On the Synchronisation Problem over Cellular Automata

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Abstract

Cellular automata are a discrete, synchronous, and uniform dynamical system that give rise to a wide range of dynamical behaviours. In this paper, we investigate whether this system can achieve synchronisation. We study the cases of classical bi-infinite configurations, periodic configurations, and periodic configurations of prime period. In the two former cases, we prove that only a "degenerated" form of synchronisation – there exists a fix-point – is possible. In the latter case, we give an explicit construction of a cellular automaton for which any periodic configuration of prime period eventually converges to cycle of two uniform configurations. Our construction is based upon sophisticated tools: aperiodic NW-deterministic tilings [7] and partitioned intervals [1].

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Introduction

Complex systems are systems compound of many "simple" components whose interactions give birth to a wide range of complex behaviours. To better grasp mechanisms behind such emerging behaviours, a classical theoretical approach is to reproduce these behaviours on a regular and simple model. Introduced to model self-replication [10], cellular automata are an example of such a basic simple discrete dynamical system. They consist in a bi-infinite line of *cells* endowed with a *state* chosen among a finite alphabet. The system evolves thanks to the uniform and synchronous application of a *local rule.* This rule gives the new state of a cell according to its previous state and the ones of its neighbours. Despite its apparent simplicity, cellular automata can exhibit a wide range of complex behaviours [11]. In this paper, we focus on one specific behaviour: *synchronisation.* as in biological cell synchronisation.

Section 1 is devoted to present the context, provide a formal definition, and give several preliminary properties. Inspired by biological cell synchronisation, synchronisation is the following: is it possible, starting from any configuration, to ensure that all cells eventually enter the same state at the same moment? This notion has also strong connections with the Firing Squad Synchronisation Problem [2, 8]. In this paper, the two main specificities are that the system is fully deterministic (no randomness is provided) and we require synchronisation for every configuration.

Results are presented in two parts: in Section 2, we prove that our framework does not allow "real" synchronisation neither over the whole configuration space nor over periodic configurations. Then, in Section 3, we present our main result: a detailed construction of a cellular automaton that – over periodic configurations of prime period – always converges toward a cycle consisting of two uniform configurations, solving the *synchronisation problem* as studied in [5]. Finally, Section 4 gives some consequences of our main result.

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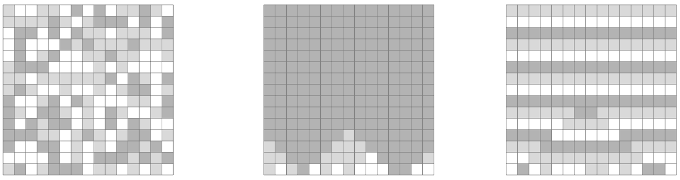


(a) No synchronisation (b) Synchronisation towards a fix-point (c) Synchronisation towards a cycle

Figure 1 Examples of partial space-time diagrams (time goes up).

1 Synchronisation

A *cellular automaton* (CA for short) is a pair (Q, f) where Q is a finite set of *states* and $f : Q^3 \to Q$ is the *local rule*. The cellular automaton acts on elements of $Q^{\mathbb{Z}}$ called *configurations*. The resulting action is called *global function* $F : Q^{\mathbb{Z}} \to Q^{\mathbb{Z}}$ and is defined by, for any $c \in Q^{\mathbb{Z}}$, $i \in \mathbb{Z}$, $F(c)_i = f(c_{i-1}, c_i, c_{i+1})$. The local function is extended to finite words $f : Q^n \to Q^{n-2}$ by, for any $w_0 w_1 \ldots w_{n-1} \in Q^n$, $f(w_0 w_1 \ldots w_{n-1}) = f(w_0, w_1, w_2) f(w_1, w_2, w_3) \ldots f(w_{n-3}, w_{n-2}, w_{n-1})$. A configuration $c \in Q^{\mathbb{Z}}$ is of *period* $p \in \mathbb{N}^+$ if, for any $i \in \mathbb{Z}$, $c_{i+p} = c_p$. Due to uniformity of global function, the image of a periodic configuration is periodic and its period divides the original period. In the rest of the paper, the periodic configuration made by the repetition of the non-empty word $u \in Q^*$ is denoted as ${}^{\omega}u^{\omega}$. When the period is 1, the configuration is called *uniform*.

