Improved Bounds for Minimal Feedback Vertex Sets in Tournaments

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Abstract

We study feedback vertex sets (FVS) in tournaments, which are orientations of complete graphs. As our main result, we show that any tournament on n nodes has at most 1.5949^n minimal FVS. This significantly improves the previously best upper bound of 1.6667^n by Fomin et al. (STOC 2016). Our new upper bound almost matches the best known lower bound of $21^{n/7} \approx 1.5448^n$, due to Gaspers and Mnich (ESA 2010). Our proof is algorithmic, and shows that all minimal FVS of tournaments can be enumerated in time $O(1.5949^n)$.

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1 Introduction

The Minimum Feedback Vertex Set (FVS) problem in directed graphs is a fundamental problem in combinatorial optimization: given a directed graph G, find a smallest set of vertices in G whose removal yields an acyclic digraph. This problem belongs to Karp's original list of 21 NP-complete problems [8].

The Minimum FVS problem remains NP-complete even in tournaments [13], which are orientations of complete undirected graphs. In other words, a tournament T is a digraph with exactly one arc between any two of its vertices. Various approaches have been suggested to solve the Minimum FVS problem on tournaments, including approximation algorithms [3, 10], fixed-parameter algorithms [4, 9] as well as exact exponential-time algorithms [4, 5, 6]. In particular, one approach that was used to find a minimum FVS is to list all inclusion-minimal FVS of a given tournament using a polynomial-delay enumeration algorithm [6, 12]. The run time of this approach is within a polynomial factor of the number $M(T)$ of minimal FVS in T. Therefore, the complexity of the Minimum FVS problem is within a polynomial factor of the maximum of $M(T)$ over all n-vertex tournaments, which we denote by $M(n)$.

The first one to provide non-trivial bounds on $M(n)$ was Moon [11], who in 1971 established that $1.4757^n \leq M(n) \leq 1.7170^n$. This was improved by Gaspers and Mnich [6] in 2010 to $1.5448^n \leq M(n) \leq 1.6740^n$. Very recently, an improvement on the upper bound was made by Fomin et al. [5], who show that $M(n) \leq 1.6667^n$. The problem of exactly determining $M(n)$ was explicitly posed by Woeginger [15].

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Table 1 State of the art for lower and upper bounds on the number of minimal FVS in tournaments.

$M(n)$	lower bound	upper bound
Moon (1971)	1.4757^n	1.7170^n
Gaspers and Mnich (ESA 2010)	$21^{n/7} \approx 1.5448^n$	1.6740^n
Fomin et al. (STOC 2016)		1.6667^n
This paper		1.5949^n
This paper, regular tournaments:		$21^{n/7} \approx 1.5448^n$

Our Contributions

In this paper we make significant progress on establishing better bounds for $M(n)$. Our main combinatorial result is as follows:

▶ **Theorem 1.** *Any tournament of order n has at most $M(n) \leq 1.5949^n$ minimal FVS.*

We also consider regular tournaments (in which all vertices have the same out-degree), because the best known lower bound on $M(n)$ is attained by regular tournaments. For regular tournaments, we show an upper bound on $M(n)$ that matches the lower bound:

▶ **Theorem 2.** *Any regular tournament of order n has at most $21^{n/7}$ minimal FVS, and this is sharp: some regular tournament of order n has exactly $21^{n/7}$ minimal FVS.*

Table 1 provides an overview on lower and upper bounds on $M(n)$.

Our proof of Theorem 1 is inspired by the one of Gaspers and Mnich [6] for their weaker upper bound. Their proof works by induction on the number n of nodes in the input tournament T. Starting with T, they consider a vertex v with maximum out-degree Δ, and depending on the value of Δ and neighbors of v, they construct subtournaments by deleting distinct vertices, such that each maximal transitive vertex set of T is contained in at least one subtournament. Applying the induction hypothesis to the subtournaments then implies their upper bound.

