L_1 Geodesic Farthest Neighbors in a Simple Polygon and Related Problems*

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Abstract

In this paper, we investigate the L_1 geodesic farthest neighbors in a simple polygon P, and address several fundamental problems related to farthest neighbors. Given a subset $S \subseteq P$, an L_1 geodesic farthest neighbor of $p \in P$ from S is one that maximizes the length of L_1 shortest path from p in P. Our list of problems include: computing the diameter, radius, center, farthest-neighbor Voronoi diagram, and two-center of S under the L_1 geodesic distance. We show that all these problems can be solved in linear or near-linear time based on our new observations on farthest neighbors and extreme points. Among them, the key observation shows that there are at most four extreme points of any compact subset $S \subseteq P$ with respect to the L_1 geodesic distance after removing redundancy.

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1 Introduction

The geometry of points in a simple polygon P has been one of the most attractive research subjects in computational geometry since the 1980s. As a metric space, P is often associated with a distance function d induced by shortest paths that stay inside P. Indeed, there are several ways to define a shortest path between two points in P, depending on which underlying metric is adopted to determine the length of a segment in P. Most common are the Euclidean and the L_1 metrics that define the Euclidean and the L_1 shortest paths, respectively, in P. The length of a shortest path between two points $p, q \in P$ is called the (Euclidean or L_1) geodesic distance $d(p, q)$.

In this paper, we are interested in fundamental problems related to geodesic farthest neighbors in P. Given a set S of points in P, a farthest neighbor of $p \in P$ from S is one that maximizes the geodesic distance $d(p, q)$ from p to every $q \in S$. Specifically, our list of problems include those of computing the following:

- The *farthest-neighbor Voronoi diagram* of S.
- The *diameter* of S: $\mathrm{diam}(S) := \max_{q \in S} \max_{q' \in S} d(q, q')$.
- The *radius* of S: $\mathrm{rad}(S) := \min_{p \in P} \max_{q \in S} d(p, q)$.
- A *center* of S: a point $c \in P$ such that $\max_{q \in S} d(c, q) = \mathrm{rad}(S)$.
- A *two-center* of S: a pair of points $c_1, c_2 \in P$ that minimizes $\max_{q \in S} \min\{d(c_1, q), d(c_2, q)\}$.

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In the Euclidean case, where d denotes the Euclidean geodesic distance, these problems have been intensively studied. Aronov et al. [2] presented an $O((n+m)\log(n+m))$-time algorithm that computes the farthest-neighbor Voronoi diagram of S if S consists of m points and P is an n-gon. Very recently, Oh et al. [11] showed that the diagram can be computed faster in $O(n \log\log n + m\log(n+m))$ time, or in $O(n\log\log n)$ time when S is the set of vertices of P. Note that computing the diameter, radius, and center of S is reduced from the farthest-neighbor Voronoi diagram in linear time. On the other hand, in a special case where $S = P$, it is known that we can compute them in linear $O(n)$ time [9, 1]. The problem of computing a two-center of S under the Euclidean geodesic distance was recently addressed by Oh et al. [12] and Oh et al. [10], resulting in two algorithms that run in $O(n^2\log^2 n)$ time when $S = P$ and in $O(m^2(m+n)\log^3(m+n))$ time when S is a set of m points in P.

The problems in the L_1 geodesic distance have attained less interest compared to those in the Euclidean case. This is probably because most of results for the Euclidean counterpart automatically hold for the L_1 geodesic distance. Note that the Euclidean shortest paths in P are also L_1 shortest paths, and the algorithm of Aronov et al. [2] can be implemented for computing the L_1 geodesic farthest-neighbor Voronoi diagram. However, it is not clear whether the approach by Oh et al. [11] can be extended to compute the L_1 diagram. Bae et al. [3] exhibited some geometric observations on the L_1 geodesic distance that are different from the Euclidean one, and exploited them to devise linear-time algorithms that compute the diameter, radius, and center of a simple polygon P, i.e., the special case where $S = P$. Prior to this work, no algorithm for the two-center of S in the L_1 geodesic distance was known in the literature.

In this paper, we reveal that farthest neighbors in the L_1 geodesic distance behave quite different from – indeed much nicer than – gthose in the Euclidean geodesic distance. Based on our new observations, we show that all the problems listed above in the L_1 geodesic distance can be computed in linear or near-linear time:

- $O(n + m\log n)$ time when S is a set of m points in P, or
- $O(n)$ time when either $S = P$ or S equals the set of vertices of P.

It is worth noting that our algorithms runs in time linear to each of n and m, while the $O(m\log n)$ term was unavoidable for evaluation of the geodesic distance $d(p,q)$. Note that, in particular, our algorithms for the farthest-neighbor Voronoi diagram and the two-center are faster than the currently best algorithms for those in the Euclidean case: roughly by a factor $\log\log n$ for the farthest-neighbor Voronoi diagram [11], and by a factor of n or of m^2 for the two-center problem [12, 10]. All these algorithmic results are based on a key observation that for any compact subset $S \subseteq P$ of P, there are at most four extreme points of S in general. Note that in the Euclidean case, there can be linearly many extreme points.

This phenomenon can be understood as an extension of the relation between the L_1 plane and the Euclidean plane. In the plane associated with the L_1 metric, there are at most four extreme points of S in the four directions corresponding to the four segments of the L_1 metric balls, while in the plane associated with the Euclidean metric, every point of S lying on the boundary of its convex hull is considered to be extreme. An immediate implication is that the farthest-neighbor Voronoi diagram in the L_1 metric consists of at most four nonempty regions and thus has $O(1)$ complexity, while this is not the case for the Euclidean metric. Similarly, an L_1 (or rectilinear) two-center of m points in the plane can be computed in $O(m)$ time [5], while the best known algorithm that computes a Euclidean two-center in the plane runs in $O(m\log^2 m(\log\log m)^2)$ deterministic time [4]. Our results thus provide a series of analogies on farthest neighbors in the L_1 plane into those in the metric space (P,d), where P is a simple polygon and d is the L_1 geodesic distance in P.

2 Preliminaries

For any subset $A \subset \mathbb{R}^2$, we denote by ∂A and $\mathrm{int}A$ the boundary and the interior of A, respectively. For $p, q \in \mathbb{R}^2$, denote by $\overline{pq}$ the line segment with endpoints p and q. For any path π in $\mathbb{R}^2$, let $|\pi|$ be the length of π under the L_1 metric, or simply the L_1 *length.* Note that $|\overline{pq}|$ equals the L_1 distance between p and q.

The following is a basic observation on the L_1 length of paths in $\mathbb{R}^2$. A path is called *monotone* if any vertical or horizontal line intersects it in at most one connected component.

