An Improved Tax Scheme for Selfish Routing*

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Abstract

We study the problem of routing traffic for independent selfish users in a congested network to minimize the total latency. The inefficiency of selfish routing motivates regulating the flow of the system to lower the total latency of the Nash Equilibrium by economic incentives or penalties. When applying tax to the routes, we follow the definition of [8] to define ePoA as the Nash total cost including tax in the taxed network over the optimal cost in the original network. We propose a simple tax scheme consisting of step functions imposed on the links. The tax scheme can be applied to routing games with parallel links, affine cost functions and single-commodity networks to lower the ePoA to at most $\frac{4}{3} - \epsilon$, where ϵ only depends on the discrepancy between the links. We show that there exists a tax scheme in the two link case with an ePoA upperbound less than 1.192 which is almost tight. Moreover, we design another tax scheme that lowers ePoA down to 1.281 for routing games with groups of links such that links in the same group are similar to each other and groups are sufficiently different.

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1 Introduction

We study the problem of routing traffic for independent selfish users in a congested network to minimize the total cost (latency). In many settings, it is very expensive or impossible to regulate the traffic precisely. In the absence of regulation, users usually only focus on minimizing his own cost measured by the total time needed to traverse his chosen route. Many works focus on the degradation in network performance measured by comparing the cost of the Nash equilibrium flow and the cost of the optimal setting. The ratio of total cost of Nash Equilibria to the minimum possible cost is defined to be the Price of Anarchy (PoA). Therefore one could consider PoA as an index of the inefficiency of the lack of regulation in a network of selfish behavior. In [25], it is proven that the PoA is $\leq \frac{4}{3}$ for

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affine latency functions, and the upper bound $\frac{4}{3}$ is tight in a well-known example called the Pigue's example [20]. There are many well-known works on the selfish routing game, such as [22, 23, 24, 21, 6, 18].

The inefficiency of selfish routing motivates regulating the flow of the system to lower the total latency of the Nash Equilibria by economic incentives or penalties. Marginal cost pricing is an ancient idea proposed in [20]. Marginal cost taxes may induce the minimum-latency flow as a flow at a Nash equilibrium, assuming all network users choose the routes to minimize the sum of latency and tax [2]. One major research is to lower price of anarchy to 1 for users having different sensitivity to tax in a single-commodity network [10], with an upper bound of tax with complexity $O(n^3)$. Several further researches improved the result above, such as generalizing the result for single commodity to multi-commodity [12, 15] and generalizing the result for giving an tax upperbound with complexity $O(n)$ [11]. In [4], optimal tax with constraints can be derived in certain circumstances. Another similar concept is the coordination mechanisms introduced in [7]. Coordination Mechanisms have been used to improve the PoA in scheduling problems for parallel and related machines [7, 14, 17] as well as for unrelated machines [1, 5].

In the above researches, the system is efficient only if the tax is returned to the users, otherwise dis-utility for users due to large tax may exist. In [16], an PoA upperbound of 2 is given if tax is included as a part of the cost. The bound becomes 5/4 particularly for affine latency case. On the other hand, it has been proven [9] that marginal tax could not help reduce total cost if tax is considered as a part of the cost for affine cost functions. It is also proven [8] that continuous tax functions yield no improvement to the total latency.

In the above modelings, the total flow r is specified as a part of the game. However, there are situations that the total flow is unknown beforehand, thus finding a good tax scheme becomes more difficult. Christodoulou *et al.* [8] studied this type of problem for single-commodity routing games with affine cost functions. They designed a tax scheme such that the PoA is at most $\frac{4}{3} - \epsilon$ over all possible amount of flow, where ϵ is a constant that approaches 0 when the number of links go to infinity. In this work, arbitrary tax function is allowed as along as the sum of tax and the original cost (latency) function is monotone increasing.

In our work, we focus on step-function congestion tolls. This type of tax scheme has been studied by transport economists to model the effects of the traffic lights on traffic regulations [13, 19]. Compared to arbitrary tax schemes, the step-function congestion tolls is more feasible in transportation regulations. This motivates us to investigate the possibility of improving ePoA using only step-function congestion tolls in settings similar to [8].

Our Result: We provide a simple tax scheme consisting of step functions imposed on the links. The tax scheme is applied to routing games with parallel links, affine cost function, single-commodity networks to lower the ePoA below $\frac{4}{3} - \epsilon$, where ϵ depends on the discrepancy between the links but not the number of links. Moreover, we consider a special case in which all links can be clustered into several groups. The latency function is similar among links in the same group and are sufficiently different between links in different groups. Each group could be seen as different transportation methods. For example, all freeway may belong to one group, and all local roads and railroad may each belong to another group. In this case, we propose a tax scheme which reduces ePoA to 1.281.

The rest of the paper is organized as follows. In Section 2 we describe the basic routing game model and the type of tax scheme that we will use. In Section 3 we define the parameters in the tax scheme more formally and prove some essential results on the relationship between the ePoA and the imposed tax. In Section 4 we show that step function tolls perform equally well to previous optimal (arbitrary) tax scheme for networks with two parallel links. In Section 5 we propose a tax scheme for $\frac{4}{3} - \epsilon$ ePoA. In Section 6 we give an 1.281 upperbound of ePoA for networks with groups of similar links.

