

Linear Kernels and Linear-Time Algorithms for Finding Large Cuts

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Abstract

The maximum cut problem in graphs and its generalizations are fundamental combinatorial problems. Several of these cut problems were recently shown to be fixed-parameter tractable and admit polynomial kernels when parameterized above the tight lower bound measured by the size and order of the graph. In this paper we continue this line of research and considerably improve several of those results:

- We show that an algorithm by Crowston et al. [ICALP 2012] for (SIGNED) MAX-CUT ABOVE EDWARDS-ERDŐS BOUND can be implemented in such a way that it runs in *linear time* $8^k \cdot O(m)$; this significantly improves the previous analysis with run time $8^k \cdot O(n^4)$.
- We give an *asymptotically optimal* kernel for (SIGNED) MAX-CUT ABOVE EDWARDS-ERDŐS BOUND with $O(k)$ vertices, improving a kernel with $O(k^3)$ vertices by Crowston et al. [COCOON 2013].
- We improve *all* known kernels for strongly λ -extendable properties parameterized above tight lower bound by Crowston et al. [FSTTCS 2013] from $O(k^3)$ vertices to $O(k)$ vertices.
- As a consequence, MAX ACYCLIC SUBDIGRAPH parameterized above Poljak-Turzík bound admits a kernel with $O(k)$ vertices and can be solved in time $2^{O(k)} \cdot n^{O(1)}$; this answers an open question by Crowston et al. [FSTTCS 2012].

All presented kernels can be computed in time $O(km)$.

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1 Introduction

A recent paradigm in parameterized complexity is to not only show a problem to be fixed-parameter tractable, but indeed to give algorithms with *optimal* run times in *both* the parameter and the input size. Ideally, we strive for algorithms that are *linear* in the input size, and optimal in the dependence on the parameter k assuming a standard hypothesis such as the Exponential Time Hypothesis [17]. New results in this direction include fixed-parameter algorithms for GRAPH BIPARTIZATION [18, 30], PLANAR SUBGRAPH ISOMORPHISM [9], DAG PARTITIONING [29] and SUBSET FEEDBACK VERTEX SET [20].

Here, we consider the fundamental MAX-CUT problem from the view-point of linear-time fixed-parameter algorithms. In this classical NP-complete problem [19], the task is to find a

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bipartite subgraph of a given graph G with the maximum number $\text{mc}(G)$ of edges. We refer to the survey [26] for an overview of the research area.

We focus on MAX-CUT *parameterized above Edwards-Erdős bound*. This parameterization is motivated by the classical result of Edwards [10, 11] that any connected graph on n vertices and m edges admits a cut of size at least

$$m/2 + (n - 1)/4 . \tag{1}$$

This lower bound is known as the *Edwards-Erdős bound*, and it is tight for cliques of every odd order n . Ngoc and Tuza [24] gave a *linear-time* algorithm that finds a cut of size at least (1).

Parameterizing MAX-CUT above Edwards-Erdős bound means, for a given connected graph G and integer k , to determine if G admits a cut that exceeds (1) by an amount of k : formally, the problem MAX-CUT ABOVE EDWARDS-ERDŐS BOUND (MAX-CUT AEE) is to determine if $\text{mc}(G) \geq |E(G)|/2 + (|V(G) - 1 + k)/4$. It was asked in a sequence of papers [5, 12, 21, 22] whether MAX-CUT AEE is fixed-parameter tractable, before Crowston et al. [7] gave an algorithm that solves instances of this problem in time $8^k \cdot O(n^4)$, as well as a kernel of size $O(k^5)$. Their result inspired a lot of further research on this problem, leading to smaller kernels of size $O(k^3)$ [4] and fixed-parameter algorithms for generalizations [23] and variants [8].

In the SIGNED MAX-CUT problem, we are given a graph G whose edges are labeled by (+) or (−), and we seek a maximum balanced subgraph H of G , where *balanced* means that each cycle has an even number of negative edges. MAX-CUT is the special case where all edges are negative. SIGNED MAX-CUT finds applications in, e.g., modeling social networks [14], statistical physics [1], portfolio risk analysis [15], and VLSI design [3]. The dual parameterization of SIGNED MAX-CUT by the number of edge deletions was also shown to be fixed-parameter tractable [16].

Poljak and Turzík [25] showed that the property of having a large cut (i.e., a large bipartite subgraph) can be generalized to many other classical graph properties, including properties of oriented and edge-labeled graphs. They defined the notion of “ λ -extendable” properties Π and generalized the lower bound (1) to tight lower bounds for all such properties; we refer to these lower bounds as the *Poljak-Turzík bound* for Π . Well-known examples of such properties include bipartite subgraphs, q -colorable subgraphs for fixed q , or acyclic subgraphs of oriented graphs. Mnich et al. [23] considered the problem ABOVE POLJAK-TURZÍK(Π) of finding subgraphs in Π with k edges above the *Poljak-Turzík bound*; they gave fixed-parameter algorithms for this problem on all “strongly” λ -extendable properties Π . A subclass of these, requiring certain technical conditions, was later shown to admit polynomial kernels [8].

1.1 Our Contributions

Linear-Time FPT. Our first result is that the fixed-parameter algorithm given by Crowston et al. [4] for the SIGNED MAX-CUT AEE problem can be implemented in such a way that it runs in linear time.