▶ **Lemma 1.** *For any path π between $p, q \in \mathbb{R}^2$, $|\pi| = |\overline{pq}|$ if and only if π is monotone.*

Let P be a simple polygon with n vertices. We regard P as a compact set in $\mathbb{R}^2$, so its boundary ∂P is contained in P. An L_1 *shortest path* between p and q is a path joining p and q that lies in P and minimizes its L_1 length. The L_1 *geodesic distance* $d(p, q)$ is the L_1 length of an L_1 shortest path between p and q. For any $p, q \in P$, let $\Pi(p, q)$ be the set of all L_1 shortest paths from p to q.

Analogously, a path lying in P minimizing its *Euclidean* length is called the *Euclidean shortest path.* It is well known that there is always a unique Euclidean shortest path between any two points in a simple polygon [7]. We let $\pi_2(p, q)$ be the unique Euclidean shortest path from $p \in P$ to $q \in P$. The following states a crucial relation between Euclidean and L_1 shortest paths in a simple polygon.

▶ **Lemma 2** (Hershberger and Snoeyink [8])**.** *For any two points $p, q \in P$, the Euclidean shortest path $\pi_2(p, q)$ is also an L_1 shortest path between p and q. That is, $\pi_2(p, q) \in \Pi(p, q)$.*

Lemma 2 enables us to exploit several structures for Euclidean shortest paths such as Guibas et al. [7] and Guibas and Hershberger [6].

Another important concept regarding the shortest paths in P is the relative convexity. A subset $A \subset P$ is called *relative convex* if $\pi_2(p, q) \subset A$ for any $p, q \in A$. For any subset $A \subset P$, the *relative convex hull* $\mathrm{rconv}(A)$ of A is the smallest relative convex set including A. If A is the set of m points in P, then its relative convex hull forms a weakly simple polygon in P with $O(m + n)$ vertices. Touissant [13] presented an $O((n + m) \log(n + m))$-time algorithm that computes $\mathrm{rconv}(A)$, and Guibas and Hershberger [6] improved it to $O(n + m \log(n + m))$.

Throughout the paper, unless otherwise stated, a shortest path and the geodesic distance always refer to an L_1 shortest path and the L_1 geodesic distance d.

3 Properties of L_1 Shortest Paths

In this section, we observe several useful properties of L_1 shortest paths in P.

We define a *chord* of P to be a maximal segment contained in P. For any $z \in P$, let h_z^- and h_z^+ be the left and right endpoints, respectively, of the horizontal chord through z, while v_z^- and v_z^+ denote the lower and upper endpoints, respectively, of the vertical chord through z. Note that the horizontal or vertical chord may intersect the boundary ∂P of P in several connected components by definition. We also consider the four segments $\overline{zh_z^-}$, $\overline{zh_z^+}$, $\overline{zv_z^-}$ and $\overline{zv_z^+}$, called the *leftward, rightward, downward*, and *upward half-chords* from z, respectively.

Let $z \in P$ be fixed and $p \in P$ be any point. We say that $\pi \in \Pi(p, z)$ *chooses* a half-chord from z if π intersects it at a point other than z. Then, Lemma 1 implies that every $\pi \in \Pi(p, z)$ chooses at most one half-chord from z or none of the four. We then observe the following.

▶ **Lemma 3.** *For any $p, z \in P$, there are no two shortest paths $\pi, \pi' \in \Pi(p, z)$ such that π chooses a half-chord from z and π' chooses its opposite half-chord from z.*

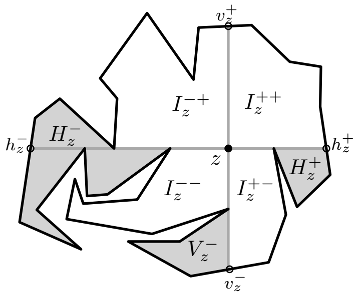


Figure 1 Partition around z.

This implies that any $p \in P$ avoids at least two half-chords that are not opposite by shortest paths to z. We thus consider the four regions of P according to the pair of excluded half-chords. More precisely, for any $\sigma_1, \sigma_2 \in \{+, -\}$, let $P_z^{\sigma_1\sigma_2} \subset P$ be the set of points $p \in P$ such that no shortest path $\pi \in \Pi(p, z)$ chooses $\overline{zh_z^{\overline{\sigma_1}}}$ or $\overline{zv_z^{\overline{\sigma_2}}}$, where $\overline{+} = -$ and $\overline{-} = +$. Lemma 3 guarantees that $P = P_z^{--} \cup P_z^{-+} \cup P_z^{+-} \cup P_z^{++}$ for any $z \in P$, while these four regions are not disjoint. Also, note that $P_z^{--} \cap P_z^{++} = \{z\}$ and $P_z^{-+} \cap P_z^{+-} = \{z\}$.

In order to gain a comprehensive understanding on the four regions $P_z^{\sigma_1\sigma_2}$, we consider the following eight subsets of P around z: Define $H_z^{\sigma_1} := P_z^{\sigma_1 -} \cap P_z^{\sigma_1 +}$, $V_z^{\sigma_2} := P_z^{-\sigma_2} \cap P_z^{+\sigma_2}$, and $I_z^{\sigma_1\sigma_2} := P_z^{\sigma_1\sigma_2} \setminus (H_z^{\sigma_1} \cup V_z^{\sigma_2})$. Observe that H_z^-, for example, is the set of points $p \in P$ such that no shortest path in $\Pi(p, z)$ chooses the downward, upward, or rightward half-chord from z, and I_z^{--} is the set of points $p \in P$ that admit two shortest paths $\pi, \pi' \in \Pi(p, z)$ such that π chooses the leftward half-chord and π' chooses the downward half-chord from z. See Figure 1 for an illustration. Note that these eight subsets $H_z^{\sigma_1}$, $V_z^{\sigma_2}$, and $I_z^{\sigma_1\sigma_2}$ form a partition of P around z. In most cases where the horizontal and vertical chords through z intersects ∂P only at their endpoints, we have $H_z^{\sigma_1} = \overline{zh_z^{\sigma_1}}$ and $V_z^{\sigma_2} = \overline{zv_z^{\sigma_2}}$. However, this is not always the case.

To be more precise, consider the complement $\mathcal{C}_z := P \setminus (\overline{h_z^- h_z^+} \cup \overline{v_z^- v_z^+})$, which in general consists of several connected components. Such a component $C \subseteq \mathcal{C}_z$ is said to be *adjacent* to a half-chord from z if its boundary ∂C intersects the half-chord at a point other than z. Note that any component of $\mathcal{C}_z$ is adjacent to at least one and at most two half-chords from z. The following describes how H_z^-, H_z^+, V_z^-, and V_z^+ are formed.