Here, we use a refined technique, that yields upper bounds on the number of inclusion-maximal vertex sets with certain properties. Namely, in addition to deleting vertices to generate subtournaments, we also keep fixed vertex sets. Within these subtournaments we only consider maximal transitive vertex sets that contain all the fixed vertices. We introduce a new function $M(n, k)$ for the maximum number of maximal transitive vertex sets in a tournament of order n containing a fixed set of k vertices, and we will show that $M(n, k) \leq 1.5949^{n-k}$ for all $0 \leq k \leq n$. A similar approach has been used by Gupta et al. [7] to bound the number of maximal r-regular induced subgraphs in undirected graphs.

Our combinatorial result has algorithmic consequences. First, our proof of Theorem 1 is algorithmic, and shows that all minimal FVS of any tournament of order n can be listed in time $O(1.5949^n)$. Second, using an algorithm by Gaspers and Mnich [6] to list all minimal FVS of a tournament with polynomial delay and in polynomial space, we directly obtain the following:

▶ **Corollary 3.** *Given any tournament T of order n, all its minimal FVS can be listed in time $M(T) \cdot n^{O(1)} = O(1.5949^n)$ with polynomial delay and in polynomial space.*

Enumerating the minimal FVS in tournaments has several interesting applications. For example, Banks [1] introduced the notion “Banks winner” in a social choice context, which is a vertex v with in-degree 0 in a subtournament induced by a maximal transitive vertex set. Brandt et al. [2] consider the problem of determining the "Banks set", which is the set of all Banks winners. As Woeginger [14] showed that deciding whether a vertex is a Banks winner is NP-complete, a feasible approach to determine the Banks set is to enumerate all minimal FVS. For this purpose, Brandt et al. [2] implemented the algorithm of Gaspers and Mnich. Thus, our new algorithm in this paper can be used to compute the Banks set of a tournament asymptotically faster.

2 Preliminaries

A *tournament* $T = (V, A)$ is a directed graph with exactly one edge between each pair of vertices. We denote the set of all tournaments with n vertices by $\mathcal{T}_n$. A *feedback vertex set* (FVS) of T is a set $F \subseteq V(T)$ such that $T - F$ is free of (directed) cycles, where $T - F$ is the induced subgraph of T after removing all vertices in F. An FVS is *minimal* if none of its proper subsets is an FVS.

Denote by $M(T)$ the number of minimal FVS in a tournament T, and define

$$M(n) = \max_{T \in \mathcal{T}_n} M(T)$$

to be the maximum number of minimal FVS in tournaments of order n.

Let $T = (V, A)$ be a tournament. For a set $V' \subseteq V$, let $T[V']$ be the subtournament of T induced by V'. For each $v \in V$, let $N^-(v) = \{u \in V \mid (u, v) \in A\}$ and let $N^+(v) = \{u \in V \mid (v, u) \in A\}$. We write $v \to u$ if $u \in N^+(v)$ and call v a *predecessor* of u and u a *successor* of v. For each $v \in V$, its *in-degree* is $d^-(v) = |N^-(v)|$ and its *out-degree* is $d^+(v) = |N^+(v)|$; call T *regular* if all its vertices have the same out-degree. Let $\Delta^+(T)$ denote the maximum out-degree over all vertices of T. Further, T is *strong* if there is a directed path from v to u for each pair of vertices $v, u \in V$; let $\mathcal{T}_n^\star$ denote the set of strong tournaments of order n. Note that any tournament can uniquely be decomposed into strong subtournaments $S_1, \ldots, S_r$ such that $v \to u$ for all $v \in V(S_i)$, $u \in V(S_j)$ for all $i < j$.

▶ **Observation 4.** *For any tournament T, we obtain $M(T) = M(S_1) \cdot \ldots \cdot M(S_r)$.*

Therefore, we can bound $M(n)$ from above by β^n for some β by considering strong tournaments of every order n.