► **Theorem 1** (★). *The (SIGNED) MAX-CUT AEE problem can be solved in time $8^k \cdot O(m)$.*

Theorem 1 considerably improves the earlier run time analysis [4, 7], which shows a run time of $8^k \cdot O(n^4)$. At the same time, our algorithm improves the very involved algorithm by Bollobás and Scott [2] that considers the weaker lower bound $m/2 + (\sqrt{8m + 1} - 1)/8$ instead of (1). Third, Theorem 1 generalizes the linear-time algorithm by Ngoc and Tuza [24] for

the special case of MAX-CUT with $k = 0$. Note that MAX-CUT AEE cannot be solved in time $2^{o(k)} \cdot n^{O(1)}$ assuming the Exponential Time Hypothesis [7].

Linear Vertex Kernels. Our second contribution is a kernel with a linear number $O(k)$ of vertices for MAX-CUT AEE and its generalization SIGNED MAX-CUT AEE.

► **Theorem 2.** *The (SIGNED) MAX-CUT AEE problem admits a kernel with $O(k)$ vertices, which can be computed in time $O(km)$.*

These results considerably improve the previous best kernel bound of $O(k^3)$ vertices by Crowston et al. [4]. Moreover, the presented kernel completely resolves the asymptotic kernelization complexity of (SIGNED) MAX-CUT AEE, since a kernel with $o(k)$ vertices would again contradict the Exponential-Time Hypothesis, as the MAX-CUT problem can be solved by checking all vertex bipartitions. On top of that, our kernelization is also *fast*. In fact, we only need to compute $O(k)$ DFS/BFS trees. The rest of the algorithm runs in time $O(m + n)$.

Extensions to Strongly λ -Extendable Properties. As mentioned, the property of graphs having large bipartite subgraphs can be generalized to λ -extendable properties as defined by Poljak and Turzík [25] (we defer the formal definitions to Section 2). For a given λ -extendable property Π , we consider the following problem.

ABOVE POLJAK-TURZÍK BOUND(Π)

Input: A connected graph G and an integer k .

Question: Does G have a spanning subgraph $H \in \Pi$ s.t. $|E(H)| \geq \lambda \cdot |E(G)| + \frac{1-\lambda}{2} \cdot (|V(G)| - 1) + k$?

Note the slight change in the definition of k compared to (SIGNED) MAX-CUT AEE, where k was divided by 4 = $\frac{2}{1-\lambda}$ for $\lambda = \frac{1}{2}$.

Crowston et al. [4] gave polynomial kernels with $O(k^3)$ or $O(k^2)$ vertices for the problem ABOVE POLJAK-TURZÍK(Π), for all strongly λ -extendable properties Π on possibly oriented and/or labeled graphs satisfying at least one of the following properties.

(P1) $\lambda \neq \frac{1}{2}$; or

(P2) $G \in \Pi$ for all graphs G whose underlying simple graph is K_3 ; or

(P3) Π is a hereditary property of simple or oriented graphs.

Our third result improves *all* these kernels for strongly λ -extendable properties to asymptotically optimal $O(k)$ vertices:

► **Theorem 3.** *Let Π be any strongly λ -extendable property of (possibly oriented and/or labeled) graphs satisfying (P1), or (P2), or (P3). Then ABOVE POLJAK-TURZÍK(Π) admits a kernel with $O(k)$ vertices, which is computable in time $O(km)$.*

Consequences for Acyclic Subdigraphs. Theorem 3 has several applications. For instance, Raman and Saurabh [27] asked for the parameterized complexity of the MAX ACYCLIC SUBDIGRAPH problem above the Poljak-Turzík bound: Given a weakly connected oriented graph G on n vertices and m arcs, does it have an acyclic sub-digraph of at least $m/2 + (n - 1)/4 + k$ arcs? For this problem, Crowston et al. [6] gave an algorithm with run time $2^{O(k \log k)} \cdot n^{O(1)}$ and showed a kernel with $O(k^2)$ vertices. They explicitly asked whether the kernel size can be improved to $O(k)$ vertices, and whether the run time can be improved to $2^{O(k)} \cdot n^{O(1)}$. Here, we answer their questions in the affirmative by using Theorem 3 and

then applying an $O^*(2^n)$ -time algorithm by Raman and Saurabh [28, Thm. 2] to our kernel with $O(k)$ vertices.

► **Corollary 4.** *The MAX ACYCLIC SUBDIGRAPH problem parameterized above Poljak-Turzík bound admits a kernel with $O(k)$ vertices and can be solved in time $2^{O(k)} \cdot n^{O(1)}$.*

Again, assuming the Exponential Time Hypothesis, the run time of this algorithm is asymptotically optimal.

Due to space constraints, proofs of statements marked by (\star) are deferred to the full version.

2 Preliminaries

We use \uplus to denote the disjoint union of sets. The term “graph” refers to finite undirected graphs without self-loops, parallel edges, edge directions, or labels. For a graph G , let $V(G)$ denote its set of vertices and let $E(G)$ denote its set of edges. In an oriented graph, each edge $e = \{u, v\}$ has one of two directions, $\vec{e} = (u, v)$ and $\overleftarrow{e} = (v, u)$; thus, an oriented graph is a digraph without 2-cycles and loops. We sometimes write an edge $e = \{u, v\}$ as $e = uv$, if no confusion arises; this way, three distinct vertices a, b, c can *induce* a triangle $abca$. In a labeled graph, each edge in $E(G)$ receives one of a constant number of labels. For an oriented and/or labeled graph G , let $\langle G \rangle$ denote the underlying simple graph obtained from omitting orientations and/or labels. Throughout the paper, we assume graphs to be encoded as adjacency lists.