2 Model

We consider single-commodity congestion games on networks, defined by a directed traffic network $G = (V, E, l)$, with vertex set V, edge set E, and cost function (or latency function such as [25]) set l. l is the set of cost function l_e for each link $e \in E$. There is only a start node and an end node in V, while each link $e \in E$ connects the start node directly to the end node, and we denote all links $E = \{e_1, e_2, \dots, e_m\}$. r is defined to be the rate of traffic or the total flow, which is independent of the network G. Unlike some previous works, r is not part of the game. We aim to lower the ePoA for any value of r, instead of choosing a different tax scheme for different value at r. A flow is a function that maps every link $e \in E$ to a non-negative real number. Given G and r, we call a flow feasible if $\Sigma_{e\in E} f_e = r$.

l_e is the cost function of link $e \in E$, which is non-decreasing, non-negative and affine. Therefore we order the links by an increasing order of the constant of the latency of the links. Without loss of generality, we let $l_{e_i}(f) = a_i \cdot f + b_i$ and $b_i \le b_j$ for any $i, j > 0$ such that $i < j$.

The concept of User Equlibrium [3] is adopted as Nash Equilibrium in this work. Formally, a flow f feasible for traffic network G and total rate r, is at User Equilibrium if and only if for every $e_1, e_2 \in E$ with $f_{e_1} > 0$, $l_{e_1}(f) \le \lim_{\epsilon\to 0} l_{e_2}(f + \epsilon\mathbf{1_{e2}} - \epsilon\mathbf{1_{e1}})$. It has been proven that for the case where all discontinuity is lower semicontinuous, the User Equilibrium exists as a theorem in [3]. The definition follows an equivalent definition in [25] when the latency function is continuous. We call the flow at Nash Equilibrium, or User equilibrium simply the Nash flow in the rest of the paper. The cost of flow f in traffic network G is $C(f, G) = \Sigma_{e\in E} f_e \cdot l_e(f_e)$. We use $C_{opt}(r, G)$ to denote the minimum cost of any flow feasible at rate r, or the cost of the optimal flow. Therefore the optimal flow is the flow that minimizes the cost of flow for given (G, r), which would be referred to as OPT. Moreover, we say that a flow uses j links when there are j links with non-zero flow-value. We use $C_N(r, G)$ to denote the cost of the Nash flow at rate r, while the uniqueness of the Nash equilibrium is guaranteed in theorem in [3]. When the context is clear, we may omit G, using $C(f)$ for $C(f, G)$, $C_{opt}(r)$ for $C_{opt}(r, G)$, $C_N(r)$ for $C_N(r, G)$. The Price of Anarchy is defined as $PoA(r) = \frac{C_N(r)}{C_{opt}(r)}$, and $PoA = \max_{r>0} PoA(r)$. It should be noted that the PoA defined here is not a function of r as in most previous works. The PoA in our work is the worst case of $PoA(r)$ among any r-value for a particular network G. In the remainder of the paper, we focus on single commodity, parallel-link networks $G = (V, E, l)$, where E consists of m links $\{e_1, \cdots, e_m\}$, and cost function of link e_i is of the form $l_{e_i}(f) = a_i \cdot f + b_i$.

2.1 Tax

On each edge, the original cost function before imposing the tax is $l_{e_i}(f) = a_i \cdot f + b_i$, the tax-modified cost function becomes $\hat{l}_{e_i}(f) = \hat{a}_i \cdot f + \hat{b}_i$, and $\hat{a} = a$. The tax scheme used in our work adds tax $\hat{b}_j - b_j$ to the cost for users using link j, where $\hat{b}_j$ is a function of total flow r.

$$\hat{b_j} = b_j + \sum_{i>j} (b_i - b_{i-1}) \cdot h_i \cdot u(r - w_i), \tag{1}$$

where $h_i < 1$ and w_i are constants to be chosen, and u is the unit step function. Note that to guarantee the existence of User Equilibrium, the unit step function is defined to be lower-semicontinuity. One point to be noted is that under this form of tax, the Nash flow accounting tax on any link is non-decreasing while total rate r increases. This is a desired property, which makes taxing feasible and efficient, since rerouting existing traffic when total traffic increases may be very costly if at all possible.

For the taxed network, we consider adding tax to be a modification to the original network. Therefore, we call $\hat{G}$ the tax-modified network obtained by imposing tax on G. All notations for the taxed network $\hat{G}$ is denoted with a hat, such as the expression $\hat{b}_j$ defined above. We specify that the $\hat{C}_N(r)$ is the total cost of the Nash equilibrium flow of the tax modified network at rate r, where the cost of each edge and the Nash flow are both affected by the tax. We formally define $\mathrm{ePoA} = \max_{r>0}\mathrm{ePoA}(r) = \max_{r>0}\mathrm{P\hat{o}A}(r) = \max_{r>0}\frac{\hat{C}_N(r)}{C_{opt}(r)}$.

