal pair with distortion at least $2(1 - 1/k)$.*

We now continue with the construction of our input graph. For some constant $s \ge 3$ and some integer k, we construct a $(s, 2)$-SS of the terminal set T and denote it by

$\mathcal{T} = \{T_1, \ldots, T_r\}$, where $r = \binom{k}{2}/\binom{s}{2} \geq 2\binom{k}{2}/s^2$. Similarly to the proof of Theorem 7, we apply the black-box reduction with $\mathcal{T}' = \mathcal{T}$, and set G^* as a star with c_1 terminals, to generate a bipartite graph G. For any constant $c_1 > 0$, we define the family of minors $\mathcal{L} := \{H : H \text{ is a minor of } G \text{ and } |V(H)| < \binom{k}{2}/c_1\}$.

▶ **Claim 17.** *Let π be any probability distribution over $\mathcal{L}$. There exists a non-terminal of G that is involved in an edge contraction with probability at least $1 - s^2/2c_1$ under π.*

Proof. Proof of Theorem 9 Let v be the non-terminal from Claim 17 and let T_i be its corresponding set of size c_1. Invoking Lemma 16 and using conditional expectations, we get that there exists a terminal pair $(t, t') \in T_i$ such that

$$\begin{aligned}\frac{\mathbb{E}_\pi[d_H(t,t')]}{d_G(t,t')} &\geq \frac{\mathbb{E}_\pi\left[d_H(t,t') \mid v \text{ is contracted}\right] \cdot \mathbb{P}_\pi\left[v \text{ is contracted}\right]}{d_G(t,t')} \\ &\geq 2\left(1 - \frac{1}{s}\right)\left(1 - \frac{s^2}{2c_1}\right) \;\geq\; 2 - \left(\frac{2}{s} + \frac{s^2}{c_1}\right),\end{aligned}$$

which can be made arbitrarily close to 2 by setting s and c_1 sufficiently large. To be precise, given any $\epsilon > 0$, by setting $s = 5/\epsilon$ and $c_1 = 2s^2/\epsilon$, the distortion is at least $2 - \epsilon$. ◀

4 Minor Construction for General Graphs

In this section we give minor constructions that present numerous trade-offs between the distortion and size of DAMs. Our results are obtained by combining the work of Coppersmith and Elkin [8] on sourcewise distance preservers with the well-known notion of spanners.

Given an undirected graph $G = (V, E, \ell)$ with terminals T, we let $\Pi_{u,v}$ denote the shortest path between u and v in G. Without loss of generality, we assume that for any pair of vertices (u, v), the shortest path connecting u and v is *unique*. This can be achieved by slightly perturbing the original edge lengths of G (see [8]).

For a graph G, let $N_G(u)$ denote the vertices incident to u in G. We say that two paths Π and Π' branch at a vertex $u \in V(\Pi) \cap V(\Pi')$ iff $|N_{\Pi \cup \Pi'}(u)| > 2$. We call such a vertex u a *branching* vertex. Let $\mathcal{P}$ denote the set of shortest paths corresponding to every pair of vertices in G. We review the following result proved in [8, Lemma 7.5].

▶ **Lemma 18.** *Any pair of shortest paths $\Pi, \Pi' \in \mathcal{P}$ has at most two branching vertices.*

▶ **Definition 19** (Terminal Path Cover). Given $G = (V, E, \ell)$ with terminals T, a set of shortest paths $\mathcal{P}' \subset \mathcal{P}$ is an $(\alpha, f(k))$-terminal path cover (abbr. $(\alpha, f(k))$-TPc) of G if

1. $\mathcal{P}'$ covers the terminals, i.e. $T \subseteq V(H)$, where $H = \bigcup_{\Pi \in \mathcal{P}'} E(\Pi)$,
2. $|\mathcal{P}'| \leq f(k)$ and $\forall t, t' \in T$, $d_G(t,t') \leq d_H(t,t') \leq \alpha \cdot d_G(t,t')$.

We remark that the endpoints of the shortest paths in $\mathcal{P}'$ are not necessarily terminals. Now we give a simple algorithm generalizing the one presented by Krauthgamer et al. [15].