A graph is *connected* if there is a path between any two of its vertices. A *connected component* of G is a maximal connected subgraph of G . A *cut vertex* of a graph G is a vertex whose removal increases the number of connected components. A graph is *2-connected* if it does not contain any cut vertices. A maximal 2-connected subgraph of a graph G is called a *block* of G . A block that contains at most one cut vertex of G is called a *leaf block* of G . A *clique tree* is a connected graph whose blocks are *cliques*, where a clique is a complete subgraph of a graph. A *clique forest* is a graph whose connected components are clique trees.¹ For an oriented and/or labeled graph G we say that G has one of the above-defined properties if $\langle G \rangle$ does.

Let G be a graph. For a vertex $v \in V(G)$, let $N_G(v) = \{u \in V(G) \mid \{u, v\} \in E(G)\}$. For signed graphs G , we define $N_G(v) = N_{\langle G \rangle}(v)$. For a vertex set $V' \subseteq V(G)$, let $N_G(V') = (\bigcup_{v \in V'} N_G(v)) \setminus V'$. For disjoint vertex sets $V_1, V_2 \subseteq V(G)$, let $E(V_1, V_2)$ denote the set of edges with one endpoint in V_1 and the other endpoint in V_2 . For signed graphs G , let $E^+(G) \subseteq E(G)$ be the edges with positive labels, and $E^-(G) = E(G) \setminus E^+(G)$ be the edges with negative labels. Define $N_G^+(v) = \{u \in V(G) \mid vu \in E^+(G)\}$ and $N_G^-(v) = \{u \in V(G) \mid vu \in E^-(G)\}$ for all $v \in V(G)$.

A *graph property* Π is simply a set of graphs. For a graph G , a Π -subgraph is a subgraph of G that belongs to Π . A graph property Π is *hereditary* if for any $G \in \Pi$ also all vertex-induced subgraphs of G belong to Π . Poljak and Turzík [25] defined the notion of “ λ -extendability” for graph properties Π , and proved a lower bound on the size of any Π -subgraph in arbitrary graphs. A related notion of “strong λ -extendability” was introduced by Mnich et al. [23]; any strongly λ -extendable property is λ -extendable, but it is unclear whether the other direction holds.

¹ Clique forests are sometimes called *block graphs*; however, there are competing definitions for this term in the literature and so we refrain from using it.

► **Definition 5.** Let \mathcal{G} be a class of (possibly labeled and/or oriented) graphs and let $\lambda \in (0, 1)$. A (graph) property Π is *strongly λ -extendable* on \mathcal{G} if it satisfies the following properties:

- (i) *inclusiveness*: $\{G \in \mathcal{G} \mid \langle G \rangle \in K_1, K_2\} \subseteq \Pi$.
- (ii) *block additivity*: $G \in \mathcal{G}$ belongs to Π if and only if each block of G belongs to Π .
- (iii) *extendability*: For any $G \in \mathcal{G}$ and any partition $U \uplus W$ of $V(G)$ for which $G[U], G[W] \in \Pi$ there is a set $F \subseteq E(U, W)$ of size $|F| \geq \lambda|E(U, W)|$ for which $G - (E(U, W) \setminus F) \in \Pi$.

The set of all bipartite graphs $\Pi_{\text{bipartite}}$ is a strongly $\frac{1}{2}$ -extendable property. Thus, MAX-CUT AEE is equivalent to ABOVE POLJAK-TURZÍK BOUND($\Pi_{\text{bipartite}}$).

Poljak and Turzík[25] showed that, given a (strongly) λ -extendable property Π , any connected graph G contains a subgraph H with at least $\lambda|E(G)| + \frac{1-\lambda}{2}(|V(G)| - 1)$ edges such that $H \in \Pi$. We denote this lower bound by $\text{pt}(G)$. Further, we define the *excess* of G over this lower bound with respect to Π as $\text{ex}(G) = \max\{|E(H)| - \text{pt}(G) \mid H \subseteq G, H \in \Pi\}$. When considering properties of labeled and/or oriented graphs, we denote by $\text{ex}(K_t)$ the minimum value of $\text{ex}(G)$ over all labeled and/or oriented graphs G with $\langle G \rangle = K_t$; here, K_t denotes the complete graph of order t . (Our definition slightly differs from the one by Crowston et al. [8].)

A strongly λ -extendable property Π *diverges on cliques* if $\text{ex}(K_j) > \frac{1-\lambda}{2}$ for some $j \in \mathbb{N}$. For example, every strongly λ -extendable property with $\lambda \neq \frac{1}{2}$ diverges on cliques [8]. We recall the following fact about diverging properties:

► **Proposition 6** ([8, Lemma 8]). *Let Π be a strongly λ -extendable property diverging on cliques, and let $j \in \mathbb{N}, a > 0$ be such that $\text{ex}(K_j) = \frac{1-\lambda}{2} + a$. Then $\text{ex}(K_i) \geq ra$ for all $i \geq rj$.*

We need the following proposition in all sections. For SIGNED MAX CUT, we will apply it with $\lambda = \frac{1}{2}$.