Our proofs will use the following well-known observation about cycles in tournaments:

▶ **Lemma 5.** *In a tournament, any vertex contained in a cycle is contained in a triangle.*

Proof. Let $v_1, \ldots, v_\ell$ be a shortest cycle containing v_1 with $\ell > 3$, $v_i \to v_{i+1}$ for all $i \in \{1, \ldots, \ell - 1\}$ and $v_\ell \to v_1$. Depending on the orientation of the arc between v_1 and v_3, either v_1, v_2, v_3 form a triangle or $v_1, v_3, v_4, \ldots, v_\ell$ is a shorter cycle containing v_1. ◀

We call a vertex set *transitive* if its induced subtournament is acyclic. Thus, a vertex set is a maximal transitive vertex set if and only if its complement is a minimal FVS. Instead of counting minimal FVS, we count maximal transitive vertex sets. The next property of maximal transitive vertex sets was already used by Moon [11] and Gaspers and Mnich [6]:

▶ **Lemma 6.** *For any tournament T, $M(T) \leq \sum_{v \in V(T)} M(d^+(v))$.*

Proof. Any maximal transitive vertex set W of T has a vertex v with in-degree 0 in $T[W]$. Hence, W is also a maximal transitive vertex set in $T[N^+(v) \cup \{v\}]$; this yields the bound. ◀

Lemma 6 allows us to effectively bound $M(T)$ in terms of a recurrence relation, in particular in combination with the next lemma that extends Lemma 3 by Gaspers and Mnich [6]:

▶ **Lemma 7.** *Let $n \in \mathbb{N}$ and let $T \in \mathcal{T}_n^\star$. Then either T is regular, or for any $d \in \mathbb{N}$ at most $2d$ vertices in T have out-degree at least $n-d-1$.*

Proof. Let $\tilde{V}$ be the set of vertices in T with out-degree at least $n-d-1$. Then any vertex in $\tilde{V}$ has in-degree at most d. Hence,

$$\sum_{v\in\tilde{V}} |N^-(v)| \leq |\tilde{V}| \cdot d \ . \tag{1}$$

We may suppose that $\tilde{V} \neq \emptyset$, for otherwise the statement of the lemma holds. We distinguish two cases.

Consider first the case that $\tilde{V} \neq V(T)$. Then, since T is strong and $\tilde{V} \neq \emptyset$, there is some arc from $V(T) \setminus \tilde{V}$ to $\tilde{V}$. There are $\binom{|\tilde{V}|}{2}$ arcs between vertices in $\tilde{V}$. Therefore, $\sum_{v\in\tilde{V}} |N^-(v)| \geq \binom{|\tilde{V}|}{2} + 1$. Combining this inequality with (1) and solving for $d \in \mathbb{N}$ yields $|\tilde{V}| \leq 2d$.

Second, consider the case that $\tilde{V} = V(T)$. We may suppose that T is not regular, for otherwise the statement of the lemma holds. Note that not every vertex of $\tilde{V} = V(T)$ can have in-degree exactly d, since T is not regular. Hence, some vertex in $\tilde{V}$ has in-degree at most $d-1$. Consequently,

$$\sum_{v\in\tilde{V}} |N^-(v)| \leq (|\tilde{V}|-1) \cdot d + (d-1) \ .$$

There are $\binom{|\tilde{V}|}{2}$ arcs between vertices in $\tilde{V}$. Thus, $\sum_{v\in\tilde{V}} |N^-(v)| \geq \binom{|\tilde{V}|}{2}$. Combining these two inequalities and solving for $d \in \mathbb{N}$ yields $|\tilde{V}| \leq 2d$. ◀

We remark that a regular tournament may have more than $2d$ vertices of out-degree at least $n-d-1$, as witnessed for instance by the triangle and $d = 1$.

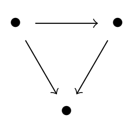
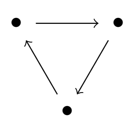
3 Improved Upper Bound on the Maximum Number of Minimal FVS

In this section we show that the maximum number $M(n)$ of minimal FVS in any tournament of order n is bounded from above by 1.5949^n. For this purpose, for a tournament T and $V' \subseteq V(T)$ let $M(T, V')$ be the number of maximal transitive vertex sets in T that contain all vertices in V'. Also, let

$$M(n,k) = \max_{T\in\mathcal{T}_n, V'\subseteq V(T), |V'|=k} M(T, V') \ .$$

Note that $M(n) = M(n, 0)$.