Algorithm 1 MinorSparsifier (graph G, terminals T, $(\alpha, f(k))$-TPc $\mathcal{P}'$ of G)

```
Set H = ∅. Then add all shortest paths from the path cover P′ to H.
while there exists a degree two non-terminal v incident to edges (v,u) and (v,w) do
    Contract the edge (u,v), then set the length of edge (u,w) to d_H(u,w).
return H
```

▶ **Lemma 20.** *For a given graph $G = (V, E, \ell)$ with terminals $T \subset V$ and an $(\alpha, f(k))$-TPc $\mathcal{P}'$ of G,* MinorSparsifier*$(G$,T,$\mathcal{P}')$ outputs an $(\alpha, f(k)^2)$-DAM of G.*

A trivial *exact* terminal path cover for any k-terminal graph is to take the union of all terminal shortest paths, which we refer to as the $(1, \mathcal{O}(k^2))$-TPc $\mathcal{P}'$ of G. Krauthgamer et al. [15] used this $(1, \mathcal{O}(k^2))$-TPc to construct an $(1, \mathcal{O}(k^4))$-DAM. Here, we study the question of whether increasing the distortion slightly allows us to obtain a cover of size $o(k^2)$. We answer this question positively, by reducing it to the well-known spanner problem. Let $q \geq 1$ be an integer and let $G = (V, E, \ell)$ be an undirected graph. A q-spanner of G is a subgraph $S = (V, E_S, \ell)$ such that $\forall u, v \in V,\ d_G(u, v) \leq d_S(u, v) \leq q \cdot d_G(u, v)$. We refer to q and $|E_S|$ as the *stretch* and *size* of spanner S, respectively.

▶ **Lemma 21** ([1])**.** *Let $q \geq 1$ be an integer. Any graph $G = (V, E, \ell)$ admits a $(2q-1)$-spanner S of size $\mathcal{O}(|V|^{1+1/q})$.*

Given a graph $G = (V, E, \ell)$ with terminals T, we compute the complete graph $Q_T = (T, \binom{T}{2}, d_G|T)$, where $d_G|T$ denotes the distance metric of G restricted to the point set T (In other words, for any pair of terminals $t, t' \in T$, the weight of the edge connecting them in Q_T is given by $w_{Q_T}(t, t') = d_G(t, t')$). Recall that all shortest paths in G are unique.

Using Lemma 21, we construct a $(2q - 1)$-spanner S of size $\mathcal{O}(k^{1+1/q})$ for Q_T. Observe that each edge of S corresponds to an unique (terminal) shortest path in G since S is a subgraph of Q_T. Thus, the set of shortest paths corresponding to edges of S form a $(2q - 1, \mathcal{O}(k^{1+1/q}))$-TPc $\mathcal{P}'$ of G. Using $\mathcal{P}'$ with Lemma 20 gives the following theorem.

▶ **Theorem 22.** *Let $q \geq 1$ an integer. Any graph $G = (V, E, \ell)$ with $T \subset V$ admits a $(2q - 1, \mathcal{O}(k^{2+2/q}))$-DAM.*

We mention two trade-offs from the above theorem. When $q = 2$, we get an $(3, \mathcal{O}(k^3))$-DAM. When $q = \log k$, we get an $(\mathcal{O}(\log k), \mathcal{O}(k^2))$-DAM. These are new distortion-size trade-offs.

The above method allows us to have improved guarantees for bounded treewidth graphs.

▶ **Theorem 23.** *Let $q \geq 1$ be an integer. Any graph $G = (V, E, \ell)$ with treewidth at most p, $T \subset V$ and $k \geq p$ admits a $(2q - 1, \mathcal{O}(p^{1+2/q}k))$-DAM.*

5 Minor Construction for Graphs that Exclude a Fixed Minor

In this section we give improved guarantees for distance approximating minors for special families of graphs. Specifically, we show that graphs that exclude a fixed minor admit an $(\mathcal{O}(1), \widetilde{\mathcal{O}}(k^2))$-DAM. This family of graphs includes, among others, the planar graphs.

The reduction to spanner in Section 4 does not consider the structure of Q_T, which is inherited from the input graph. We exploit this structure, together with the use of the randomized Steiner Point Removal Problem, which is equivalent to finding an $(\alpha, 0)$-rDAM.

We will make use the following theorem due to Englert et al. [9, Theorem 14].