▶ **Lemma 4.** *Let $z \in P$ and $\sigma_1, \sigma_2 \in \{+, -\}$. Then, $H_z^{\sigma_1}$ is equal to the union of $\overline{zh_z^{\sigma_1}}$ and the components of $\mathcal{C}_z$ that are adjacent to $\overline{zh_z^{\sigma_1}}$ but to none of the others. Analogously, $V_z^{\sigma_2}$ is equal to the union of $\overline{zv_z^{\sigma_2}}$ and the components of $\mathcal{C}_z$ that are adjacent to $\overline{zv_z^{\sigma_2}}$ only.*

Thus, any component C of $\mathcal{C}_z$ that is adjacent to exactly one half-chord is included into the corresponding subset $H_z^{\sigma_1}$ or $V_z^{\sigma_2}$ for some $\sigma_1, \sigma_2 \in \{+, -\}$. On the other hand, if a component C of $\mathcal{C}_z$ is adjacent to two half-chords from z, then the boundary of C must contain z. Thus, there are at most four such components of $\mathcal{C}_z$, and each of them forms $I_z^{\sigma_1\sigma_2}$ for some $\sigma_1, \sigma_2 \in \{+, -\}$. Lemma 4 and the above discussion imply the following corollary.

▶ **Corollary 5.** *Suppose that $H_z^{\sigma_1} \setminus \overline{zh_z^{\sigma_1}} \neq \emptyset$ for $\sigma_1 \in \{+, -\}$. Then, either $z \in \{v_z^-, v_z^+\}$ or there exists a vertex u of P lying on $\overline{zh_z^-}$ such that for all $p \in H_z^{\sigma_1} \setminus \overline{zh_z^{\sigma_1}}$ any shortest path $\pi \in \Pi(p, z)$ passes through u. An analogous claim also holds for the set $V_z^{\sigma_2}$ with $\sigma_2 \in \{+, -\}$.*

We then prove the following properties of L_1 shortest paths in terms of the partition around a point $z \in P$.

▶ **Lemma 6.** *Let $p, q, z \in P$. If $p, q \in A$, then every $\pi \in \Pi(p, q)$ is contained in A, where A is equal to one of the following sets: $H_z^{\sigma_1}$, $V_z^{\sigma_2}$, $I_z^{\sigma_1\sigma_2}$, $I_z^{\sigma_1\sigma_2} \cup \partial I_z^{\sigma_1\sigma_2}$, $I_z^{\sigma_1\sigma_2} \cup H_z^{\sigma_1}$, $I_z^{\sigma_1\sigma_2} \cup V_z^{\sigma_2}$, and $P_z^{\sigma_1\sigma_2}$ for any $\sigma_1, \sigma_2 \in \{+, -\}$.*

▶ **Lemma 7.** *Let $p, q, z \in P$ be any three points. Then, $d(p, q) = d(p, z) + d(z, q)$ if and only if $p \in P_z^{\sigma_1\sigma_2}$ and $q \in P_z^{\overline{\sigma_1}\,\overline{\sigma_2}}$ for some $\sigma_1, \sigma_2 \in \{+, -\}$.*

4 L_1 Geodesic Farthest Neighbors and Extreme Points

Let $S \subseteq P$ be a nonempty, compact subset of P. We are interested in farthest neighbors of each $p \in P$ from S. For each $p \in P$, let $\Phi_S(p) := \max_{q \in S} d(p, q)$. This is well defined since S is a compact set. We call such a $q \in S$ with $d(p, q) = \Phi_S(p)$ an L_1 *geodesic farthest neighbor of p from S*, or shortly a *farthest neighbor of p* when there is no confusion. There can be several farthest neighbors of $p \in P$ from S. We denote by $F_S(p)$ the set of all farthest neighbors of p from S. In order pick a representative among them, we impose a total order $\prec$ on S, such as the lexicographical order. We then define $f_S(p) \in F_S(p)$ to be the least with respect to the order $\prec$ among the farthest neighbors of p in $F_S(p)$. We call $q \in S$ an *(L_1 geodesic) extreme point* of S if $q = f_S(p)$ for some $p \in P$.

There are two fundamental quantities defined by farthest neighbors in P: the *(L_1 geodesic) diameter* $\operatorname{diam}(S) := \max_{q \in S} \Phi_S(q)$ and the *(L_1 geodesic) radius* $\operatorname{rad}(S) := \min_{p \in P} \Phi_S(p)$ of S. The diameter and radius of S are well defined since P and S are compact sets. A pair of points $q, q' \in S$ is called *diametral* if $d(q, q') = \operatorname{diam}(S)$, while a point $c \in P$ is called an *(L_1 geodesic) center* of S if $\Phi_S(c) = \operatorname{rad}(S)$. Let $\operatorname{cen}(S)$ be the set of all centers $c \in P$ of S.

In this section, we fully reveal the behavior of the L_1 farthest neighbors and extreme points of any compact set S in P, and finally prove the following theorem.

▶ **Theorem 8.** *In a simple polygon P, there are at most four extreme points of any compact subset $S \subseteq P$ with respect to the L_1 geodesic distance.*

In order to prove Theorem 8, we consider farthest neighbors constrained in regions. For $\sigma_1, \sigma_2 \in \{+, -\}$, define $f_S^{\sigma_1\sigma_2}(p)$ to be the farthest neighbor of p from $S \cap P_p^{\sigma_1\sigma_2}$ that is the least with respect to $\prec$. In the case where $S \cap P_p^{\sigma_1\sigma_2} = \emptyset$, $f_S^{\sigma_1\sigma_2}(p)$ is undefined. Then observe that $f_S(p)$ is the farthest one that is the least with respect to $\prec$ among the four candidates $f_S^{++}(p)$, $f_S^{-+}(p)$, $f_S^{--}(p)$, and $f_S^{+-}(p)$.

We first gather some useful properties of farthest neighbors.

▶ **Lemma 9.** *Given any $p \in P$, suppose that $f_S^{\sigma_1\sigma_2}(p) \in F_S(p)$ for $\sigma_1, \sigma_2 \in \{+, -\}$. Then, $f_S^{\sigma_1\sigma_2}(p) \in F_S(p')$ for any $p' \in P_p^{\overline{\sigma_1}\,\overline{\sigma_2}}$, and $f_S(p') = f_S^{\sigma_1\sigma_2}(p)$ for any $p' \in I_p^{\overline{\sigma_1}\,\overline{\sigma_2}}$. Moreover, if $f_S(p) = f_S^{\sigma_1\sigma_2}(p)$, then $f_S(p') = f_S^{\sigma_1\sigma_2}(p)$ for any $p' \in P_p^{\overline{\sigma_1}\,\overline{\sigma_2}}$.*

▶ **Lemma 10.** *For any $p \in P$, let $z \in \pi$ be a point on a shortest path $\pi \in \Pi(p, f_S(p))$ Then, for any $\sigma_1, \sigma_2 \in \{+, -\}$ with $p \in P_z^{\sigma_1\sigma_2}$ and $f_S(p) \in P_z^{\overline{\sigma_1}\,\overline{\sigma_2}}$, it holds that $f_S(p) = f_S^{\overline{\sigma_1}\,\overline{\sigma_2}}(z)$.*

Note that such $\sigma_1, \sigma_2 \in \{+, -\}$ with $p \in P_z^{\sigma_1\sigma_2}$ and $f_S(p) \in P_z^{\overline{\sigma_1}\,\overline{\sigma_2}}$ in Lemma 10 always exist by Lemma 7 since z is a point on a shortest path from p to $f_S(p)$, so $d(p, f_S(p)) = d(p, z) + d(z, f_S(p))$.