Example. To clarify the definition, we compute $M(3,1)$. Precisely, we show that $M(3,1) = 2$. There are two non-isomorphic tournaments for $n = 3$:

T_1:  T_2: 

The tournament T_1 is acyclic and thus has only a single maximal transitive vertex set, $V(T_1)$. Thus, $M(T_1, \{v\}) = 1$ for all $v \in V(T_1)$. The tournament T_2 has three maximal transitive vertex sets, each consisting of exactly two vertices. Thus, each vertex of T_2 is contained in exactly two maximal transitive vertex sets. This yields $M(T_2, \{v\}) = 2$ for all $v \in V(T_2)$. Summarizing, we get $M(3, 1) = 2$. ◀

Henceforth, fix $\beta = 1.5949$. We will show that $M(n, k) \leq \beta^{n-k}$ for all $n \in \mathbb{N}$ and $k \in \{0, \dots, n\}$. To this end, ideally we would like to prove the following statements:

(I) *It holds $M(n, k) \leq \beta^{n-k}$ for all $n \geq k > 0$.*

(II) *It holds $M(n, 0) \leq \beta^n$.*

Unfortunately, we are unable to do prove these directly. The reason is that our proof of Statement (I) for a fixed pair (n, k) with $n \geq k > 0$ *depends* on the validity of Statement (II) for values $\tilde{n} < n$. Vice-versa, our proof of the validity of Statement (II) for fixed $n \in \mathbb{N}$ depends on the validity of Statement (I).

We will therefore establish the following two lemmas:

▶ **Lemma 8.** *Let $n \in \mathbb{N}$. If $M(\tilde{n}) \leq \beta^{\tilde{n}}$ and $M(\tilde{n}, \tilde{k}) \leq \beta^{\tilde{n}-\tilde{k}}$ holds for all $0 < \tilde{k} \leq \tilde{n} < n$, then $M(n, k) \leq \beta^{n-k}$ for $0 < k \leq n$.*

The proof of Lemma 8 is given in Sect. 4.

▶ **Lemma 9.** *Let $n \in \mathbb{N}$. If $M(\tilde{n}) \leq \beta^{\tilde{n}}$, $M(\tilde{n}, \tilde{k}) \leq \beta^{\tilde{n}-\tilde{k}}$ and $M(n, \tilde{k}) \leq \beta^{n-\tilde{k}}$ for all $0 < \tilde{k} \leq \tilde{n} < n$, then $M(n) \leq \beta^n$.*

The proof of Lemma 9 consists of a lengthy case analysis; we thus defer it to the full version of this paper.

We are ready to prove Theorem 1.

Proof of Theorem 1. We show that for all $n \in \mathbb{N}$, it holds $M(n) \leq 1.5949^n$. Clearly, $M(1) \leq 1 \leq 1.5949$ and $M(1, k) \leq 1 \leq 1.5949^{1-k}$ for all $k \in \{0, 1\}$. This yields our induction hypothesis. Lemma 8 and Lemma 9 yield our inductive step and prove the desired bound on $M(n)$ for all $n \in \mathbb{N}$. ◀

4 Proof of Lemma 8

In this section we prove Lemma 8. For sake of contradiction, suppose that the statement of the lemma does not hold. Let (T, V') be a minimum counterexample, that is, T is a tournament and $V' \subseteq V(T)$ such that $|V(T)| - |V'|$ is minimum and $M(T, V') > \beta^{|V(T)|-|V'|}$. Throughout this section, write $n = |V(T)|$ and $k = |V'| > 0$.

We will distinguish several cases and show that $M(T, V') \leq \beta^{n-k}$ for each of them; this yields the desired contradiction (and hence the truth of the statement of the lemma). In each case, we will use the minimality of (T, V') to bound $M(T, V')$ from above.

Case 1: Three vertices in V' form a triangle. Then, as no transitive vertex set contains all of these three vertices, $M(T, V') = 0 \leq \beta^{n-k}$.