▶ **Theorem 24** ([9])**.** *Let $\alpha = \mathcal{O}(1)$. Given a graph that excludes a fixed minor $G = (V, E, \ell)$ with $T \subset V$, there is a probability distribution π over its minors $H = (T, E', \ell')$, such that $\forall\, t, t' \in T,\ \mathbb{E}_{H \sim \pi}[d_H(t, t')] \leq \alpha \cdot d_G(t, t')$ and for all H in support of π, $d_H(t, t') \geq d_G(t, t')$.*

Given a graph G that excludes a fixed minor, any minor H of G only on the terminals also excludes the fixed minor. Thus H has $\mathcal{O}(k)$ edges [23]. This leads to the corollary below.

▶ **Corollary 25.** *Let $\alpha = \mathcal{O}(1)$. Given a graph that excludes a fixed minor $G = (V, E, \ell)$ with $T \subset V$ and Q_T as defined in Section 4, there exists a probability distribution π over subgraphs $H = (T, E', \ell')$ of Q_T, each having at most $\mathcal{O}(k)$ edges, such that for all $t, t' \in T,\ \mathbb{E}_{H \sim \pi}[d_H(t, t')] \leq \alpha \cdot d_{Q_T}(t, t')$.*

Proof. Let π be the distribution over minors of G from Theorem 24, then every minor in its support is clearly a subgraph of Q_T with $\mathcal{O}(k)$ edges. Since during the construction of these minors we may assume that $\forall (t,t') \in E',\ \ell'(t,t') = d_G(t,t')$, the corollary follows. ◀

▶ **Lemma 26.** *Given a graph that excludes a fixed minor $G = (V, E, \ell_G)$ with $T \subset V$, and Q_T as defined in Section 4, there exists an $\mathcal{O}(1)$-spanner S of size $\mathcal{O}(k \log k)$ for Q_T.*

Proof. Let π be the probability distribution over subgraphs H from Corollary 25. Set $S = \emptyset$. First, we sample independently $q = 3 \log k$ subgraphs $H_1, \ldots, H_q$ from π. We then add the edges from all these subgraphs to the graph S, i.e., $E_S = \bigcup_{i=1}^{q} E_{H_i}$. Fix an edge (t,t') from Q_T and a subgraph H_i. By Corollary 25 and the Markov inequality, $\mathbb{P}[d_{H_i}(t,t') \geq 2\alpha \cdot d_{Q_T}(t,t')] \leq 2^{-1}$, and hence

$$\mathbb{P}[d_S(t,t') \geq 2\alpha \cdot d_{Q_T}(t,t')] = \prod_{i=1}^{q} \mathbb{P}[d_{H_i}(t,t') \geq 2\alpha \cdot d_{Q_T}(t,t')] \leq 2^{-q} = k^{-3}.$$

Applying union bound overall all edges from Q_T yields

$$\mathbb{P}[\text{there exists an edge } (t,t') \in Q_T \text{ s.t. } d_S(t,t') \geq 2\alpha \cdot d_{Q_T}(t,t')] \leq k^2 \cdot k^{-3} = k^{-1}.$$

Hence, for all edges (t,t') from Q_T, with probability at least $1 - 1/k$, we preserve the shortest path distance between t and t' up to a factor of $2\alpha = \mathcal{O}(1)$ in S. Since S is a subgraph of Q_T, this implies that there exists a $\mathcal{O}(1)$-spanner S of size $\mathcal{O}(k \log k)$ for Q_T. ◀

Similar to the last section, the set of shortest paths corresponding to edges of S is an $(\mathcal{O}(1), \mathcal{O}(k \log k))$-TPc $\mathcal{P}'$ of G. Using $\mathcal{P}'$ with Lemma 20 gives the following theorem.

▶ **Theorem 27.** *Any graph that excludes a fixed minor $G = (V, E, \ell)$ with $T \subset V$ admits an $(\mathcal{O}(1), \widetilde{\mathcal{O}}(k^2))$-DAM.*

6 Minor Construction for Planar Graphs

In this section, we show that for planar graphs, we can improve the constant guarantee bound on the distortion to 3 and $1+\varepsilon$, respectively, without affecting the size of the minor. Our work builds on existing techniques used in the context of approximate distance oracles, thereby bypassing our previous spanner reduction. Both results use essentially the same ideas and rely heavily on the fact that planar graphs admit separators with special properties.

We say that a graph $G = (V, E, \ell)$ admits a λ-separator if there exists a set $R \subseteq V$ whose removal partitions G into connected components, each of size at most λn, where $1/2 \leq \lambda < 1$. Lipton and Tarjan [17] showed that every planar graph has a 2/3-separator R of size $\mathcal{O}(\sqrt{n})$. Later on, Gupta et al. [12] and Thorup [24] independently observed that one can modify their construction to obtain a 2/3-separator R, with the additional property that R consists of vertices belonging to shortest paths from G (note that this R is not guaranteed to be small). We briefly review the construction of such *shortest path separators*.