The evolution of a configuration $c \in Q^{\mathbb{Z}}$ is often depicted by piling up the successive iterations $(c, F(c), F^2(c), \ldots)$. Such a representation is called *space-time diagram*; examples of such a representation can be seen in Figure 1. Intuitively, synchronisation is achieved when, at some time, all cells reach an agreement, i.e. a uniform configuration (see Figure 1).

▶ **Definition 1** (Synchronisation). A cellular automaton (Q, f) is *synchronizing* if, for any configuration $c \in Q^{\mathbb{Z}}$, there exist $n_c \in \mathbb{N}^*$ and $a \in Q$, such that $F^{n_c}(c) = {}^{\omega}a^{\omega}$.

Moreover, the cellular automaton is *fully synchronizing* if a does not depend on c – *i.e.*, there exists $a \in Q$ such that, for any configuration $c \in Q^{\mathbb{Z}}$, there exists $n_c \in \mathbb{N}^*$ such that $F^{n_c}(c) = {}^{\omega}a^{\omega}$.

A careful reader may notice that our definition allows cases where the synchronisation does not bring meaningful information: for example, when a configuration converges to a uniform fix-point as in Figure 1b. The solution to exclude these cases is to forbid the existence of such a fix-point.

▶ **Definition 2.** A synchronizing CA is *strongly* synchronizing if it is synchronizing and does not have a uniform fix-point (formally, there does not exist $b \in Q$ such that $F({}^{\omega}b^{\omega}) = {}^{\omega}b^{\omega}$).

Intuitively, synchronizing CA correspond to CA whose attractors are a finite set of cycles (a unique cycle for fully synchronizing). Strongly synchronizing CA are the specific case where none of these attractors is reduced to a single configuration.

Several results are known in slightly different contexts. In particular, M. Delacourt's construction in [4] can be used to construct a cellular automaton which converges toward a uniform cycle of length two for *almost all* configurations or for *almost all* periodic configurations [3]. In the case of probabilistic cellular automata, N. Fatès has shown a wide range of rules displaying full strong synchronisation [5].

2 On general and periodic configurations

We now show that, over the set of all configurations and over the set of periodic configurations, strong synchronisation is not possible. Moreover, in the general case, the notion of synchronisation corresponds exactly to the well-known notion of nilpotency (see [7] for more information about this notion).

▶ **Lemma 3.** *A cellular automaton (Q, f) is synchronizing over the set of all configurations if and only if it is nilpotent (i.e., $\exists a \in Q$ and $N \in \mathbb{N}$ such that, $\forall c \in Q^{\mathbb{Z}}$, $F^N(c) = {}^{\omega}a^{\omega}$).*

Proof. Trivially, nilpotency implies (full) synchronisation and existence of a fix-point.

For the other direction, the basic idea is to consider a *universe* configuration $c_\Omega \in Q^{\mathbb{Z}}$ containing all finite words of Q^* as factors. By definition, if the automaton is synchronizing, there exist $n_\Omega \in \mathbb{N}$ and $a \in Q$ such that $F^{n_\Omega}(c_\Omega) = {}^{\omega}a^{\omega}$. This implies that, for any word $u \in Q^{2n_\Omega+1}$, it holds $f^{n_\Omega}(u) = a$. Hence, for any configuration $c \in Q^{\mathbb{Z}}$, we have $F^{n_\Omega}(c) = {}^{\omega}a^{\omega}$. ◀

▶ **Lemma 4.** *Any synchronizing cellular automaton (Q, f) over the set of periodic configurations has a fix-point.*