Case 2: Two vertices in V' form a triangle with some vertex $v \in V(T) \setminus V'$. Any transitive vertex set that contains all vertices in V' does not contain v. Hence,

$$M(T, V') = M(T - \{v\}, V') \leq M(n - 1, k) \leq \beta^{n-k-1} \leq \beta^{n-k} \ .$$

Case 3: There is a vertex $v \in V'$ that is not contained in any cycle of T. Then, a set $W \supseteq V'$ is a maximal transitive vertex set of T if and only if $W \setminus \{v\} \supseteq V' \setminus \{v\}$ is a maximal transitive vertex set of $T - v$. This yields

$$M(T, V') = M(T - \{v\}, V' \setminus \{v\}) \leq M(n-1, k-1) \leq \beta^{n-k} \ .$$

▶ Remark. We remark that it is this case where we rely on the validity of Lemma 9, namely that $M(\tilde{n}) < \beta^{\tilde{n}}$ for $\tilde{n} < n$. The reason is that possibly $V' \setminus \{v\} = \emptyset$, in which case $k - 1 = 0$ and we need that $M(n-1, 0) \leq \beta^{n-1}$.

Henceforth, consider pairs (T, V') to which Cases 1–3 do not apply.

▶ **Observation 10.** *If Cases 1–3 do not apply to* (T, V')*, then* (i) *any vertex of* V' *is contained in at least one triangle (by Lemma 5), and* (ii) *any triangle contains at most one vertex of* V'*.*

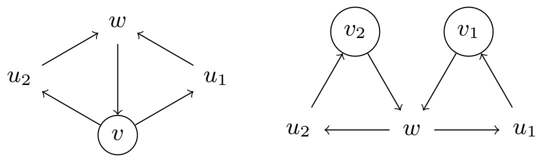
▶ Remark. We remark that with Case 1–3 we can already show a bound of $M(T, V') \leq \beta_0^{n-k}$ for $\beta_0 = 1.6181$ (under the conditions imposed by the lemma). By Observation 10, there is a vertex $v \in V'$ that forms a triangle with two vertices $w_1, w_2 \notin V'$. Any maximal transitive vertex set $W \supseteq V'$ (and thus containing v) cannot contain both w_1 and w_2. Therefore, $w_1 \in W$ implies $w_2 \notin W$ and we get

$$\begin{aligned} M(T, V') &\leq M(T - \{w_1\}, V') + M(T - \{w_2\}, V' \cup \{w_1\}) \\ &\leq M(n-1, k) + M(n-1, k+1) \leq \beta_0^{n-k-1} + \beta_0^{n-k-2} \ , \end{aligned}$$

which is bounded by β_0^n for $\beta_0 = 1.6181$.

The subsequent cases allow us to improve $\beta_0 = 1.6181$ to $\beta = 1.5949$.

Case 4: There is a vertex $w \notin V'$ that is contained in two distinct triangles, both of which contain a vertex from V' (possibly shared by both triangles). Then we are in one of two cases, where vertices in V' are circled:

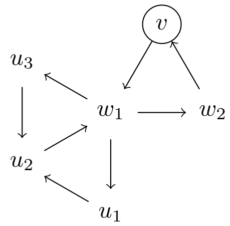


Let $(w, u_1, v_1), (w, u_2, v_2)$ be distinct triangles containing w, such that $v_1, v_2 \in V'$ where possibly $v_1 = v_2$. Let W be a maximal transitive vertex set of T containing V'. Then either $w \notin W$ or $w \in W$. Clearly, if $w \in W$ then $u_1, u_2 \notin W$. We therefore have

$$\begin{aligned} M(T, V') &\leq M(T - \{w\}, V') + M(T - \{u_1, u_2\}, V' \cup \{w\}) \\ &\leq M(n-1, k) + M(n-2, k+1) \leq \beta^{n-k-1} + \beta^{n-k-3}. \end{aligned}$$

The last expression on the right-hand side is at most β^{n-k}, since $\beta \geq 1.4656$.