Let $G = (V, E, \ell)$ be a triangulated planar graph (the triangulation is guaranteed by adding infinity edge lengths among the missing edges). Further, let us fix an arbitrary shortest path tree A rooted at some vertex r. Then, it can be inferred from the work of Lipton and Tarjan [17] that there always exists a non-tree edge $e = \{u, v\}$ of A such that the fundamental cycle $\mathcal{C}$ in $A \cup \{e\}$, formed by adding the non-tree edge e to A, gives a 2/3-separator for G. Because A is a tree, the separator will consist of two paths from the $\text{lca}(u, v)$ to u and v. We denote such paths by P_1 and P_2, respectively. Both paths are shortest paths as they belong to A. We will show how to use such separators to obtain terminal path covers. Before proceeding, we give the following preprocessing step.

Given a planar graph $G = (V, E, \ell)$ with $T \subset V$, the algorithm MinorSparsifier(G, T, $\mathcal{P}'$) with $\mathcal{P}'$ being the $(1, \mathcal{O}(k^2))$-TPc of G, produces an $(1, \mathcal{O}(k^4))$-DAM for G. Thus, without loss of generality, we may assume that G has at most $\mathcal{O}(k^4)$ vertices.

Stretch-3 Guarantee. When solving a graph problem, it is often that the problem can be more easily solved for simpler graph instances, e.g., trees. Driven by this, it is desirable to reduce the problem from arbitrary graphs to one or several tree instances, possibly allowing a small loss in the quality of the solution. Along the lines of such an approach, Gupta et al. [12] gave the following definition in the context of shortest path distances.

▶ **Definition 28** (Forest Cover)**.** Given a graph $G = (V, E, \ell)$, a forest cover (with stretch α) of G is a family $\mathcal{F}$ of subforests $\{F_1, F_2, \dots, F_k\}$ of G such that for every $u, v \in V$, there is a forest $F_i \in \mathcal{F}$ such that $d_G(u, v) \leq d_{F_i}(u, v) \leq \alpha \cdot d_G(u, v)$.

If we restrict our attention to planar graphs, Gupta et al. [12] used shortest path separators (as described above) to give a divide-and-conquer algorithm for constructing forest covers with small guarantees on the stretch and size. Here, we slightly modify their construction for our purpose. Before proceeding to the algorithm, we give the following useful definition.

▶ **Definition 29.** Let t be a terminal and let P be a shortest path in G. Then $t^P_{\min}$ denotes the vertex of P that minimizes $d_G(t, p)$, for all $p \in P$, breaking ties lexicographically.

Algorithm 2 ForestCover (planar graph G, terminals T)

if $|V(G)| \leq 1$ **then return** $V(G)$
Compute a 2/3-separator $\mathcal{C}$ consisting of shortest paths P_1 and P_2 for G.
for $i = 1, 2$ **do**
  Contract P_i to a single vertex p_i and compute a shortest path tree L_i from p_i.
  Expand back the contracted edges in L_i to get the tree L'_i.
  for every terminal $t \in T$ **do**
    Add $t^{P_i}_{\min}$ as a terminal in the tree L'_i.
Let (G_1, T_1) and (G_2, T_2) be the resulting connected graphs from $G \setminus \mathcal{C}$,
where T_1 and T_2 are disjoint subsets of the terminals T induced by $\mathcal{C}$.
`// Note that all distances involving terminals from` $\mathcal{C}$ `are taken care of.`
return $\bigcup_{i=1}^{2} L'_i \cup \bigcup_{i=1}^{2}$ ForestCover(G_i, T_i).

Algorithm 3 PlanarTPc-1 (planar graph G, terminals T)

Set $\mathcal{P}' = \emptyset$. Set $\mathcal{F} =$ ForestCover(G, T).
for every forest $F_i \in \mathcal{F}$ **do**
  Let R_i be the terminal set of F_i and let $\mathcal{P}'_i$ be the (trivial) $(1, \mathcal{O}(k^2))$-TPc of F_i;
  Compute $F'_i =$ MinorSparsifier(F_i, R_i, $\mathcal{P}'_i$).
  Add the shortest paths corresponding to the edges of F'_i to $\mathcal{P}'$.
return $\mathcal{P}'$

▶ **Theorem 30** ([12], Theorem 5.1)**.** *Given a planar graph $G = (V, E, \ell)$ with $T \subset V$,* ForsetCover(G, T) *produces a stretch-3 forest cover with $\mathcal{O}(\log |V|)$ forests.*

We note that the original construction does not consider terminal vertices, but this does not worsen neither the stretch nor the size of the cover. The only difference here is that we need to add at most k new terminals to each forest compared to the original number of terminals in the input graph. This modification affects our bounds only by a constant factor.