3 Useful Inequalities on the PoA and Tax

Before proving the main results, we need to prove some lemmas on the cost of the Nash equilibrium and OPT.

▶ **Definition 1.** We follow notations in previous works. Given a traffic network G, let $\lambda_j = 1/a_j$, $\gamma_j = b_j/a_j$, $\Lambda_j = \Sigma_{i=1}^{j}\lambda_i$, $\Gamma_j = \Sigma_{i=1}^{j}\gamma_j$ and $r_j = \Sigma_{i=1}^{j-1}(b_{i+1} - b_i)\Lambda_i$. We also define $u_j = r_j/r_{j-1}$ and $v_j = \Lambda_j/\Lambda_{j-1}$.

Intuitively, r_j is the amount of flow at which the $(j+1)$-th edge starts to have non-zero Nash flow. PoA is locally maximized at each r_j. The tax schemes we design also seeks to reduce PoA near these values.

Cost of the Nash flow and the OPT on this type of traffic network has been well studied, and closed-form expressions were given [8]. We restate some essential results in Lemma 2.

▶ **Lemma 2** ([8]). *The Nash flow uses link j for $r > r_j$ and the OPT uses link j for $r > r_j/2$. If the OPT uses exactly j links at rate r then*

$$C_{opt}(r) = \frac{1}{\Lambda_j}(r^2 + \Gamma_j r) - C_j, \quad \textit{where} \quad C_j = \Big(\sum_{h=1}^{j}\sum_{i=1}^{h}(b_h - b_i)^2\lambda_h\lambda_i\Big)/(4\Lambda_j).$$

If the Nash flow uses exactly j links at rate r then

$$C_N(r) = \frac{1}{\Lambda_j}(r^2 + \Gamma_j r).$$

If $s < r$ and OPT uses exactly j links at s and r then

$$C_{opt}(r) = C_{opt}(s) + \frac{1}{\Lambda_j}((r-s)^2 + (\Gamma_j + 2s)(r-s)).$$

If $s < r$ and the Nash flow uses exactly j links at s and r then

$$C_N(r) = C_N(s) + \frac{1}{\Lambda_j}((r-s)^2 + (\Gamma_j + 2s)(r-s)).$$

Directly from Lemma 2, we know that both the OPT and the Nash flow start to use links with the same b-value simultaneously because $r_i = r_j$ if $b_i = b_j$.

▶ **Lemma 3.** *Given a traffic network G, if there exists an index i such that $b_i = b_{i+1}$, we can find a network G' having one less link than G such that $C_N(r, G) = C_N(r, G'), C_{opt}(r, G) = C_{opt}(r, G')$ for all r.*

Using Lemma 3, given a traffic network G, we can replace all links with the same b values by one link and let the cost of the Nash and the OPT remain the same. Furthermore, if we apply tax in the new game, we can apply the same tax on every corresponding links in the old game, as a result, we only consider traffic networks such that $b_i \neq b_j, \forall i \neq j$ in the rest of the paper.

Informally, the tax scheme we design works in the following way. For every flow value r_i which corresponds to a local maximum in the PoA-r curve, we add a set of step functions which reduces the tax in the flow range $[\alpha r_i, \beta r_i]$ if the original PoA at flow r_i is greater than a certain threshold. This set of step functions has no effect on PoA when the total flow is less than αr_i but increases PoA marginally when the total flow is greater than βr_i. A tax scheme can be described by a set of parameters (T, A, B), where T is the threshold, $A = \{\alpha_1, \cdots, \alpha_m\}$, $B = \{\beta_1, \cdots, \beta_m\}$ describes the range of flow in which PoA is supressed. When the tax is imposed on a flow value r_i, a step function are added onto the original cost functions for the first i links, where the heights and positions of those step functions are chosen such that the Nash flow on these i links stop increasing when the total flow r is between $[\alpha r_i, \beta r_i]$, causing the Nash flow to use new links. The detailed definition of the tax scheme being used is the following:

▶ **Definition 4.** Given a traffic network G, let G_j be an identical network of G with links e_1 to e_{j-1} removed. Let f_j be the Nash flow on a given a network G_j and rate r, let $C_{Nj}(r)$ be the cost of f_j on G_j.

▶ **Definition 5.** Given a traffic network G, constants T, α_i and β_i such that $\alpha_i < 1$, $\beta_i > 1$ for $1 \leq i \leq m$, Let $A = \{\alpha_1, \cdots, \alpha_m\}$, $B = \{\beta_1, \cdots, \beta_m\}$, $S(T)$ be the set of all index i such that PoA$(r_i) > T$ and $\hat{G}$ be the network obtained from applying tax(T, A, B) to G. The parameters h_j and w_j in equation (1) (Section 2.1), which correspond to the heights and the locations of the step functions are chosen as following,

$$h_j = \begin{cases} \left(\frac{C_{Nj}((\beta_j-\alpha_j)\cdot r_j)}{(\beta_j-\alpha_j)\cdot r_j} - \frac{\hat{C_N}(\alpha_j\cdot r_j)}{\alpha_j\cdot r_j}\right)/(b_j - b_{j-1}), & \text{if } j \in S(T) \\ 0, & \text{otherwise.} \end{cases}$$

$$w_j = \alpha_j \cdot r_j.$$

We also set two parameters, $h_{max} = \max_i h_i$, $v_{min} = \min_{i \in S(T)} v_i$.