Case 5: There are vertices $v \in V'$ and $w_1, w_2 \in V(T) \setminus V'$ that form a triangle, such that w_1 also belongs to triangles $(w_1, u_1, u_2), (w_1, u_2, u_3)$ for some $u_1, u_2, u_3 \in V(T) \setminus \{v, w_2\}$.



Then we can assume that $u_1, u_2, u_3 \in V(T) \setminus V'$, as otherwise Case 2 or Case 4 would apply. Any transitive vertex set $W \supseteq V'$ either contains w_1 or not. If $w_1 \in W$ then $w_2 \notin W$. Moreover, $w_1 \in W$ implies that either $u_2 \notin W$, or $u_2 \in W$ but $u_1, u_3 \notin W$. Thus,

$$\begin{aligned} M(T, V') &\leq M(T - \{w_1\}, V') + M(T - \{w_2\}, V' \cup \{w_1\}) \\ &\leq M(T - \{w_1\}, V') + M(T - \{w_2, u_2\}, V' \cup \{w_1\}) \\ &\quad + M(T - \{w_2, u_1, u_3\}, V' \cup \{w_1, u_2\}) \\ &\leq M(n-1, k) + M(n-2, k+1) + M(n-3, k+2) \\ &\leq \beta^{n-k-1} + \beta^{n-k-3} + \beta^{n-k-5} \ . \end{aligned}$$

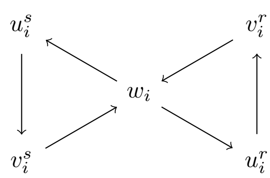
The last expression on the right-hand side is at most β^{n-k}, since $\beta \geq 1.5702$.

Henceforth, we assume that Cases 1-5 do not apply to (T, V'). Then some vertex $v_0 \in V'$ forms a triangle with some $w_1, w_2 \in V(T) \setminus V'$, as Cases 1-3 do not apply. For $i = 1, 2$, let Δ_i be the set of triangles $t_i = (u_i, v_i, w_i)$ that are disjoint from w_{3-i} and for which $T[\{u_i, v_i, v'\}]$ is acyclic for all $v' \in V'$. Consequently, all triangles in $\Delta_1 \cup \Delta_2$ are disjoint from V', as Case 4 does not apply. Further, all triangles in Δ_i are pairwise edge-disjoint (as Case 5 does not apply), and therefore intersect only in w_i.

To prove an upper bound on $M(T, V')$, we again distinguish the maximal transitive vertex sets that contain w_1 or w_2, from those that do not contain either of them. Let W be a maximal transitive vertex set of T containing V'.

First consider that $w_1, w_2 \notin W$. Then, $T[W \cup \{w_i\}]$ contains a cycle for $i = 1, 2$, by maximality of W. Thus, by Lemma 5, there is a triangle $t = (w_i, z_1, z_2)$ for some $z_1, z_2 \in W$. We have that $t \in \Delta_i$, since z_1, z_2 do not form a triangle with any $v' \in V'$ as $z_1, z_2 \in W$. Thus, those W with $w_1, w_2 \notin W$ can be partitioned into $|\Delta_i|$ classes, where the r-th class contains the sets W that contain the two vertices of the r-th triangle in Δ_i.

To use this argument effectively, we need some further observations about the relation among triangles in $\Delta_1 \cup \Delta_2$. Consider two triangles $t_i^r = (u_i^r, v_i^r, w_i), t_i^s = (u_i^s, v_i^s, w_i) \in \Delta_i$:



Since all triangles that contain w_i are pairwise edge-disjoint (as Case 5 does not apply), the edge between u_i^r and v_i^s has to be directed from v_i^s to u_i^r; else, w_i, u_i^r, v_i^s would form a triangle that is not edge-disjoint from the triangle w_i, u_i^r, v_i^r. Likewise, the edge between u_i^s and v_i^r has to be directed from v_i^r to u_i^s. Ignoring symmetries obtained by swapping the roles of t_i^r and t_i^s, there are only two possibilities how the two remaining edges (between u_i^r, u_i^s and v_i^r, v_i^s) can be oriented:

We refer to the situation in the left figure as **Case A**, and to the situation in the right figure as **Case B**. Note that in Case A, (u_i^r, u_i^s, v_i^s) and (v_i^r, u_i^s, v_i^s) form triangles; while in Case B, triangles are formed by (u_i^r, u_i^s, v_i^s) and (u_i^r, v_i^r, v_i^s).