► **Proposition 7** ([8, Lemma 6]). *Let Π be a strongly λ -extendable property, let G be a connected graph and let $U_1 \uplus U_2$ be a partition of $V(G)$ into non-empty sets U_1, U_2 . For $i = 1, 2$ let c_i be the number of connected components of $G[U_i]$. If $\text{ex}(G[U_i]) \geq k_i$ for some $k_i \in \mathbb{R}$ and $i = 1, 2$, then $\text{ex}(G) \geq k_1 + k_2 - \frac{1-\lambda}{2}(c_1 + c_2 - 1)$.*

3 Linear-Time Fixed-Parameter Algorithms and Linear Vertex Kernels for Signed Max Cut

In this section we consider the SIGNED MAX-CUT AEE problem. We show that the fixed-parameter algorithm given by Crowston et al. [4] can be implemented in such a way that it runs in time $8^k \cdot O(|E(G)|)$. That is, given a connected graph G whose edges are labeled either positive (+) or negative (-), and an integer k , we can decide in time $8^k \cdot O(|E(G)|)$ whether G has a balanced subgraph of size $|E(G)|/2 + (|V(G)| - 1 + k)/4$. This will prove Theorem 1. In the second part of the section we will show how to obtain a kernel with $O(k)$ vertices and thus prove Theorem 2.

Let us first reformulate the SIGNED MAX-CUT AEE problem.

► **Proposition 8** (Harary [13]). *A signed graph G is balanced if and only if there exists a partition $V_1 \uplus V_2 = V(G)$ such that all edges in $G[V_1]$ and $G[V_2]$ are positive and all edges $E(V_1, V_2)$ between V_1 and V_2 are negative.*

3.1 Linear-Time Fixed-Parameter Algorithm

The algorithm by Crowston et al. [4] starts by applying the following seven reduction rules. We restate them here, as they are crucial for our results. A reduction rule is *1-safe* if, on input

(G, k) it returns a pair (G', k') such that (G, k) is a “yes”-instance for SIGNED MAX-CUT AEE if (G', k') is. (Note that the converse direction does not have to hold.) In a signed graph G we call a triangle *positive* if its number of negative edges is even. In the description of the rules, G is always a connected signed graph and C is always a clique that does not contain a positive triangle.

► **Reduction Rule 9.** *If $abca$ is a positive triangle such that $G - \{a, b, c\}$ is connected, then mark a, b, c , delete them, and set $k' = k - 3$.*

► **Reduction Rule 10.** *If $abca$ is a positive triangle such that $G - \{a, b, c\}$ has exactly two connected components C and Y , then mark a, b, c , delete them, delete C , and set $k' = k - 2$.*

► **Reduction Rule 11.** *Let C be a connected component of $G - v$ for some vertex $v \in V(G)$. If there exist $a, b \in V(C)$ such that $G - \{a, b\}$ is connected and there is an edge av but no edge bv , then mark a, b , delete them, and set $k' = k - 2$.*

► **Reduction Rule 12.** *Let C be a connected component of $G - v$ for some vertex $v \in V(G)$. If there exist $a, b \in C$ such that $G - \{a, b\}$ is connected and $vabv$ is a positive triangle, then mark a, b , delete them, and set $k' = k - 4$.*

► **Reduction Rule 13.** *If there is a vertex $v \in V(G)$ such that $G - v$ has a connected component C such that $G[V(C) \cup \{v\}]$ is a clique that does not contain a positive triangle, then delete C . If $|V(C)|$ is odd, then set $k' = k - 1$. Otherwise, set $k' = k$.*

► **Reduction Rule 14.** *If abc is a vertex-induced path in G for some vertices $a, b, c \in V(G)$ such that $G - \{a, b, c\}$ is connected, then mark a, b, c , delete them, and set $k' = k - 1$.*

► **Reduction Rule 15.** *Let C, Y be the connected components of $G - \{v, b\}$ for some vertices $v, b \in V(G)$ such that $vb \notin E(G)$. If $G[V(C) \cup \{v\}]$ and $G[V(Y) \cup \{b\}]$ are cliques that do not contain a positive triangle, then mark v, b , delete them, delete C , and set $k' = k - 1$.*

We slightly changed Rule 13. Crowston et al. [4] always set $k' = k$, whereas we set $k' = k - 1$ when $|V(C)|$ is odd. In this case, $\text{pt}(G[V(C) \cup \{v\}])$ cannot be integral because $|V(C) \cup \{v\}|$ is even, and thus $\text{ex}(G[V(C) \cup \{v\}]) \geq \frac{1}{4}$. Therefore our change for k is 1-safe due to the following result.

► **Proposition 16** ([4, Lemma 2]). *Let G be a connected signed graph and Z be a connected component of $G - v$ for some $v \in V(G)$. Then $\text{ex}(G) = \text{ex}(G - Z) + \text{ex}(G[V(Z) \cup \{v\}])$.*

We subsume the results by Crowston et al. [4] in the following proposition.

► **Proposition 17** ([4]). *Rules 9–15 are 1-safe. To any connected signed graph with at least one edge, one of these rules applies and the resulting graph is connected. If S is the set of vertices marked during the exhaustive application of Rules 9–15 on a connected signed graph G , then $G - S$ is a clique forest. If $|S| > 3k$, then (G, k) is a “yes”-instance.*

Following Crowston et al. [4, Corollary 3], we assume – without loss of generality – from now on that the resulting clique forest $G - S$ does not contain a positive edge.