Follow the definition, we can describe the cost of the Nash flow on $\hat{G}$ with Lemma 6.

▶ **Lemma 6.** *Given a traffic network G, constants T, α_i and β_i such that $\alpha_i < 1$, $\beta_i > 1$ for $1 \leq i \leq m$, and tax(T, A, B) imposed on G,*

$$\hat{C_N}(r) = \hat{C_N}(\alpha_j \cdot r_j) + C_{Nj}(r - \alpha_j \cdot r_j) \quad \textit{for } r \in [\alpha_j \cdot r_j, \beta_j \cdot r_j] \textit{ and } j \in S(T).$$

If the Nash flow uses j links on $\hat{G}$ at rate r,

$$\hat{C_N}(r) = \frac{1}{\Lambda_j}(r^2 + r \cdot \hat{\Gamma}_j(r)) \quad \textit{for } r \notin (\alpha_j \cdot r_j, \beta_j \cdot r_j) \forall j \in S(T),$$

where $\hat{\Gamma}_j(r) = \Sigma_{i=1}^j \hat{b}_i(r)/\hat{a}_i = \Sigma_{i=1}^j \hat{b}_i(r)/a_i$. If the Nash flow uses exactly n links on G_j at rate r,

$$C_{Nj}(r) = \frac{1}{\Lambda_{j+n-1} - \Lambda_{j-1}}(r^2 + (\Gamma_{j+n-1} - \Gamma_{j-1}) \cdot r).$$

Proof. Equations can be derive directly from Lemma 2. ◀

In this paper, all tax schemes are designed in a way that after the tax is being applied, the ePoA is determined by the Nash/OPT costs at total flow $\alpha_i r_i$ or $\beta_i r_i$ for some i. In order to have a good estimate of the ePoA, we first derive Theorem 7 which gives us a good estimate of the original PoA at total flow $\alpha_i r_i$ and $\beta_i r_i$. In this theorem, the first inequality gives a good upper bound on PoA at $\beta_i r_i$ and the second inequality gives a good upper bound on the PoA at $\alpha_i r_i$. All upper bounds are described using parameters Λ_j since these values play an important role in determining the PoA [8].

▶ **Theorem 7.** *If the Nash flow uses exactly j links and the OPT uses exactly h links at rate r then*

$$PoA(r) \leq max\Big\{\frac{4r}{4r - r_{j-1}}, \frac{r^2\Lambda_j^{-1} + r \cdot r_j(\Lambda_{j-1}^{-1} - \Lambda_j^{-1})}{r^2\Lambda_{j-1}^{-1} - \Sigma_{i=j}^{h}(r - r_i/2)^2 \cdot (\Lambda_{i-1}^{-1} - \Lambda_i^{-1})}\Big\}.$$

If the Nash flow uses exactly j-1 links and the OPT uses exactly h links at rate r then

$$PoA(r) \leq max\Big\{\frac{4r}{4r - r_{j-1}}, \frac{r^2\Lambda_{j-1}^{-1}}{r^2\Lambda_{j-1}^{-1} - \Sigma_{i=j}^{h}(r - r_i/2)^2 \cdot (\Lambda_{i-1}^{-1} - \Lambda_i^{-1})}\Big\}.$$

The proof is omitted due to space constraints.

In the PoA-r curve, local maximum only exists at $r = r_j$. The following lemma gives an upper bound on PoA which will be used to show that PoA in the region $[\beta_i r_i, \alpha_{i+1} r_{i+1}]$ is bounded by the PoA of this region's two endpoints.

▶ **Lemma 8.** *Given a traffic network G, if the Nash flow uses exactly j links at rate s and t for $s < t$, then*

$$PoA(r) \leq max\{PoA(s), PoA(t)\}, \forall r \in [s, t]\,.$$

Most of our proof relies on Theorem 7 and Lemma 8, first with Lemma 8 to bound the PoA for total flow far away from the peak values r_i, then with Theorem 7 to provide a good bound for total flow close to these peak values.

As previously mentioned, the step functions that decrease PoA near r_i will increase PoA when the total flow is greater than $\beta_i r_i$. Lemma 9 shows that our tax will only increase the total cost by a constant factor.