4.1 Proof of Theorem 8

Now, we give a proof of Theorem 8. The case where S consists of at most one point is trivial, so we assume that S consists of more than one point. For a center $c \in \operatorname{cen}(S)$ of S, we consider the set $F_S(c)$ of its farthest neighbors. Since c is a center and S consists of at least two points, we have $|F_S(c)| \geq 2$ and $d(c, \chi) = \Phi_S(c) = \operatorname{rad}(S)$ for any $\chi \in F_S(c)$.

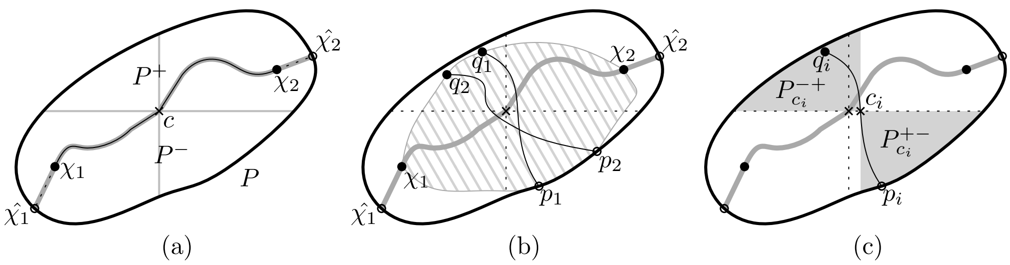


Figure 2 (a) The path $\hat{\pi}$ through $c \in \text{cen}(S)$ partitions P into P^+ and P^-. (b) Illustration to Claims 13 and 14. Shaded region represents $\text{rconv}(S \cup \{p_1, p_2\})$. (c) Illustration to Claim 15.

▶ **Lemma 11.** *For any $c \in \text{cen}(S)$, there exist $\chi_1, \chi_2 \in F_S(c)$ satisfying the following: (i) $d(\chi_1, \chi_2) = d(\chi_1, c) + d(c, \chi_2)$, and (ii) $f_S(\chi_1) = \chi_2$ and $f_S(\chi_2) = \chi_1$.*

From now on, we fix any two farthest neighbors $\chi_1, \chi_2 \in F_S(c)$ of c with the property of Lemma 11. Note that χ_1 and χ_2 are extreme points of S. Since $d(\chi_1, \chi_2) = d(\chi_1, c)+d(c, \chi_2)$, we have $\chi_1 \in P_c^{\sigma_1\sigma_2}$ and $\chi_2 \in P_c^{\overline{\sigma_1}\,\overline{\sigma_2}}$ for some $\sigma_1, \sigma_2 \in \{+, -\}$ by Lemma 7. Without loss of generality, we assume that $\sigma_1 = \sigma_2 = -$, so $\chi_1 \in P_c^{--}$ and $\chi_2 \in P_c^{++}$.

Let $\pi := \pi_2(\chi_1, c) \cup \pi_2(c, \chi_2)$ be a path from χ_1 to χ_2. Since $d(\chi_1, \chi_2) = d(\chi_1, c)+d(c, \chi_2)$, π is a shortest path from χ_1 to χ_2, that is, $\pi \in \Pi(\chi_1, \chi_2)$. Then, by Lemma 10, we have $\chi_1 = f_S^{--}(c)$ and $\chi_2 = f_S^{++}(c)$, as $c \in \pi$. This further implies that $f_S(p) = \chi_2$ for any $p \in I_c^{--}$, and $f_S(p) = \chi_1$ for any $p \in I_c^{++}$ by Lemma 9 since $\chi_1, \chi_2 \in F_S(c)$.

We will need the following lemma, which rephrases the Ordering Lemma by Aronov et al. [2]. Note that every extreme point of S appears on the boundary of the relative convex hull $\text{rconv}(S)$ of S.

▶ **Lemma 12** (Aronov et al. [2]). *Suppose that there are three distinct extreme points χ_1, χ_2, χ_3 of S in the counter-clockwise order along $\partial\text{rconv}(S)$. Let $p_i \in \partial P$ be a point on the boundary of P such that $f_S(p_i) = \chi_i$ for each $i \in \{1, 2, 3\}$. Then, p_1, p_2, p_3 appear in this order along ∂P in the counter-clockwise direction.*

Consider the extension of the last segment of $\pi_2(c, \chi_i)$ for each $i \in \{1, 2\}$ until it hits the first boundary point $\hat{\chi}_i \in \partial P$. Let $\hat{\pi}$ be the shortest path from $\hat{\chi}_1$ to $\hat{\chi}_2$ obtained by these extensions from π; that is, $\hat{\pi} = \pi_2(\hat{\chi}_1, c) \cup \pi_2(c, \hat{\chi}_2)$. Note that $f_S(\hat{\chi}_1) = \chi_2$ and $f_S(\hat{\chi}_2) = \chi_1$ by Lemma 9. The path $\hat{\pi}$ partitions P into two parts P^- and P^+, where ∂P^- consists of $\hat{\pi}$ and the chain along ∂P from $\hat{\chi}_1$ to $\hat{\chi}_2$ in the counter-clockwise direction, and ∂P^+ consists of $\hat{\pi}$ and the chain along ∂P from $\hat{\chi}_2$ to $\hat{\chi}_1$ in the counter-clockwise direction. See Figure 2(a) for an illustration. In the following, we show that there are at most one more extreme point of S, other than χ_1 and χ_2, in each of P^- and P^+. Recall that an extreme point of S is $q \in S$ such that $q = f_S(p)$ for some $p \in P$.

Suppose to the contrary that there are two extreme points q_1, q_2 of S such that $q_1, q_2 \in P^+$ and the four points χ_1, χ_2, q_1, q_2 are all distinct. Then there exist two boundary points $p_1, p_2 \in \partial P$ such that $f_S(p_1) = q_1$ and $f_S(p_2) = q_2$. Such boundary points p_1, p_2 are guaranteed to exist by Lemma 9. Our proof will be done by a contradiction based on the following four claims.

▶ **Claim 13.** *Both p_1 and p_2 lie in I_c^{+-}.*

In the following, we assume that the four points χ_2, q_1, q_2, χ_1 appear in this order along $\partial\text{rconv}(S)$ in the counter-clockwise direction. Then, Lemma 12 implies the following.

▶ **Claim 14.** *The four points p_1, p_2, q_1, q_2 appear in this order along $\partial\mathrm{rconv}(S \cup \{p_1, p_2\})$.*

See Figure 2(b) for an illustration to the above two claims. Note that Claim 13 implies that $p_1, p_2 \in P^-$ as $I_c^{+-} \subset P^-$, so any shortest path from p_i to q_i crosses $\hat{\pi}$. On the other hand, Claim 14 implies that $\pi_2(p_1, q_1)$ and $\pi_2(p_2, q_2)$ cross each other.