Proof. Let us first consider the sequence of states $(q_n)_{n\in\mathbb{N}}$ defined by $q_0 = q_1 = q_2 = q \in Q$ and for any $n \geq 2$, $q_{n+1} = f(q_{n-2}, q_{n-1}, q_n)$. Since Q is finite, there exists $x, y > 2, x \neq y$ such that $(q_{x-1}, q_x, q_{x+1}) = (q_{y-1}, q_y, q_{y+1})$. Let us denote as $\sigma : Q^{\mathbb{Z}} \to Q^{\mathbb{Z}}$ the *shift* defined by, for any $c \in Q^{\mathbb{Z}}$ and $p \in \mathbb{Z}$, $\sigma(c)_p = c_{p+1}$. By construction, $F({}^{\omega}(q_x q_{x+1} \dots q_{y-1})^{\omega}) = \sigma^2({}^{\omega}(q_x q_{x+1} \dots q_{y-1})^{\omega})$ and, $\forall n \in \mathbb{N}$, $F^n({}^{\omega}(q_x q_{x+1} \dots q_{y-1})^{\omega}) = \sigma^{2n}({}^{\omega}(q_x q_{x+1} \dots q_{y-1})^{\omega})$. Since the cellular automaton is synchronizing, there exists $a \in Q$ such that ${}^{\omega}(q_x q_{x+1} \dots q_{y-1})^{\omega} = {}^{\omega}a^{\omega}$ which implies $F({}^{\omega}a^{\omega}) = \sigma^2({}^{\omega}a^{\omega}) = {}^{\omega}a^{\omega}$. ◀

In fact, the previous proof can also prove the following: if a cellular automaton does not have a fix-point, then there exists a non-uniform periodic configuration on which it behaves as a power of the shift.

Using the uniform fix-point as an sink-hole, it is easy to construct a wide range of possible behaviours including non fully synchronizing cellular automata. For example, the automaton on $Q = \{e, 0, 1\}$ with the rule $f(0, 0, 0) = 1$, $f(1, 1, 1) = 0$ and f outputs e for any other case gives a synchronizing CA on periodic configurations which either stays in the cycle ${}^{\omega}0^{\omega}$, ${}^{\omega}1^{\omega}$ (starting from one of these configurations) or goes into ${}^{\omega}e^{\omega}$ otherwise.

3 Our main result

At this point, our goal is to find a fully and strongly synchronizing cellular automaton. Indeed, we give in this section an example for the set of automata working over finite configurations of prime period.

▶ **Theorem 5.** *There exists a fully and strongly synchronizing cellular automaton over the set of periodic configurations of prime period (or unit period).*

The rest of the section is devoted to construct a cellular automaton (Q, f) for which any periodic configuration (of prime period) converges toward the length two cycle containing the two configurations ${}^{\omega}0^{\omega}$ and ${}^{\omega}1^{\omega}$ $(0, 1 \in Q)$.

The construction is done by using three layers of set of states $Q = N \times K \times C$. At first, Section 3.1 gives a non-deterministic automaton (N, f), on partially synchronized growing intervals. This automaton is extended into a deterministic cellular automaton on set of

Table 1 Set N of states. Symbol on top and left defines subset of the state set for convenience.

	L (left border)	I (interior)	R (right border)	D (dual)
B (before)	[→	→		
Sr (right signal)	[⟨ , [⟨⟨ , [⟨⟨⟨	╱ , ╱╱ , ╱╱╱		
Sl (left signal)	[╲	╲ , ╲╲ , ╲╲╲ , ⋱	⟩] , ⟩⟩] , ⟩⟩⟩] , ˋ]	╲□
Er (erasure)	[´			⌞□ , □
A (after)		←	←]	
V (values)		0 , 1		

states $N \times K \times C$ where K and C are layers controlling which of the possible rules of f the N-layer chooses from. In Section 3.2, we add the K-layer based on an aperiodic tiling to ensure that any configuration will contain only full intervals. In Section 3.3, we define the C-layer controlling disappearing behaviour of intervals, using (at last) primality, to achieve the result.

3.1 Intervals

The central element of our construction is the notion of interval that contains an increasing part of locally synchronized computation. In our cellular automaton, an interval is based on a portion of the configuration between a *left border* depicted by symbol [, and a *right border*] , containing *interior*. This interval is endowed with a *signal* going left S_l (╲ , ╲╲ and ╲╲╲) and right S_r (╱ , ╱╱ and ╱╱╱). To ensure that there is exactly one signal in each interval, all cells in the interval, not containing the signal, indicate whether they are *before* → or *after* ← the signal. The states resulting from the product of these two elements are detailed in Table 1 alongside the subsets of states used in the rest of the paper.

▶ **Definition 6** (Interval)**.** An *interval* is a word of the form $\left(LI^*R\right) \bigcap \left(B^*(Sl \cup Sr \cup Er)A^*\right)$ or an element of D.