▶ **Lemma 9.** *Given a traffic network G, constants T, α_i and β_i such that $\alpha_i < 1$, $\beta_i > 1$ for $1 \leq i \leq m$, let $\hat{G}$ be the traffic network obtained by imposing tax(T, A, B) on G, then*

$$\frac{\hat{C_N}(r)}{C_N(r)} \leq 1 + \frac{h_{max}}{v_{min}}, \quad \textit{for} \quad r \quad \textit{such that} \quad r \notin (\alpha_j \cdot r_j, \beta_j \cdot r_j) \quad \textit{for all} \quad j \in S(T).$$

Proof. For total flow $r \in [r_{j-1}, r_j]$ and $r \notin (\alpha \cdot r_i, \beta \cdot r_i)$ for all $i \in S(T)$, let k be the largest $i \in S(T)$ such that $i < j$, from Lemma 2,

$$\frac{\hat{C_N}(r)}{C_N(r)} \leq \frac{r + \hat{\Gamma}_{j-1}(r)}{r + \Gamma_{j-1}} = 1 + \frac{\sum_{i=1}^{j-1}(\hat{b}_i(r) - b_i)\lambda_i}{r + \Gamma_{j-1}}.$$

The largest possible tax added to a link when $r \in [r_{j-1}, r_j]$ is $h_{max} \cdot b_k$, and only link 1 to k-1 have non-zero tax added rate r,

$$\frac{\hat{C_N}(r)}{C_N(r)} \leq 1 + \frac{\sum_{i=1}^{k-1} h_{max} \cdot b_k \lambda_i}{r + \Gamma_{j-1}} \leq 1 + \frac{h_{max} \cdot b_k \Lambda_{k-1}}{r_{j-1} + \Gamma_{j-1}} = 1 + \frac{h_{max} \cdot b_k \Lambda_{k-1}}{b_{j-1}\Lambda_{j-1}}.$$

Since $k < j$, $b_k \leq b_{j-1}$ and $\Lambda_k \leq \Lambda_{j-1}$,

$$\frac{\hat{C_N}(r)}{C_N(r)} \leq 1 + \frac{h_{max}}{\Lambda_k/\Lambda_{k-1}} = 1 + \frac{h_{max}}{v_k} \leq 1 + \frac{h_{max}}{v_{min}}.$$ ◀

4 The ePoA for Two-Link Networks

In this section, we study the networks with two parallel links. In this special case, we give an upperbound of ePoA for the step function tolls which is 1.192. This result shows that applying step function tolls is as powerful as arbitrary tax scheme proposed in [8]. In fact, when the total flow is between 0 and βr_1, our step function tax is exactly identical to the tax scheme in [8]. When the total flow is greater than βr_1, the previous tax scheme remove the previously added step-function tax and does not impose tax on any link. In this paper, removing the step functions is not allowed. We prove that even though these step functions only increase PoA when the total flow is greater than βr_1, the influence is marginal and the maximum value always happen at total flow βr_1. The proof is omitted due to space constraints.

▶ **Theorem 10.** *Given a two link traffic network G, there always exist a pair of $\alpha \in (\frac{1}{2}, 1), \beta \in (1, \infty)$ such that if $tax(T = 1.192, \{\alpha\}, \{\beta\})$ is imposed, then $ePoA \leq 1.192$.*

5 Upperbound of the ePoA for Multiple Parallel-Link Networks

In this section we consider parallel-link networks. Given a traffic netowrk G, we consider that ratio between two adjacent peak values $\frac{r_i}{r_{i-1}}$. Let $\epsilon = \min(\min_{i>1} u_i, 2) - 1 = \min(\min_{i>1} \frac{r_i}{r_{i-1}}, 2) - 1$. We prove that the ePoA has an upper bound less than $\frac{4}{3} - \frac{1}{3}(\frac{\epsilon}{3})^3$. Notice that in this case, ϵ only depends on the discrepancy between the links and is independent of the number of links in the network. The main result of this section is the following theorem.

▶ **Theorem 11.** *Given a traffic network G, and $tax(T = \frac{4}{3} - (\frac{\epsilon}{3})^3, \{\alpha_1 = \cdots = \alpha_m = 1 - 2(\frac{\epsilon}{3})^3\}, \{\beta_1 = \cdots = \beta_m = 1 + 3(\frac{\epsilon}{3})^3\})$ is imposed. Then*

$$ePoA < \frac{4}{3} - \frac{1}{3}(\frac{\epsilon}{3})^3.$$

Notice that $\frac{r_j}{r_{j-1}}$ is less than $\frac{b_j}{b_{j-1}}$ and increases when the difference between a_j and a_{j-1} increases, and thus is a good indicator of the discrepancy between the links.

In order to prove Theorem 11, we need the following lemmas. Intuitively, we first use Lemma 14 and 15 to prove that the PoA of the original network is at most T when the total flow is αr_j of βr_j. Combining with Lemma 8 and 9, we know that ePoA$\leq T(1 + \frac{h_{max}}{v_{min}})$ for all $r \notin (\alpha r_j, \beta r_j)$. Lemma 16 shows that when the total flow is between αr_j and βr_j, the ePoA is also bounded. Plug in the value of h_{max} and v_{min} from Lemma 12 and 13 to finish the proof. For the constants T, α_i, β_i chosen, we can bound all related parameters needed in Theorem 11 with some straightforward calculations.