Let $\beta := \partial I_c^{+-} \setminus \partial P$. Note that β is a subset of the union of the rightward half-chord $\overline{ch_c^+}$ and the downward half-chord $\overline{cv_c^-}$ from c. By Claim 13, the paths $\pi_2(p_1, q_1)$ and $\pi_2(p_2, q_2)$ must cross over β as $p_1, p_2 \in I_c^{+-}$ and $q_1, q_2 \in P^+$. For each $i \in \{1, 2\}$, let c_i be the first intersection point of $\pi_2(p_i, q_i) \cap \beta$ when walking from p_i to q_i along $\pi_2(p_i, q_i)$. We then observe the following.

▶ **Claim 15.** *For $i \in \{1, 2\}$, we have $p_i \in P_{c_i}^{+-}$ and $q_i \in P_{c_i}^{-+}$*

See Figure 2(c) for an illustration to Claim 15. Our last claim to prove Theorem 8 is the following.

▶ **Claim 16.** *There exists $c' \in P$ such that $p_1, p_2 \in P_{c'}^{+-}$ and $q_1, q_2 \in P_{c'}^{-+}$.*

Now, we are ready to achieve the final contradiction. Let $c' \in P$ be such a point described in Claim 16. Then, we have $d(p_i, q_i) = d(p_i, c') + d(c', q_i)$ for $i \in \{1, 2\}$ by Lemma 7. Since $f_S(p_i) = q_i$ and $q_i \in P_{c'}^{-+}$, we have $f_S^{-+}(c') = q_i$ for each $i \in \{1, 2\}$ by Lemma 10. This leads to a contradiction since $f_S^{-+}(c')$ is uniquely determined by definition.

Consequently, there are no two disticnt extreme points $q_1, q_2 \in S$ of S such that $q_1, q_2 \in P^+$, implying that there is at most one extreme point of S in P^+ or P^-. This completes the proof of Theorem 8. ◀

5 L_1 Geodesic Center

In this section, we investigate the set $\mathrm{cen}(S)$ of L_1 geodesic centers of S in P. Recall that an L_1 geodesic center c of S minimizes $\Phi_S(c')$ over all $c' \in P$, so $\Phi_S(c) = \mathrm{rad}(S)$. Another remarkable consequence from the discussions in the previous section is the following.

▶ **Lemma 17.** *For any nonempty compact subset $S \subseteq P$, there is a diametral pair (χ_1, χ_2) of S such that $f_S(\chi_1) = \chi_2$ and $f_S(\chi_2) = \chi_1$.*

The above lemma and its proof indeed show the following.

▶ **Corollary 18.** *For any nonempty compact subset $S \subseteq P$, it holds that $\mathrm{rad}(S) = \mathrm{diam}(S)/2$.*

Bae et al. [3] considered a special case where $S = P$, and proved that $\mathrm{diam}(P) = 2\mathrm{rad}(P)$ by using a Helly-type theorem: any family of L_1 geodesic balls has Helly number at most two. It is worth noting that we generalize it to any compact subset S of P with a relatively direct argument in terms of extreme points of S.

For $p \in P$ and $r \in \mathbb{R}$, let $\mathrm{B}_p(r) := \{x \in P \mid d(x, p) \leq r\}$ be the L_1 *geodesic ball* at p with radius r. Bae et al. [3] also exhibited several basic properties of the L_1 geodesic balls; among them is that $\mathrm{B}_p(r)$ is relative convex for any $p \in P$ and $r \in \mathbb{R}$.

We fully characterize the set $\mathrm{cen}(S)$ of all centers of S by using those known results.

▶ **Lemma 19.** *For any nonempty compact subset $S \subseteq P$, $\mathrm{cen}(S)$ is equal to the intersection of at most four geodesic balls $\bigcap_{\chi \in X} \mathrm{B}_\chi(\mathrm{rad}(S))$, where X is the set of extreme points of S.*

In the following, we assume that S consists of at least two points. Then, the set X of extreme points of S consists of at least two and at most four points. Let (χ_1, χ_2) be a diametral pair such that $f_S(\chi_1) = \chi_2$ and $f_S(\chi_2) = \chi_1$. Such a diametral pair exists by Lemma 17. Since $f_S(\chi_1) = \chi_2$ and $f_S(\chi_2) = \chi_1$, both χ_1 and χ_2 are extreme points of S, that is, $\chi_1, \chi_2 \in X$. By Corollary 18, we know that $\mathrm{diam}(S) = 2\mathrm{rad}(S)$. Thus, $\mathrm{B}_{\chi_1}(\mathrm{rad}(S))$ and $\mathrm{B}_{\chi_2}(\mathrm{rad}(S))$ intersect only in their boundaries. Let $B := \mathrm{B}_{\chi_1}(\mathrm{rad}(S)) \cap \mathrm{B}_{\chi_2}(\mathrm{rad}(S))$. Since any L_1 geodesic ball is relative convex, as shown in [3], we observe that $B \cap \partial P$ is either $\emptyset$, a single point, or two points. Again by the relative convexity, B already forms a line segment of slope 1 or -1, since the boundary of any L_1 geodesic ball in the interior of P consists of line segments of slope 1 or -1. By Lemma 19, it holds that $\mathrm{cen}(S) \subseteq B$. This implies the the following.

▶ **Corollary 20.** *The set* $\mathrm{cen}(S)$ *of centers of any nonempty compact subset* $S \subseteq P$ *forms a line segment of slope* 1 *or* -1*, unless it is a point.*

Let $c_1, c_2 \in P$ be the endpoints of the segment $\mathrm{cen}(S)$. By Lemma 19, we know that $\mathrm{cen}(S) = B \cap \bigcup_{\chi \in X \setminus \{\chi_1, \chi_2\}} \mathrm{B}_\chi(\mathrm{rad}(S))$. Thus, if $|X| \geq 3$, then a third extreme $\chi_3 \in X$ with $\chi_3 \neq \chi_1, \chi_2$ determines an endpoint c_1 or c_2 of $\mathrm{cen}(S)$ as the intersection $B \cap \partial \mathrm{B}_{\chi_3}(\mathrm{rad}(S))$. More precisely, we observe the following.