▶ **Observation 11.** *In Case A, $u_i^s, v_i^s \in W$ implies that $u_i^r, v_i^r \notin W$. In Case B, $u_i^r, v_i^r \in W$ implies that $v_i^s \notin W$; and $u_i^s, v_i^s \in W$ implies that $u_i^r \notin W$.*

Thus, for each $t_i^r = (u_i^r, v_i^r, w_i) \in \Delta_i$ let $V_{t_i^r}$ be the set of vertices that are excluded from those W with $u_i^r, v_i^r \in W$ due to Observation 11.

In Lemma 12, we will show that any two triangles in Δ_1 and Δ_2 are vertex-disjoint. Therefore, for each $t_i^r \in \Delta_i$, every vertex in $V_{t_i^r}$ is not contained in any triangle of Δ_{3-i}. This implies that for any pair of triangles $t_1 \in \Delta_1, t_2 \in \Delta_2$ the sets V_{t_1}, V_{t_2} are disjoint. Altogether, this means that we can bound the number of maximal transitive vertex sets $W \supseteq V'$ not containing w_1, w_2 from above by

$$\begin{aligned}&\sum_{t=(w_1,u_1,u_2)\in\Delta_1}\ \sum_{t'=(w_2,u'_1,u'_2)\in\Delta_2} M(T-\{w_1,w_2\}-V_t-V_{t'}, V'\cup\{u_1,u_2,u'_1,u'_2\})\\ \le\ &\sum_{t\in\Delta_1}\sum_{t'\in\Delta_2}\beta^{n-2-|V_t|-|V_{t'}|-(k+4)} \le \beta^{n-k-6}\underbrace{\sum_{t\in\Delta_1}\beta^{-|V_t|}\sum_{t'\in\Delta_2}\beta^{-|V_{t'}|}}_{(\star)}\ . \qquad (2)\end{aligned}$$

Thus, our goal is now to bound $(\star)$. Fix $i \in \{1,2\}$. Let $t_i^1, \ldots, t_i^{|\Delta_i|}$ be an ordering of the triangles in Δ_i such that $|V_{t_i^r}| \le |V_{t_i^s}|$ for $1 \le r < s \le |\Delta_i|$. Then for any pair $r, s \in \{1, \ldots, \Delta_i\}$ with $r \ne s$, Observation 11 implies

$$|V_{t_i^s} \cap \{u_i^r, v_i^r\}| + |V_{t_i^r} \cap \{u_i^s, v_i^s\}| = 2\ .$$

Thus, for any $r < s$, since $\beta \ge 1$, we get

$$\beta^{-|V_{t_i^s}|} + \beta^{-|V_{t_i^r}|} \quad\le\quad \beta^{-(|V_{t_i^s} \cup \{u_i^r, v_i^r\}|)} + \beta^{-(|V_{t_i^r} \setminus \{u_i^s, v_i^s\}|)}\ .$$

Thus, we can bound $(\star)$ by the case where for any $r < s$,

$$|V_{t_i^s} \cap \{u_i^r, v_i^r\}| = 2 \wedge |V_{t_i^r} \cap \{u_i^s, v_i^s\}| = 0\ .$$

Hence, we can assume that $|V_{t_i^r}| = 2(r-1)$ for all $r = 1, \ldots, |\Delta_i|$. We obtain

$$\sum_{t\in\Delta_i}\beta^{-|V_t|} \le \sum_{r=0}^{|\Delta_i|-1}\beta^{-2r} \le \sum_{r=0}^{\infty}\beta^{-2r} = \frac{\beta^2}{\beta^2-1}\ .$$

Consequently, $(\star)$ is bounded by $(\frac{\beta^2}{\beta^2-1}) \cdot (\frac{\beta^2}{\beta^2-1}) = \frac{\beta^4}{(\beta^2-1)^2}$.