▶ **Lemma 12.** *Given a traffic network G, a constant T such that $T > \frac{4+4\epsilon}{3+4\epsilon}$.*

$$v_{min} = min_{i \in S(T)} v_i = min_{i \in S(T)} \frac{\Lambda_i}{\Lambda_{i-1}} > \frac{(2\epsilon - \epsilon^2)T}{4 - 3T}.$$

Proof. By definition of set S, $\mathrm{PoA}(r_i) > T$ for all $i \in S(T)$. From Theorem 7,

$$\mathrm{PoA}(r_j) \le \max\Big\{\frac{4r_j}{4r_j - r_{j-1}}, \frac{r_j^2 \Lambda_{j-1}^{-1}}{r_j^2 \Lambda_{j-1}^{-1} - \Sigma_{j \le i \le h}(r_j - r_i/2)^2 \cdot (\Lambda_{i-1}^{-1} - \Lambda_i^{-1})}\Big\}.$$

Since $r_h > r_{h-1} > \cdots > r_{j+1} \ge (1+\epsilon) r_j \ge (1+\epsilon)^2 r_{j-1}$ and $\Lambda_h^{-1} > 0$,

$$\mathrm{PoA}(r_j) \le \max\Big\{\frac{4+4\epsilon}{3+4\epsilon}, \frac{4}{3 + (2\epsilon - \epsilon^2)\Lambda_{j-1}/\Lambda_j}\Big\}.$$

From condition of T,

$$T < \mathrm{PoA}(r_j) \le \frac{4}{3 + (2\epsilon - \epsilon^2)\Lambda_{j-1}/\Lambda_j} \quad \forall j \in S(T).$$ ◀

▶ **Lemma 13.** $h_j \le \Big(\frac{1}{v_j - 1}(\beta_j - \alpha_j) + (1 - \alpha_j)\Big) \cdot \frac{r_j}{r_j - r_{j-1}}$.

▶ **Lemma 14.** *Given a traffic network G, constants T and α such that $\alpha \le \frac{T + (T^2 - T)^{\frac{1}{2}}}{2}$, $\frac{4\alpha \cdot r_j}{4\alpha \cdot r_j - r_{j-1}} \le T$ and $1 < T < \frac{4}{3}$, then $PoA(\alpha \cdot r_j) \le T$.*

▶ **Lemma 15.** *Given a traffic network G, constants T and β such that $2 > \beta \ge \frac{T}{4(T-1)}$ and $1 < T < \frac{4}{3}$, then $PoA(\beta \cdot r_j) \le T$.*

▶ **Lemma 16.** *Given a traffic network G, constants T, α_i and β_i such that $\alpha_i < 1$, $\beta_i > 1$ for $1 \le i \le m$, and $tax(T, A, B)$ imposed on G,*

$$ePoA(r) \le max\Big\{ePoA(\alpha_j \cdot r_j), (\beta_j - \alpha_j)\frac{\Lambda_j}{\Lambda_j - \Lambda_{j-1}} + (1 - \beta_j + \alpha_j)\Big\}$$
$$\text{for } r \in [\alpha_j \cdot r_j, \beta_j \cdot r_j] \text{ and } j \in S(T).$$

The Proof of Lemma 13 to 16 are omitted due to space constraints.

Proof of Theorem 11. Let $\alpha = \alpha_1 = \cdots = \alpha_m = 1 - 2(\frac{\epsilon}{3})^3$, $\beta = \beta_1 = \cdots = \beta_m = 1 + 3(\frac{\epsilon}{3})^3$. First consider the case when total flow $r \notin (\alpha \cdot r_j, \beta \cdot r_j) \quad \forall j \in S(T)$. Since $\beta \cdot r_j < \alpha \cdot r_{j+1} \quad \forall j$, we can apply the result of Lemma 8,

$$\mathrm{PoA}(r) \le \max\Big\{\max_{i \notin S(T)} \mathrm{PoA}(r_i), \max_{i \in S(T)} \mathrm{PoA}(\alpha \cdot r_i), \max_{i \in S(T)} \mathrm{PoA}(\beta \cdot r_i)\Big\}.$$

From Lemma 14, 15 and the definition of $S(T)$, all terms above are bounded by the threshold T,

$$\mathrm{PoA}(r) \le T \text{ for } r \notin (\alpha \cdot r_j, \beta \cdot r_j) \quad \forall j \in S(T).$$

$\mathrm{ePoA}(r)$ is bounded by $\mathrm{PoA}(r)$ times the ratio between cost of the Nash flow on $\hat{G}$ and G, From Lemma 9,

$$\mathrm{ePoA}(r) = \mathrm{PoA}(r) \cdot \frac{\hat{C_N}(r)}{C_N(r)} \le T(1 + \frac{h_{max}}{v_{min}}) \quad \text{for} \quad r \notin (\alpha \cdot r_j, \beta \cdot r_j) \quad \forall j \in S(T). \tag{2}$$

We then consider $\mathrm{ePoA}(r)$ when total flow $r \in [\alpha \cdot r_j, \beta \cdot r_j]$, and $j \in S(T)$. From Lemma 16,

$$\mathrm{ePoA}(r) \le \max\Big\{\mathrm{ePoA}(\alpha \cdot r_j), (\beta - \alpha)\frac{\Lambda_j}{\Lambda_j - \Lambda_{j-1}} + (1 - \beta + \alpha)\Big\}.$$