► **Lemma 18** (\star). *Let G be a connected signed graph, let X be a leaf block of G , and let $r \in V(G)$ such that $V(X) \setminus \{r\}$ does not contain a cut vertex of G . Then we can apply one of the Rules 9–15 to G deleting and marking only vertices from X in time $O(|E(X)|)$.*

Given an instance (G, k) , we can thus compute in time $O(k \cdot |E(G)|)$ a vertex set S that either proves that (G, k) is a “yes”-instance or $G - S$ is a clique forest. We now show that, if a partition for the vertices in S is already given, we can in time $O(|E(G)|)$ compute an optimal extension to G . We use the following problem, which goes back to Crowston et al. [7].

MAX-CUT EXTENSION

Input: A clique forest G_S with weight functions $w_i : V(G_S) \rightarrow \mathbb{N}_0$ for $i = 0, 1$.

Task: Find an assignment $\varphi : V(G_S) \rightarrow \{0, 1\}$ maximizing $\sum_{xy \in E(G_S)} |\varphi(x) - \varphi(y)| + \sum_{i=0}^1 \sum_{x: \varphi(x)=i} w_i(x)$.

► **Lemma 19** (\star). MAX-CUT EXTENSION can be solved in time $O(|V(G_S)| + |E(G_S)|)$ on a clique forest G_S .

We now give a proof sketch for Theorem 1. Lemma 18 allows us to find the set S from Proposition 17 in time $O(km)$ (the case that k is not decreased can only take $O(m)$ total time). Guess one of the at most 2^{3k} partitions on S and solve the corresponding MAX-CUT EXTENSION problem with Lemma 19.

3.2 A Linear Vertex Kernel for Signed Max-Cut AEE

For the whole section, let G^0 be the original graph, let S be the set of marked vertices during the exhaustive application of Rules 9–15 on G^0 , and let G^r be the resulting graph after the exhaustive application of our kernelization Rules 20–21 (to be defined later) on G^0 .

If there is a (unique by Proposition 17) remaining vertex v left after the exhaustive application of Rules 9–15, then add a path vw to G , i.e., define $G' = (V(G) \cup \{w, x\}, E(G) \cup \{vw, wx\})$. Then $(G', k + 2)$ is an instance of MAX-CUT AEE that is due to Proposition 16 equivalent to (G, k) because the excess of a path of length 2 is $2/4$. This implies that we can w.l.o.g. assume that every vertex gets removed during the exhaustive application of the reduction rules because we can assume we finish with deleting the new path with Rule 14. Furthermore, as Rule 13 can then not be applied last, we can assume that at least one of the vertices that are removed last is contained in S .

We will use two-way reduction rules which are similar to the two-way reduction rules by Crowston et al. [4]. However, our two-way reduction rules have the property that connected components of $G - S$ cannot fall apart, i.e., two blocks in $G^r - S$ are reachable from each other if and only if the corresponding blocks in $G^0 - S$ are reachable from each other. We can thus show that Rules 9–15 can behave “equivalently” on G^r as on G^0 (Lemma 24), i.e., that the same set S of vertices can also be marked in G^r . This is the crucial idea which allows us to obtain better kernelization results than previous papers, as it allows the following analysis.

To show size bounds for our kernel G^r , we (hypothetically) change the set of rules in such a way that whenever a vertex $s \in S$ is about to be removed, we additionally remove internal vertices from different blocks of $G^r - S$ that are all adjacent to s . This means that for every $s \in S$, we find a star-like structure Y_s such that Y_s is removed together with s , and the excess on Y_s grows linearly in $|Y_s|$. We can distribute the internal vertices from $G - S$ in such a way to the different Y_s that all generated graphs are still connected. Then the large excess of the different Y_s translates to a large excess of G^r through Proposition 7.

We use this approach twice to first bound the number of special blocks (Lemma 25) and then the number of internal vertices in special blocks (Lemma 27) to $O(k)$. On the other

hand, due to Rules 20–21 a constant fraction of vertices in $G^r - S$ must be adjacent to S . This completes the proof.

Let C be a block in the clique forest $G - S$. Define $C_{\text{int}} = \{v \in V(C) \mid N_{G-S}(v) \subseteq V(C)\}$ as the *interior* of C , and $C_{\text{ext}} = V(C) \setminus C_{\text{int}}$ as the *exterior* of C . The block C is called *special* if $C_{\text{int}} \cap N_G(S)$ is non-empty. Let \mathcal{B} be the set of blocks and \mathcal{B}_s be the set of special blocks in $G^r - S$. A block C is a Δ -*block* if it is not special, contains exactly three vertices, and $|C_{\text{ext}}| \leq 2$.

We now give our two-way reduction rules, which on input (G, k) produce an instance (G', k) of SIGNED MAX-CUT AEE. Note that the parameter k does not change. We call a rule *2-safe* if (G, k) is a “yes”-instance if and only if (G', k) is. The first rule is again due to Crowston et al. [4], who showed it to be 2-safe. The run time analysis is our work. Recall our assumption that (without loss of generality) $G - S$ does not contain any positive edges.