Let us now prove that indeed any triangle in Δ_1 is disjoint from every triangle in Δ_2.

▶ **Lemma 12.** *Let v_0, w_1, w_2, Δ_1 and Δ_2 be defined as before. Then any triangle in Δ_1 is vertex-disjoint from every triangle in Δ_2.*

Proof. First note that V' is a transitive set, as Case 1 does not apply. Thus, the vertices in V' admit a topological order such that $v'_x \to v'_y$ for all $v'_x, v'_y \in V'$ with $x > y$. Second, for each vertex $z \in V(T) \setminus V'$ the set $V' \cup \{z\}$ is a transitive set, as Case 2 does not apply. Therefore, the vertices of $V(T) \setminus V'$ can be partitioned into layers $Z_1, \ldots, Z_\ell$ such that for each $z \in Z_r$, $z \to v'_s$ if and only if $s < r$.

We claim that for $i = 1, 2$, the vertices of any triangle $(u_i^r, v_i^r, w_i) \in \Delta_i$ all belong to the same layer. This implies in particular that for $i = 1, 2$, *all* vertices in triangles of Δ_i belong to the same layer. Since w_1 and w_2 are in different layers (as $v_0 \to w_1$, $w_2 \to v_0$), this shows that any triangle in Δ_1 is vertex-disjoint from any triangle in Δ_2.

To show the claim, let $i \in \{1, 2\}$ and let $(u_i^r, v_i^r, w_i) \in \Delta_i$ be a triangle with $w_i \to u_i^r, u_i^r \to v_i^r, v_i^r \to w_i$. Suppose that $u_i^r \in Z_u, v_i^r \in Z_v, w_i \in Z_w$ for some $u, v, w \in \{1, \ldots, l\}$. So we must show that $u = v = w$ to prove the claim.

If $u < w$ then w_i, u_i^r, v'_u form a triangle, contradicting that Case 4 does not apply. If $v > w$ then w_i, v_i^r, v'_v form a triangle, again contradicting that Case 4 does not apply. Hence, $v \leq w \leq u$ holds. If $v < u$ then u_i^r, v_i^r, v'_v form a triangle, contradicting the definition of Δ_i. So indeed $u = v = w$, and the claim holds. ◀

To complete the proof of Lemma 8, we must also consider those $W \supseteq V'$ that contain exactly one of w_1, w_2 (recall that at most one of w_1, w_2 belongs to W as $v_0 \in W$, so $w_i \in W$ implies $w_{3-i} \notin W$ for $i = 1, 2$). Overall, if Cases 1–5 do not apply, with the obtained bound on $(\star)$, by (2) we have

$$\begin{aligned} M(T, V') &\leq M(T - \{w_1\}, V' \cup \{w_2\}) + M(T - \{w_2\}, V' \cup \{w_1\}) + \beta^{n-6-k} \cdot \frac{\beta^4}{(\beta^2-1)^2} \\ &\leq 2 \cdot M(n-1, k+1) + \beta^{n-6-k} \cdot \frac{\beta^4}{(\beta^2-1)^2} \leq 2 \cdot \beta^{n-k-2} + \frac{\beta^{n-k-2}}{(\beta^2-1)^2} . \end{aligned}$$

The last expression on the right-hand side is at most β^{n-k}, since $\beta \geq 1.5703$.

This completes the proof of Lemma 8. ◀

5 Discussion

In this paper we narrowed the gap between the lower and upper bounds for the maximum number $M(n)$ of minimal FVS in n-vertex tournaments, to $1.5448^n \leq M(n) \leq 1.5949^n$. It remains to determine the growth of $M(n)$ exactly—Gaspers and Mnich [6] conjectured that $M(n) \leq 21^{n/7} \approx 1.5448^n$ for all $n \in \mathbb{N}$, and we re-pose this conjecture here.

In a different direction, it would be interesting to prove non-trivial upper bounds of the form c^n for some constant $c < 2$, on the number of minimal FVS in general directed graphs. As far as we know, currently only a bound of $2^n/\sqrt{n}$ is known, implied by Sperner's Lemma.

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