For the second term above, since $j \in S(T)$, the ratio of Λ_j and Λ_{j-1} is bounded, from Lemma 12,

$$\begin{aligned}
&(\beta-\alpha)\frac{\Lambda_j}{\Lambda_j-\Lambda_{j-1}}+(1-\beta+\alpha)\\
\le\,&(\beta-\alpha)\frac{v_{min}}{v_{min}-1}+(1-\beta+\alpha)\\
\le\,&(5(\frac{\epsilon}{3})^3)\cdot\frac{(2\epsilon-\epsilon^2)(\frac{4}{3}-(\frac{\epsilon}{3})^3)}{(2\epsilon-\epsilon^2)(\frac{4}{3}-(\frac{\epsilon}{3})^3)-3(\frac{\epsilon}{3})^3)}+(1-5(\frac{\epsilon}{3})^3)\\
<\,&\frac{4}{3}-(\frac{\epsilon}{3})^3=T.
\end{aligned}$$

From previous case, we know that $\mathrm{ePoA}(\alpha\cdot r_j)\le T(1+\frac{h_{max}}{v_{min}})$, therefore

$$\mathrm{ePoA}(r)\le T(1+\frac{h_{max}}{v_{min}})\quad\text{for}\quad r\in[\alpha\cdot r_j,\beta\cdot r_j]\quad\text{if}\quad j\in S(T). \tag{3}$$

Combine (2) and (3), we have an upperbound of ePoA(r) for all $r>0$,

$$\mathrm{ePoA}\le T(1+\frac{h_{max}}{v_{min}}).$$

From Lemma 12 and 13,

$$\mathrm{ePoA}<\frac{4}{3}-\frac{\epsilon^3}{27}\Big(1-(\frac{2(2\epsilon-\epsilon^2)(\frac{4}{3}-\frac{\epsilon^3}{27})+3\epsilon^3/9}{(2\epsilon-\epsilon^2)(\frac{4}{3}-\frac{\epsilon^3}{27})-\epsilon^3/9})\cdot\frac{\epsilon+\epsilon^2}{9(2-\epsilon)}\Big)<\frac{4}{3}-\frac{\epsilon^3}{81}.$$ ◀

6 Networks with Groups of Similar Links

In previous sections, we have given an upperbound of ePoA when it is strictly less than $\frac{4}{3}$. In this section, we study a special case in which the links can be classified int many groups. Links in the same group all have similar r_i and thus similar cost functions. This special case is closely related to the case in which there are many types of transportation methods, or just many types of roads (such as freeways and local roads). We give an upper bound of ePoA for a specific case of groups of similar link defined below.

▶ **Definition 17.** A traffic network G_c is a *network with clustered latencies* if and only if there exists N intervals $[L_1,R_1],\dots,[L_N,R_N]$, and $\frac{R_i}{L_i}<=1.05$ for $i\in[1,2,\dots,N]$, and $\frac{L_{i+1}}{R_i}\ge 20$ and any r_j for $j\ge 2$ is in one of the intervals $[L_i,R_i]$.

The main result of this section is ePoA≤ 1.281 for a traffic network G_c with clustered latencies. Before proving the main result, we introduce the following transformation, and several lemmas.

▶ **Definition 18.** Given any traffic network G_c with clustered latencies, we define the *aggregated network of* G_c, G_a as the following. For all r_i in G_c, inside a certain interval $[L_k,R_k]$, we re-label the index i to be $k_1,k_2,\dots,k_{n_k}$ so that $L_k\le r_{k_1}\le r_{k_2}\le\dots\le r_{k_{n_k}}\le R_k$. An intermediate network G_{temp} is obtained by increasing the constant of the cost functions b_{k_i} to $\tilde{b_{k_i}}=b_{k_{n_k}}$ for all $i<n_k$. Now all links e_i with r_i in the same interval in G_c has the same b-value, which is $b_{k_{n_k}}$. Thus, by Lemma 3, these links can be merged through a transformation of graph without changing either the Nash flow or the OPT. After the merge, the resulting network is G_a. The transformation Tr is the combination of increasing the constants of links in G_c to get G_{temp}, and merging edges of G_{temp} to get G_a.

▶ **Lemma 19.** *In a traffic networks G_1 with r_{j-1} and r_j where , b_{j-1} is increased to b_j, and the two links are merged to index i_{new}, as stated in Definition 18 then the position of $r_{i_{new}}$ is betweeen (r_{j-1}, r_j).*

Proof. By the basic equation in Definition 1, $r_j = \Sigma_{i=1}^{j-1}(b_{i+1} - b_i)\Lambda_i$.

$$r_j - r_{i_{new}} = (b_j - b_{j-1})(\Lambda_{j-1} - \Lambda_{j-2}) > 0$$
$$r_{i_{new}} - r_{j-1} = (b_j - b_{j-1})(\Lambda_{j-1}) > 0$$

Thus, $r_{j-1} \leq r_{i_{new}} \leq r_j$. ◀

Following Definition 18, with Lemma 19 used recursively, we see that the resulting r_k after merging all links in section k lies in $[L_k, R_k]$. Therefore, after the transformation, the resulting traffic network G_a has min $\frac{r_{j+1}}{r_j} \geq 20$, which is directly from the fact that $\frac{L_{i+1}}{R_i} \geq 20$. The ratio of the optimal cost between the network after the transformation and before the transformation is less than the ratio of the largest $\frac{\hat{b_i}}{b_i} \leq 1.05$. The formal lemma and proof are below.