► **Reduction Rule 20.** *Let C be a block in $G - S$. If there exists $X \subseteq C_{\text{int}}$ such that $|X| > \frac{|V(C)| + |N_G(X) \cap S|}{2} \geq 1$, $N_G^+(x) \cap S = N_G^+(X) \cap S$ and $N_G^-(x) \cap S = N_G^-(X) \cap S$ for all $x \in X$, then delete two arbitrary vertices $x_1, x_2 \in X$.*

► **Reduction Rule 21.** *Let C_1, C_2 be two Δ -blocks in $G - S$ which share a common vertex v . Make a block out of $V(C_1) \cup V(C_2)$, i.e., add negative edges $\{\{u, w\} \mid u \in V(C_1) \setminus \{v\}, w \in V(C_2) \setminus \{v\}\}$ to G .*

► **Lemma 22** (\star). *Rules 20–21 are 2-safe. If they are applied to a connected graph G , then the resulting graph G' is also connected.*

► **Lemma 23** (\star). *Given S , Rules 20–21 can be applied exhaustively to G^0 in total time $O(m+n)$.*

► **Lemma 24** (\star). *Rules 9–15 can be applied exhaustively to the graph G^r in such a way that the set S' of marked vertices is equal to S . Moreover, if only the Rules 11/13/14/15 are applied to G^0 , the same set of rules is applied to G^r .*

The last part of the lemma will be needed later in Section 4.2.

► **Lemma 25** (\star). *If $G^r - S$ has more than $11k$ special blocks, then (G^r, k) is a “yes”-instance of SIGNED MAX-CUT AEE.*

► **Lemma 26** (\star). *If $G^r - S$ has more than $48k$ blocks, then (G^r, k) is a “yes”-instance of SIGNED MAX-CUT AEE. Otherwise, $G^r - S$ has at most $48k$ external vertices, and $\sum_{B \in \mathcal{B}} |B_{\text{ext}}| \leq 96k$.*

► **Lemma 27** (\star). *If there are more than $117k$ internal vertices in special blocks in $G^r - S$, then (G^r, k) is a “yes”-instance of SIGNED MAX-CUT AEE.*

We are now ready to prove Theorem 2.

Proof of Theorem 2. Let (G^0, k) be an instance of SIGNED MAX-CUT AEE. Like in Section 3.1, apply Rules 9–15 exhaustively to (G^0, k) in time $O(k \cdot |E(G^0)|)$, producing an instance (G', k') and a vertex set S of marked vertices. If $k' \leq 0$, then (G', k') and thus also (G, k) is a “yes”-instance.

Now apply Rules 20–21 exhaustively to (G^0, k) in time $O(|E(G)|)$ (Lemma 23) to obtain an equivalent instance (G^r, k) . Check whether (G^r, k) is a “yes”-instance due to Lemma 26 or Lemma 27. If this is not the case, then there are at most $3k$ vertices in S , at most $48k$ external vertices in $G^r - S$ and at most $117k$ internal vertices in special blocks. If there

were more internal than external vertices in a non-special block, we could apply Rule 20 to this block. Thus, the number of internal vertices in non-special blocks is bounded by $96k$ according to Lemma 26. Hence, the total number of vertices in G^r is bounded by $3k + 48k + 117k + 96k = 264k$. ◀

4 Linear Vertex Kernels for λ -Extendable Properties

In this section we extend our linear kernels for SIGNED MAX-CUT to all strongly λ -extendable properties satisfying (P1), or (P2), or (P3). Henceforth, fix a strongly λ -extendable property Π , and let (G^0, k) be an instance of ABOVE POLJAK-TURZÍK BOUND(Π). For notational brevity, we assume the empty graph to be in Π .

As in the previous section, we use a set of 1-safe reduction rules devised by Mnich et al. [23] to find a set S such that $G^0 - S$ is a clique forest; the difference compared to SIGNED MAX-CUT is the different change of k . Since we change the reduction rules slightly in the next section, we refrain from stating the rules by Mnich et al. here.

► **Lemma 28** ([23]). *There is an algorithm that, given a connected graph G and $k \in \mathbb{N}$, either decides that $\text{ex}(G) \geq k$, or finds a set S of at most $\frac{6k}{1-\lambda}$ vertices such that $G - S$ is a clique forest. This also holds for all strongly λ -extendable properties of oriented and/or labeled graphs.*

The detection which of the reduction rules can be applied to a graph G is completely analogous to the SIGNED MAX-CUT reduction rules. Hence, it follows immediately from Lemma 18 that the rules can be applied exhaustively in time $O(km)$.

4.1 Linear Kernel for Properties Diverging on Cliques

We show that ABOVE POLJAK-TURZÍK BOUND(Π) admits kernels with $O(k)$ vertices for all strongly λ -extendable properties Π that are diverging on cliques and for which $\text{ex}(K_i) > 0$ for all $i \geq 2$.