The main idea of the construction is to divide the configuration into growing intervals that will "compete" with each other until only one is left. In this section, we firstly assert the behaviour of intervals on their own.

First and foremost, the transition rule is only defined on sequences of intervals separated by portion of values. When the neighbourhood is not locally valid, the rule outputs a new *initial interval* ╲□.

To achieve growth (Figure 2a), the signal inside the interval goes back and forth three times. The third time the signal reaches the left border, the interval grows of one to the left by moving its left border.

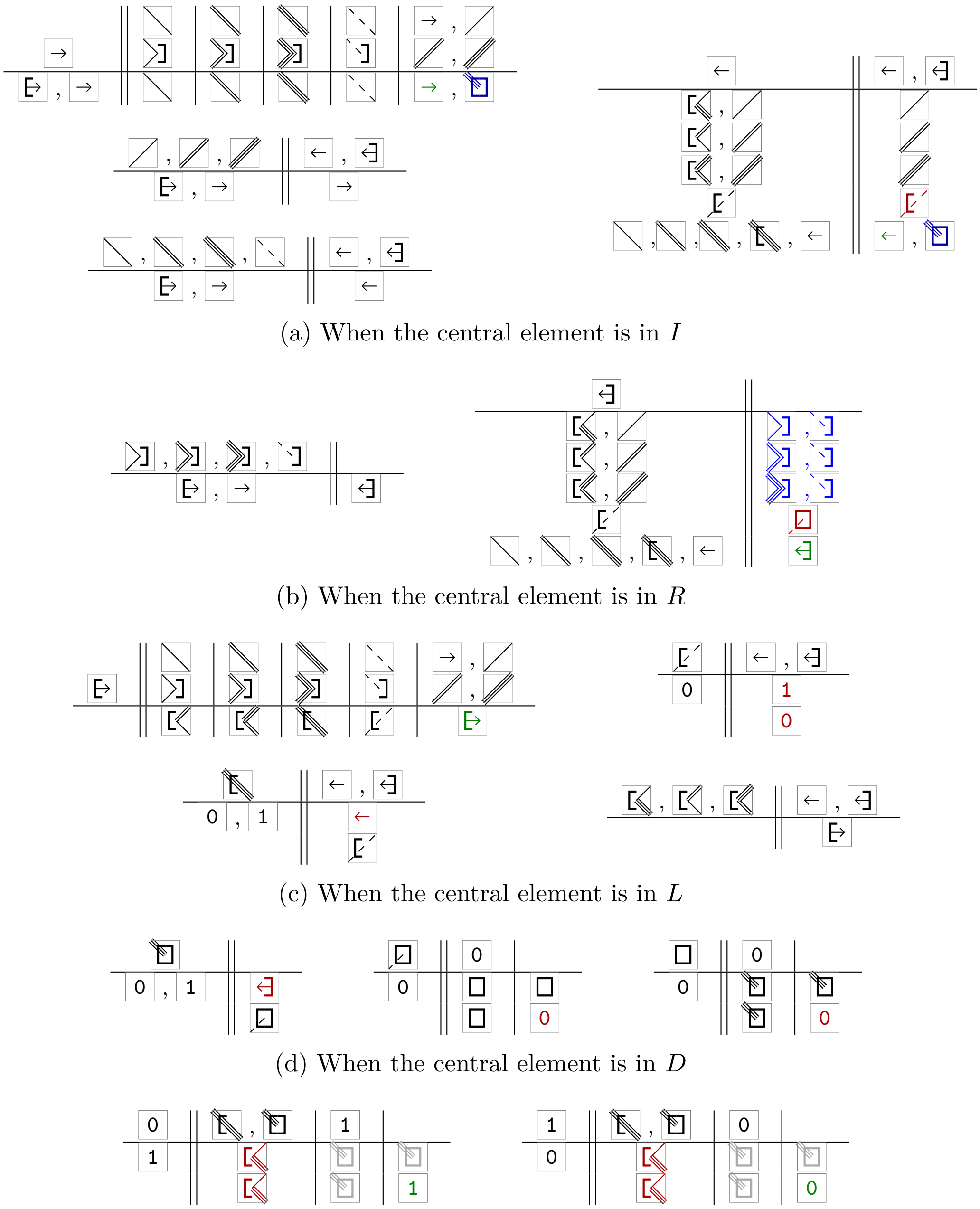
Moreover, on conditions which will be detailed in Section 3.3, the signal can decide to go into *erasure* mode (see Figure 2b) when bouncing on the right border. In this case, it goes back to the left border using the ⋱ signal and then erases the interval from left to right by pushing the left border with [´ and filling the liberated space with 0 or 1 . This last event can also occur if a grow cannot be achieved due to an obstruction on the left on the interval (see Figure 2c).