▶ **Lemma 31.** *Given a planar graph* $G = (V, E, \ell)$ *with* $T \subset V$*,* PLANARTPC-1(G, T) *produces an* $(3, \mathcal{O}(k \log k))$-TPc $\mathcal{P}'$ *for* G*.*

Proof. We first review the following simple fact, whose proof can be found in [15].

▶ **Fact 32.** *Given a forest* $F = (V, E, \ell)$ *with terminals* $T \subset V$ *and* $\mathcal{P}'$ *being the (trivial)* $(1, \mathcal{O}(k^2))$*-TPc of* F*, the procedure* MINORSPARSIFIER$(F, T, \mathcal{P}')$ *outputs an* $(1, k)$*-DAM.*

Let us proceed with the analysis. Observe that from the Preprocessing Step our input graph G has at most $\mathcal{O}(k^4)$ vertices. Thus, applying Theorem 30 on G gives a stretch-3 forest cover $\mathcal{F}$ of size $\mathcal{O}(\log k)$. In addition, recall that all shortest paths are unique in G.

Next, let F_i by any forest from $\mathcal{F}$. By construction, we note that each tree belonging to F_i has the nice property of being a concatenation of a given shortest path with another shortest path tree. We will exploit this in order to show that every edge of the minor F_i' for F_i corresponds to the (unique) shortest path between its endpoints in G.

To this end, let $e' = (u, v)$ be an edge of F_i' that does not exist in F_i. Since F_i' is a minor of F_i, we can map back e' to the path $\Pi_{u,v}$ connecting u and v in F_i. Because of the additional terminals $u_{\min}^{P_i}$ added to F_i, we claim that $\Pi_{u,v}$ is entirely contained either in some shortest path tree L_j or some shortest path separator P_j. Using the fact that subpaths of shortest paths are shortest paths, we conclude that the length of the path $\Pi_{u,v}$ (or equivalently, the length of edge e') corresponds to the unique shortest path connecting u and v in G. The same argument is repeatedly applied to every such edge of F_i'.

By construction we know that F_i has at most $2k$ terminals. Using Fact 32 we get that F_i' contains at most $4k$ edges. Since there are $\mathcal{O}(\log k)$ forests, we conclude that the terminal path cover $\mathcal{P}'$ consists of $\mathcal{O}(k \log k)$ shortest paths. The stretch guarantee follows directly from that of cover $\mathcal{F}$, since F_i' exactly preserves all distances between terminals in F_i. ◀

▶ **Theorem 33.** *Any planar graph* $G = (V, E, \ell)$ *with* $T \subset V$ *admits a* $(3, \widetilde{\mathcal{O}}(k^2))$-DAM.

Indeed, there is another distance oracle that yields better distortion $(1 + \varepsilon)$ (see [24, 14]). Similar to the above, we prove the following theorem; its proof is deferred to the full version.

▶ **Theorem 34.** *Any planar graph* $G = (V, E, \ell)$ *with* $T \subset V$ *admits an* $(1 + \varepsilon, \widetilde{\mathcal{O}}((k/\varepsilon)^2)$-DAM.

7 Discussion and Open Problems

We note that there remain gaps between some of the best upper and lower bounds, e.g., for general graphs and distortion $3 - \epsilon$, the lower bound is $\Omega(k^{6/5})$, while for distortion 3, our upper bound is $\mathcal{O}(k^3)$. Improving the bounds is an interesting open problem.

Our techniques for showing upper bounds rely heavily on the spanner reduction. For planar graphs, Krauthgamer et al. [15] showed that to achieve distortion $1 + o(1)$, $\Omega(k^2)$ non-terminals are needed; we bypass the spanner reduction to construct an $(1 + \varepsilon, \widetilde{\mathcal{O}}(k/\varepsilon)^2)$-DAM, which is tight up to a poly-logarithmic factor. It is an interesting open question on whether similar guarantees can be achieved for general graphs.

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