► **Lemma 29** (\star). *Let Π be a strongly λ -extendable property diverging on cliques, and suppose that $\text{ex}(K_i) > 0$ for all $i \geq 2$. Then ABOVE POLJAK-TURZÍK BOUND(Π) admits a kernel with $O(k)$ vertices.*

► **Theorem 30**. *Let Π be a strongly λ -extendable property. If $\lambda \neq \frac{1}{2}$ or $G \in \Pi$ for every G with $\langle G \rangle = K_3$, then ABOVE POLJAK-TURZÍK BOUND(Π) has a kernel with $O(k)$ vertices.*

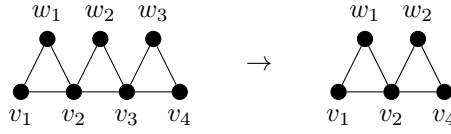
Proof. Lemmas 24–26 from Crowston et al. [8] show that if $\lambda \neq \frac{1}{2}$ or $K_3 \in \Pi$, then Π diverges on cliques and $\text{ex}(K_i) > 0$ for all $i \geq 2$. Therefore, we can apply Lemma 29. ◀

4.2 Strongly $\frac{1}{2}$ -Extendable Properties on Oriented Graphs

We now turn to strongly $\frac{1}{2}$ -extendable properties Π on oriented graphs. First of all we modify the reduction rules by Mnich et al. [23] in such a way that they are compliant with Rules 9–15. Let G always be a connected graph.

► **Reduction Rule 31**. *Let C be a connected component of $G - v$ for some vertex $v \in V(G)$ such that $G[V \cup \{v\}]$ is a clique. Delete C and set $k' = k$.*

► **Reduction Rule 32**. *Let C be a connected component of $G - v$ for some vertex $v \in V(G)$ such that C is a clique. If there exist $a, b \in V(C)$ such that $G - \{a, b\}$ is connected and $av \in E(G)$, but $bv \notin E(G)$, then mark a, b , delete them, and set $k' = k - \frac{1}{2}$.*



■ **Figure 1** Illustration of Rule 38.

► **Reduction Rule 33.** Let abc be a vertex-induced path for some vertices $a, b, c \in V(G)$ such that $G - \{a, b, c\}$ is connected. Mark a, b, c , delete them, and set $k' = k - \frac{1}{4}$.

► **Reduction Rule 34.** Let $v, b \in V(G)$ such that $vb \notin E(G)$ and $G - \{v, b\}$ has exactly two connected components C, Y . If $G[V(C) \cup \{v\}]$ and $G[V(C) \cup \{b\}]$ are cliques, then mark v, b , delete them, delete C , and set $k' = k - \frac{1}{4}$.

Rules 31–34 are exactly Rules 13/11/14/15 for SIGNED MAX-CUT AEE with all edges negative.

► **Lemma 35** (\star). Rules 31–34 are 1-safe. To any connected graph with at least one edge, one of the rules applies and the resulting graph is connected. If S is the set of marked vertices, then $G - S$ is a clique forest. If $|S| > 12k$, then (G, k) is a “yes”-instance.

Like Crowston et al. [8], we restrict ourselves to hereditary properties. Let \vec{K}_3 be the orientation of K_3 which is an oriented cycle, and let \overleftarrow{K}_3 be the only (up to isomorphisms) other orientation of K_3 . Crowston et al. [8] showed that if $\vec{K}_3 \in \Pi$, then also $\overleftarrow{K}_3 \in \Pi$, and thus Theorem 30 applies. We now consider the case that $\vec{K}_3 \notin \Pi$ together with $\overleftarrow{K}_3 \in \Pi$.

► **Proposition 36** ([8]). Let Π be a hereditary strongly $\frac{1}{2}$ -extendable property on oriented graphs with $\overleftarrow{K}_3 \in \Pi$. Then $\text{ex}(K_i) > 0$ for all $i \geq 4$ and Π diverges on cliques.

Following this lemma, the conditions of Lemma 29 are almost satisfied. The only oriented cliques without positive excess are K_1 and \vec{K}_3 , because $\text{ex}(K_2) = \frac{1}{4}$ for $\frac{1}{2}$ -extendable properties. Blocks isomorphic to K_1 can only occur as isolated vertices in $G - S$. We can bound these like in the previous section. Hence, we only need reduction rules to bound the number of blocks B in a clique forest with $B \cong \vec{K}_3$.

Let Π be a hereditary strongly $\frac{1}{2}$ -extendable property on oriented graphs with $\overleftarrow{K}_3 \in \Pi$. Let (G^0, k) be an instance of ABOVE POLJAK-TURZÍK(Π). Lemma 35 either proves that (G^0, k) is a “yes”-instance, or it finds a set S of at most $12k$ vertices such that $G^0 - S$ is a clique forest. Starting with (G^0, k) , we apply the following reduction rules, which on input (G, k) produce an equivalent instance (G', k) .

► **Reduction Rule 37.** Delete B_{int} of leaf blocks B in $G - S$ with $B \cong \vec{K}_3$ and $N_G(S) \cap B_{\text{int}} = \emptyset$.

► **Reduction Rule 38.** Let B_1, B_2, B_3 be non-leaf-blocks in $G - S$ and $v_1, \dots, v_4 \in V(G)$ be such that (i) $v_i, v_{i+1} \in (B_i)_{\text{ext}}$ for all $i \in \{1, 2, 3\}$; (ii) $B_i \cong \vec{K}_3$ for all $i \in \{1, 2, 3\}$; and (iii) $N_G(\{v_2, v_3, w_1, w_2, w_3\}) = \{v_1, v_4\}$, where w_i is the internal vertex of B_i . Delete v_3 and w_3 . Add edges v_2v_4 and w_2v_4 .