▶ **Corollary 21.** *Suppose that* $X = \{\chi_1, \chi_2, \ldots, \chi_k\}$ *with* $3 \leq k \leq 4$*, and* (χ_1, χ_2) *is a diametral pair of* S *with* $f_S(\chi_1) = \chi_2$ *and* $f_S(\chi_2) = \chi_1$*. Then, for each* $3 \leq i \leq k$*,* $\partial \mathrm{B}_{\chi_1}(\mathrm{rad}(S)) \cap \partial \mathrm{B}_{\chi_2}(\mathrm{rad}(S)) \cap \partial \mathrm{B}_{\chi_i}(\mathrm{rad}(S))$ *determines an endpoint of* $\mathrm{cen}(S)$*.*

6 L_1 Geodesic Farthest-Neighbor Voronoi Diagram

We then turn our attention to the L_1 geodesic farthest-neighbor Voronoi diagram. Given a set S of sites in P, its L_1 geodesic farthest-neighbor Voronoi diagram $\mathsf{FVD}(S)$ is a partition of P into regions according to the farthest-neighbor relation between P and S. A common degenerate case of Voronoi diagrams occurs when a point $p \in P$ has four or more equidistant sites in S. There are two popular approaches in the literature to resolve such a degenerate case: assume a general position or impose a total order $\prec$ on S. We take the latter as done so far to give a precise definition of $\mathsf{FVD}(S)$.

The L_1 *geodesic farthest-neighbor Voronoi region* $FR(q, S)$ for each $q \in S$ is defined to be

$$FR(q, S) := \{p \in P \mid f_S(p) = q\}.$$

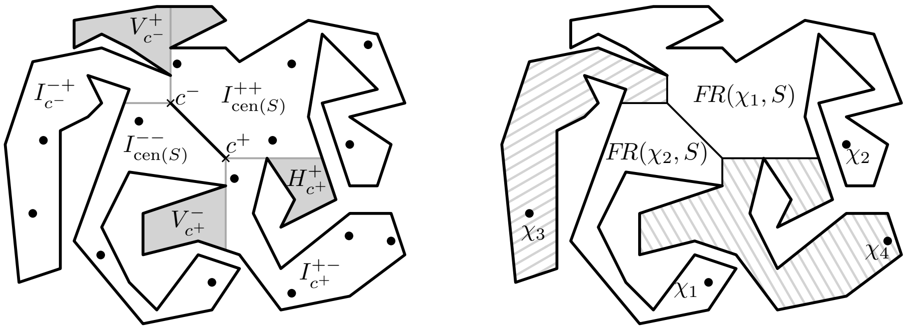
Then, the L_1 *geodesic farthest-neighbor Voronoi diagram* $\mathsf{FVD}(S)$ is defined to be

$$\mathsf{FVD}(S) := \bigcup_{q \in S} \partial FR(q, S) \setminus \partial P,$$

the union of the boundaries of each farthest-neighbor Voronoi region, except ∂P.

By definition, the Voronoi region $FR(q, S)$ for $q \in S$ is nonempty if and only if q is an extreme point of S. By Theorem 8, this implies that $\mathsf{FVD}(S)$ coincides with the diagram of at most four points in S. This enables us to define the Voronoi diagram $\mathsf{FVD}(S)$ for any nonempty compact subset $S \subseteq P$, even if S consists of an infinite number of points.

Let $X \subseteq S$ be the set of extreme points of S. Lemma 17 guarantees the existence of a diametral pair (χ_1, χ_2) of S with $f_S(\chi_1) = \chi_2$ and $f_S(\chi_2) = \chi_1$, so $\chi_1, \chi_2 \in X$. We first observe the following property of such a diametral pair.



(a) (b)

Figure 3 (a) The partition around the center $\mathrm{cen}(S)$. Points in S are depicted as dots. (b) Illustration to $\mathsf{FVD}(S)$. Shaded regions depict $FR(\chi_3, S)$ and $FR(\chi_4, S)$, respectively. In this example, S has four extreme points $\chi_1, \chi_2, \chi_3, \chi_4$ and we assume that $\chi_3 \prec \chi_1 \prec \chi_4 \prec \chi_2$.

▶ **Lemma 22.** *Let (χ_1, χ_2) be any diametral pair of S with $f_S(\chi_1) = \chi_2$ and $f_S(\chi_2) = \chi_1$. Then, there exist $\sigma_1, \sigma_2 \in \{+, -\}$ such that $\chi_1 = f_S^{\sigma_1\sigma_2}(c)$ and $\chi_2 = f_S^{\overline{\sigma_1}\,\overline{\sigma_2}}(c)$ for all $c \in \mathrm{cen}(S)$. Moreover, if $\mathrm{cen}(S)$ forms a line segment of positive length, then $\sigma_1 = \sigma_2$ when $\mathrm{cen}(S)$ is of slope -1, or $\sigma_1 = \overline{\sigma_2}$, otherwise.*

Note that if there are two distinct such pairs (χ_1, χ_2) and (χ_1', χ_2'), then Lemma 22 implies that the four points $\chi_1, \chi_2, \chi_1', \chi_2'$ must be all distinct. Since $|X| \le 4$ by Theorem 8, this implies that there are at most two such pairs.

We then observe the following.

▶ **Lemma 23.** *Suppose that $\mathrm{cen}(S)$ forms a line segment of slope $(\sigma 1)$ for $\sigma \in \{+, -\}$ or a point. Let c^- and c^+ be the left and right endpoints of $\mathrm{cen}(S)$. Then, the following hold:*

- *$I_c^{-\sigma} \subseteq FR(f_S^{+\overline{\sigma}}(c), S)$ and $I_c^{+\overline{\sigma}} \subseteq FR(f_S^{-\sigma}(c), S)$ for any $c \in \mathrm{cen}(S)$.*
- *$I_{c^-}^{-\overline{\sigma}} \subseteq FR(f_S^{+\sigma}(c^-), S)$ and $I_{c^+}^{+\sigma} \subseteq FR(f_S^{-\overline{\sigma}}(c^+), S)$.*
- *$H_{c^-}^{-} \setminus \{c^-\} \subseteq FR(f_S^{+\sigma_2}(c^-), S)$ and $V_{c^-}^{\overline{\sigma}} \setminus \{c^-\} \subseteq FR(f_S^{\sigma_1\sigma}(c^-), S)$ where $\sigma_1, \sigma_2 \in \{+, -\}$.*
- *$H_{c^+}^{+} \setminus \{c^+\} \subseteq FR(f_S^{-\sigma_2'}(c^+), S)$ and $V_{c^+}^{\sigma} \setminus \{c^+\} \subseteq FR(f_S^{\sigma_1'\overline{\sigma}}(c^+), S)$ where $\sigma_1', \sigma_2' \in \{+, -\}$.*