▶ **Lemma 20.** *For any traffic network G_c with clustered latencies and its corresponding aggregated network G_a, $\frac{C_{opt}(r,G_a)}{C_{opt}(r,G_c)} \leq 1.05$.*

The following lemma is similar to Lemma 9, for a slightly different situation.

▶ **Lemma 21.** *In a traffic network G with rate r and $\frac{r_{i+1}}{r_i} \geq 20$, where constants T, $\alpha_i < 1$, $\beta_i > 1$. When tax(T, A, B) is imposed on the network G. We have*

$$\frac{\hat{C_N}(r)}{C_N(r)} \leq 1 + \frac{h_{max}}{20 \times s} \tag{4}$$

for any s satisfying $\alpha_j \times r_j \geq r \geq s \times r_j$, and $s \times r_j \geq \beta_{j-1} r_{j-1}$, $s \leq 1$. Similarly, we have

$$\frac{\hat{C_N}(r)}{C_N(r)} \leq 1 + \frac{h_{max}}{v_j} \tag{5}$$

for $\alpha_{j+1} \times r_{j+1} \geq r \geq \beta_j \times r_j$.

We now introduce the tax scheme and upper bound the corresponding ePoA for an aggregated network. The tax scheme chooses different values of α_i, β_i, with different regions of v_i.

▶ **Lemma 22.** *For any traffic network G_c with clustered latencies and its corresponding aggregated network G_a, there exists a tax scheme G_a such that ePoA of $G_a \leq 1.22$ when the tax is applied to G_a.*

Proof. The tax scheme tax(1.198, A, B), where α_j , β_j are decided according to the value of v in Table 1 satisfies the requirement.

The proof is omitted due to space constraints. ◀

With Lemma 20 and 22 we prove the main theorem in this section by simply multiplying 1.22 and 1.05.

▶ **Theorem 23.** *The ePoA is at most 1.281 for a traffic network with clustered latencies.*

Table 1 Corresponding α, β with different values of v, note that tax is not imposed in case 0.

cases	v_j	α_j	β_j
0	$[0, 2.95]$	any	any
1	$[2.95, 3]$	0.985	1.51
2	$[3.0, 3.2]$	0.981	1.51
3	$[3.2, 3.5]$	0.964	1.51
4	$[3.5, 4.0]$	0.9428	1.5108
5	$[4.0, 4.8]$	0.9175	1.524
6	$[4.8, 7.0]$	0.89	1.55
7	$[7.0, 11.0]$	0.87	1.79
8	$[11.0, \infty]$	0.83	1.90

Proof. For any traffic network G_c with clustered latencies and its corresponding aggregated network G_a, with Lemma 20 and 22 we know that ePoA of $G_a \leq 1.22$, and $\frac{C_{opt}(r,G_a)}{C_{opt}(r,G_c)} \leq 1.05$. We view the transformation Tr on G_c as tax T_1, and the tax imposed on G_a as tax T_2. The final tax scheme imposed on G_c is $T_1 + T_2$. While the tax scheme imposed on G_a is T_2. Now we prove the theorem

$$\begin{aligned} ePoA &= \frac{\hat{C}_N(r, G_c)}{C_{opt}(r, G_c)} \\ &= \frac{\hat{C}_N(r, G_a)}{C_{opt}(r, G_a)} \cdot \frac{C_{opt}(r, G_a)}{C_{opt}(r, G_c)} \\ &\leq 1.22 \cdot 1.05 = 1.281 \end{aligned}$$

◀

A point to be noted is that the lower bound of PoA is proved in [8] to be 1.191 for two edge network, therefore that proving ePoA ≤ 1.22 is clearly close to optimal since additional tax is further accounted while in [8] the tax could be retrieved and that $\frac{r_{j+1}}{r_j}$ is ∞, where in Lemma 22 the restriction is much stricter, while only increasing the ePoA by less than 3 percent.

7 Open Problems

The goal of this work is to design a taxing scheme with unit step function which is able to be applied to general networks. In the case of parallel links in our study, we have demonstrated different possible approaches to bound the ePoA. We have proved a tight upperbound of the two link case in Section 4, given an upperbound of ePoA less than $\frac{4}{3}$ depending on the discrepancy between links in Section 5, and give an upperbound when the links are clustered while the discrepancy between links in each cluster are not limited in Section 6. However, it remains an open question whether there is a upperbound less that $\frac{4}{3}$ independent of both the discrepancy between the links and the number of links in the network. A combination of the previous methods could be a possible approach.

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