Intuitively speaking, Rule 38 takes three blocks in $G - S$ that form a “path” and are all isomorphic to \vec{K}_3 . If all vertices except the “endpoints” v_1 and v_4 are not adjacent to S , then it is safe to delete one block. For an illustration, see Fig. 1.

► **Lemma 39** (★). *Let Π be a hereditary strongly $\frac{1}{2}$ -extendable property on oriented graphs with $\vec{K}_3 \in \Pi$. Then Rules 37–38 are 2-safe. The resulting graphs are connected.*

From now on, let G^r be the resulting graph after the exhaustive application of Rules 37–38 on G^0 . Rules 37–38 are special cases of Rules 20–21. Because Rules 31–34 are Rules 13/11/14/15 for SIGNED MAX-CUT AEE with all edges negative, the next lemma follows from Lemma 24.

► **Lemma 40**. *Rules 31–34 can be applied exhaustively on the graph G^r in such a way that the set S' of vertices removed by their application is equal to S .*

Let \mathcal{B}^+ be the set of blocks of $G^r - S$ with positive excess, and let \mathcal{B}^- be the other blocks, i.e., the blocks B with $B \cong \vec{K}_3$ or $B \cong K_1$. Let $R \subseteq V(G) \setminus S$ be the set of vertices that are only contained in exactly two blocks $B_1, B_2 \in \mathcal{B}^-$ such that $(B_1)_{\text{int}} = (B_2)_{\text{int}} = \emptyset$. Further, let $V^+ \subseteq V(G) \setminus S$ be the set of vertices in blocks with positive excess, V^- be the set of vertices in blocks from \mathcal{B}^- , and let $V_{\text{int}}^- \uplus V_{\text{ext}}^- = V^-$ be the set of internal and external vertices of blocks $B \in \mathcal{B}^-$, respectively. Note that V^+ and V^- may intersect.

► **Lemma 41** (★). *It holds $|V^-| = O(|(R \cup V_{\text{int}}^-) \cap N_{G^r}(S)|)$. Furthermore, if $|(R \cup V_{\text{int}}^-) \cap N_{G^r}(S)| > 48k$, then (G^r, k) is a “yes”-instance.*

Using the same approach as in Section 4.1, one can show that $|V^+| = O(k)$ or (G^r, k) is a “yes”-instance. As Lemma 41 bounds $|V^-| = O(k)$ for every “no”-instance, and $V^+ \cup V^- \cup S = V(G^r)$, this suffices to prove the following result.

► **Theorem 42** (★). *Let Π be a hereditary strongly $\frac{1}{2}$ -extendable property on oriented graphs with $\vec{K}_3 \in \Pi$. Then ABOVE POLJAK-TURZÍK BOUND(Π) admits a kernel with $O(k)$ vertices.*

Proof of Theorem 3. Let $\lambda \in (0, 1)$ and let Π be a strongly λ -extendable property of (possibly oriented and/or labeled) graphs. If $\lambda \neq \frac{1}{2}$ or $G \in \Pi$ for every G with $\langle G \rangle = K_3$, we can use Theorem 30. Otherwise, we only have to consider the case that Π is a hereditary property of simple or oriented graphs.

Consider the case that $\vec{K}_3 \in \Pi$ or $\vec{K}_3 \in \Pi$. If $\vec{K}_3 \in \Pi$, then Crowston et al. [8] show that $\vec{K}_3 \in \Pi$, i.e., we can use Theorem 30. And if $\vec{K}_3 \in \Pi$, we use Theorem 42.

Now we may suppose that $G \notin \Pi$ for every G with $\langle G \rangle = K_3$. Then Crowston et al. [8] show that Π is the set of all bipartite graphs. Hence, in the case of simple graphs as well as if $\vec{K}_3, \vec{K}_3 \notin \Pi$ for oriented graphs, we can use Theorem 2 to obtain a linear vertex kernel.

It is easy to see that Rules 37–38 can be applied exhaustively in time $O(m)$. As λ is constant and we can apply every other reduction rule in linear time, it follows a total run time of $O(\lambda \cdot km) = O(km)$. ◀

5 Discussion

For the classical (SIGNED) MAX-CUT problem, and its wide generalization to strongly λ -extendable properties, parameterized above the classical Poljak-Turzík bound, we improved the run time analysis for a known fixed-parameter algorithm to $8^k \cdot O(m)$. We further improved all known kernels with $O(k^3)$ vertices for these problems to asymptotically optimal $O(k)$ vertices. We did not try to optimize the hidden constants, as the analysis is already quite cumbersome.

It remains an interesting question whether all positive results presented here extend to edge-weighted graphs, where each edge receives a positive integer weight and the number m of edges in the Edwards-Erdős bound (1) is replaced by the total sum of the edge weights.

Further, Mnich et al. [23] showed fixed-parameter tractability of ABOVE POLJAK-TURZÍK BOUND(Π) for *all* strongly λ -extendable properties Π . However, the polynomial kernelization results by Crowston et al. [8] as well as in this paper do not seem to apply to the special case of non-hereditary $\frac{1}{2}$ -extendable properties. Such properties Π exist; e.g., $\Pi = \{G \in \mathcal{G} \mid C \not\cong K_3 \text{ for all 2-connected components } C \text{ of } G\}$. Also, for $\frac{1}{2}$ -extendable properties on labeled graphs we only showed a polynomial kernel for the special case of SIGNED MAX-CUT. It would be desirable to avoid these restrictions.

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