Lemma 23 fully describes the farthest-neighbor Voronoi diagram $\mathsf{FVD}(S)$, according to the shape of $\mathrm{cen}(S)$. Assume without loss of generality that $\mathrm{cen}(S)$ forms a line segment of slope -1 or a point. Let $I_{\mathrm{cen}(S)}^{--} := \bigcup_{c \in \mathrm{cen}(S)} I_c^{--}$ and $I_{\mathrm{cen}(S)}^{++} := \bigcup_{c \in \mathrm{cen}(S)} I_c^{++}$. Then, observe that the eight subsets $I_{\mathrm{cen}(S)}^{--}$, $V_{c^+}^{-}$, $I_{c^+}^{+-}$, $H_{c^+}^{+}$, $I_{\mathrm{cen}(S)}^{++}$, $V_{c^-}^{+}$, $I_{c^-}^{-+}$, and $H_{c^-}^{-}$ form a partition of P around $\mathrm{cen}(S)$. See Figure 3 for an illustration. Lemma 23 describes to which Voronoi region each of these eight subsets of P belongs. Note that each of the four subsets $I_{\mathrm{cen}(S)}^{--}$, $I_{c^+}^{+-}$, $I_{\mathrm{cen}(S)}^{++}$, and $I_{c^-}^{-+}$ may be empty, when c^- or c^+ lies on ∂P. If $\mathrm{cen}(S) = \{c\}$ consists of a single point, then we have $c^+ = c^- = c$, $I_{\mathrm{cen}(S)}^{--} = I_c^{--}$, and $I_{\mathrm{cen}(S)}^{++} = I_c^{++}$. Otherwise, $\mathrm{cen}(S)$ forms a line segment of positive length. Then, since $\mathrm{cen}(S)$ is of slope -1, by Lemma 22, we have $\chi_1, \chi_2 \in X$ such that $\chi_1 = f_S^{--}(c)$ and $\chi_2 = f_S^{++}(c)$ for any $c \in \mathrm{cen}(S)$. Lemma 23 tells us that $I_{\mathrm{cen}(S)}^{++} \subseteq FR(\chi_1, S)$ and $I_{\mathrm{cen}(S)}^{--} \subseteq FR(\chi_2, S)$. On the other hand, if $I_{c^{\sigma'}}^{\sigma'\overline{\sigma'}} \neq \emptyset$ for any $\sigma' \in \{+, -\}$, then the endpoint $c^{\sigma'}$ is not a boundary point in ∂P. In particular, if $f_S^{\overline{\sigma'}\sigma'}(c^{\sigma'}) \notin \{\chi_1, \chi_2\}$, then we have a third extreme point $\chi_3 = f_S^{\overline{\sigma'}\sigma'}(c^{\sigma'})$ as described in Corollary 21.

Since the boundaries of any two of the eight subsets around $\mathrm{cen}(S)$ always intersect in a subset of a half-chord from c^- or from c^+, we conclude the following.

▶ **Theorem 24.** *For any compact subset $S \subseteq P$ with $|S| \geq 2$, its L_1 geodesic farthest-neighbor Voronoi diagram* $\mathsf{FVD}(S)$ *consists of* $\mathrm{cen}(S)$ *and a subset of the following four segments: two segments that are subsets of half-chords from each endpoint of* $\mathrm{cen}(S)$.

7 Algorithms

Now, we are ready to describe our algorithms that compute the extreme points X of S, the diameter, radius, center of S and the farthest-neighbor Voronoi diagram $\mathsf{FVD}(S)$. We keep the generality by setting S to be any nonempty compact subset of P, while an operation that computes $f_S(p)$ for any $p \in P$ is supposed to be processed in at most T time as a black box.

We first describe how to compute the set X of extreme points of S. Pick any $q_0 \in S$. Let $q_i := f_S(q_{i-1})$ for $i \geq 0$, and compute q_i until we have $q_{k+1} = q_{k-1}$ for some $k \geq 2$. By Theorem 8, this ends up with $k \leq 4$. If $k = 4$, then let $\chi_i := q_i$ for each $i \in \{1, 2, 3, 4\}$, and we are done as $X = \{\chi_1, \chi_2, \chi_3, \chi_4\}$ by Theorem 8. Otherwise, we let $\chi_1 := q_{k-1}$ and $\chi_2 := q_k$. Note that $f_S(\chi_1) = \chi_2$ and $f_S(\chi_2) = \chi_1$. Let $r := d(\chi_1, \chi_2)/2$. Then, we compute $\mathrm{B}_{\chi_1}(r)$ and $\mathrm{B}_{\chi_2}(r)$, and their intersection $\mathrm{B}_{\chi_1}(r) \cap \mathrm{B}_{\chi_2}(r)$. Since $r = d(\chi_1, \chi_2)/2$, $\mathrm{B}_{\chi_1}(r) \cap \mathrm{B}_{\chi_2}(r)$ forms a line segment $\overline{z^- z^+}$ of slope 1 or -1, where z^- is to the left of z^+, possibly being a point $z^- = z^+$. Without loss of generality, assume that $\overline{z^- z^+}$ is of slope -1. For each $\sigma \in \{+, -\}$, let $p^\sigma \in P$ be any point in $I_{z^\sigma}^{\sigma\overline{\sigma}}$ if $I_{z^\sigma}^{\sigma\overline{\sigma}}$ is nonempty, or let $p^\sigma := z^\sigma$, otherwise, if $I_{z^\sigma}^{\sigma\overline{\sigma}} = \emptyset$. Let $\chi_3 := f_S(p^-)$ and $\chi_4 := f_S(p^+)$. Then, we have $X = \{\chi_1, \chi_2, \chi_3, \chi_4\}$.

▶ **Lemma 25.** *Let $S \subseteq P$ be a given compact subset, and suppose that $f_S(p)$ for any $p \in P$ can be computed in T time. The above algorithm correctly computes the set X of extreme points of S in $O(n + T)$ time.*

The diameter, radius, center, and farthest-neighbor Voronoi diagram of S can be computed in the same time bound.

▶ **Lemma 26.** *Let $S \subseteq P$ be a given compact subset, and suppose that the set of extreme points of S is known. Then, the following can be computed in $O(n)$ time:* $\mathrm{diam}(S)$, $\mathrm{rad}(S)$, $\mathrm{cen}(S)$, *and* $\mathsf{FVD}(S)$.

Proof. Let X be the set of extreme points of S. Note that $\mathrm{diam}(S) = \max_{\chi, \chi' \in X} d(\chi, \chi')$. Thus, $\mathrm{diam}(S)$ and a diametral pair can be computed in additional $O(n)$ time [7], as $|X| \leq 4$ by Theorem 8. By Corollary 18, we have $\mathrm{rad}(S) = \mathrm{diam}(S)/2$. The set $\mathrm{cen}(S)$ can be computed by intersecting at most four geodesic balls $\mathrm{B}_\chi(\mathrm{rad}(S))$ for $\chi \in X$ by Lemma 19. This can be done in additional $O(n)$ time by computing the shortest path maps [7]. After computing $\mathrm{cen}(S)$, the farthest-neighbor Voronoi diagram $\mathsf{FVD}(S)$ can be found by considering the eight subsets around $\mathrm{cen}(S)$ by Lemma 23. As $\mathsf{FVD}(S)$ consists of at most five segments, it can be found in additional $O(n)$ time. ◀

Now, we describe the subprocedure that computes $f_S(p)$ for any $p \in P$. Here, we assume that S is a finite set of m points.

▶ **Lemma 27.** *Let S be a set of m points in P. Then, $f_S(p)$ for any $p \in P$ can be computed in $O(n + m \log n)$ time. If the order of $S \cap \partial\mathrm{rconv}(S)$ along $\partial\mathrm{rconv}(S)$ is provided, then $O(n + m)$ time is sufficient.*

Proof. As a preprocessing, we build in $O(n)$ time the data structure of Guibas and Hershberger [6] that evaluates $d(p, q)$ for any $p, q \in P$ in $O(\log n)$ time. Given any $p \in P$, we compute $d(p, q)$ for all $q \in S$, and gather the set $F_S(p)$ of farthest neighbors of p in $O(m \log n)$

time. And pick the least element in $F_S(p)$ with respect to the total order $\prec$, and report it as $f_S(p)$. This takes $O(n + m \log n)$ time.

If the order of $S \cap \partial\text{rconv}(S)$ along $\partial\text{rconv}(S)$ is known, then we can apply the fast matrix search technique of Hershberger and Suri [9]. This takes $O(n + m)$ time. ◀

Another interesting case is when $S = P$. Since $\text{rconv}(P) = P$, in this case, we know the order of points $P \cap \partial\text{rconv}(P) = \partial P$. Moreover, since $f_P(p)$ is always a vertex of P, we have the following corollary.

▶ **Corollary 28.** *For any $p \in P$, $f_P(p)$ can be computed in $O(n)$ time.*

Combining all these results, we obtain the following theorems.

▶ **Theorem 29.** *Let P be a simple n-gon and S be a set of m points in P. Then, the set of L_1 geodesic extreme points of S, $\text{diam}(S)$, $\text{rad}(S)$, $\text{cen}(S)$, and $\mathsf{FVD}(S)$ can be computed in $O(n + m \log n)$ time. If the order of $S \cap \partial\text{rconv}(S)$ along $\partial\text{rconv}(S)$ is provided, then $O(n + m)$ time is sufficient.*

▶ **Theorem 30.** *Let P be a simple n-gon. Then, the set of L_1 geodesic extreme points of P, $\text{diam}(P)$, $\text{rad}(P)$, $\text{cen}(P)$, and $\mathsf{FVD}(P)$ can be computed in $O(n)$ time.*

8 L_1 Geodesic Two-Center

In this section, we address the two-center problem for any compact subset $S \subseteq P$ under the L_1 geodesic distance. The L_1 *geodesic two-center problem* asks a pair of points $c_1, c_2 \in P$ that minimize $\max_{q \in S} \min\{d(q, c_1), d(q, c_2)\}$. Such a pair (c_1, c_2) is called an L_1 *geodesic two-center* of S in P, or shortly a *two-center* of S. Let $\text{rad}_2(S) := \max_{q\in S} \min\{d(q, c_1), d(q, c_2)\}$ be the optimal objective value for the problem, called the *two-radius* or 2*-radius* of S. A two-center (c_1, c_2) induces a bipartition (S_1, S_2) of S such that $S_1 = S \cap \mathrm{B}_{c_1}(\text{rad}_2(S))$ and $S_2 = S \setminus S_1$. Conversely, a bipartition (S_1, S_2) of S is called *optimal* if $\text{rad}_2(S) = \max\{\text{rad}(S_1), \text{rad}(S_2)\}$. Note that in general we have $\max\{\text{rad}(S_1), \text{rad}(S_2)\} \geq \text{rad}_2(S)$ if $S_1 \cup S_2 = S$. Given an optimal bipartition (S_1, S_2) of S, observe that any $c_1 \in \text{cen}(S_1)$ and $c_2 \in \text{cen}(S_2)$ form a two-center (c_1, c_2) of S. Thus, the two-center problem is equivalent to finding an optimal bipartition of S.

Another closely related problem is the *minmax-diameter bipartition problem* that asks a bipartition (S_1, S_2) of S such that $\max\{\text{diam}(S_1), \text{diam}(S_2)\}$ is minimized. Thus, this problem is to compute the 2*-diameter* $\text{diam}_2(S)$ of S defined to be the minimum value of $\max\{\text{diam}(S_1), \text{diam}(S_2)\}$ over all possible bipartitions (S_1, S_2) of S. In the L_1 geodesic case, the two-center problem is equivalent to the minmax-diameter bipartition problem.

▶ **Lemma 31.** *For any compact subset $S \subseteq P$, it holds that $\text{rad}_2(S) = \text{diam}_2(S)/2$.*

Thus, if (S_1, S_2) is the optimal solution to the minmax-diameter bipartition problem, then it is an optimal bipartition for the two-center problem.

In the following, we let X be the set of extreme points of S.

▶ **Lemma 32.** *There exists an optimal bipartition (S_1, S_2) of S such that for each $\chi \in X$, $\chi \in S_1$ if and only if $f_S(\chi) \in S_2$.*

The following is our key lemma.

▶ **Lemma 33.** *There exists an optimal bipartition (S_1^*, S_2^*) of S such that*

$$S_1^* = S \cap \bigcup_{\chi \in X \cap S_2^*} FR(\chi, S) \quad \text{and} \quad S_2^* = S \cap \bigcup_{\chi \in X \cap S_1^*} FR(\chi, S).$$

Our algorithm that computes a two-center of S is described as follows: First, compute the set X of extreme points of S, and the farthest-neighbor Voronoi diagram $\mathsf{FVD}(S)$. For each bipartition (X_1, X_2) of X that satisfies the property of Lemma 32, let $S_1 := S \cap \bigcup_{\chi \in X_2} FR(\chi, S)$ and $S_2 := S \cap \bigcup_{\chi \in X_1} FR(\chi, S)$. Then, compute $\mathrm{diam}(S_1)$ and $\mathrm{diam}(S_2)$, and keep the minimum of $\max\{\mathrm{diam}(S_1), \mathrm{diam}(S_2)\}$ for all such bipartitions of X.

Let (S_1^*, S_2^*) be the bipartition of S with a minimum value of $\max\{\mathrm{diam}(S_1^*), \mathrm{diam}(S_2^*)\}$. Then, (S_1^*, S_2^*) is an optimal bipartition and $\mathrm{diam}_2(S) = \max\{\mathrm{diam}(S_1^*), \mathrm{diam}(S_2^*)\}$ by Lemmas 31 and 33. A two-center (c_1, c_2) of S can be found by choosing any $c_1 \in \mathrm{cen}(S_1^*)$ and any $c_2 \in \mathrm{cen}(S_2^*)$.

The above algorithm works properly when S is a finite set of points in P.

▶ **Theorem 34.** *Let S be a set of m points in a simple n-gon P. Then, an L_1 geodesic two-center of S can be computed in $O(n + m \log n)$ time.*

Another interesting special case is when $S = P$.

▶ **Theorem 35.** *An L_1 geodesic two-center of a simple n-gon can be computed in $O(n)$ time.*

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