This behaviour is implemented with rules depicted in Table 2.

In this section, we will focus on the set of *pre-image* $f^{-1} : N^n \to N^{n+2}$ of finite factors of intervals.

▶ **Lemma 7.** *Any pre-image of an interval, except the initial one* ╲□ *, contains an interval.*

Table 2 Local function of the cellular automaton (N, f).



(a) When the central element is in I

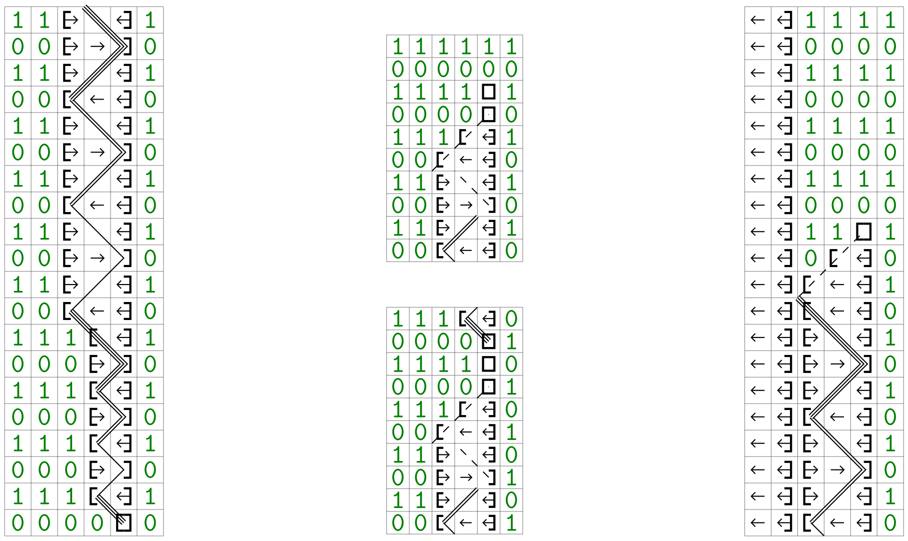
(b) When the central element is in R

(c) When the central element is in L

(d) When the central element is in D

(e) When the central element is in V

The local function is given on the form $\frac{c}{l}\Big\|\frac{r}{f(l,c,r)}$.
Any undefined case outputs ▧ . When l or r is missing, it indicates all states locally valid that have not been defined elsewhere.
Green indicates when the central element is unchanged; red indicates when the status of the central element is changed. Blue indicates non-deterministic transitions which will be treated in Section 3.2 for dark blue and in Section 3.3 for light blue. Gray indicates cases which are in fact unused after the initial step due to Lemma 10.



(a) Growth of an interval (b) Erasure of an interval (c) left blocked

Figure 2 Desired basic behaviour for interval and its implementation (note that this behaviour is not currently deterministic).

Proof. Let $(c_i)_{1\leq i\leq n} \in N^n$ be an interval and $(b_i)_{0\leq i\leq n+1} \in N^{n+2}$ be one element of its pre-image as depicted below. Cases where $n = 1$ ($c =$ ◲ or $c =$ □) can be checked directly looking at the Table 2. For the other cases, we must have $c_1 \in L$, $c_n \in R$ and $(c_i)_{2\leq i\leq n-1} \in I$.

	c_1	c_2	$\cdots$	c_{n-1}	c_n	
b_0	b_1	b_2	$\cdots$	b_{n-1}	b_n	b_{n+1}

Since $f(x)_i \in R$ if and only if $x_i \in R \cup D$ (see Table 2), then among b, only $b_n \in R$. Moreover, $f(x)_i \in L$ implies x_{i-1}, x_i or $x_{i+i} \in L$. At last, since c does not contain the state ⧅ , it must use the transitions explicitly defined in Figure 2 which force by construction the word between the two latter positions to be an interval. ◀

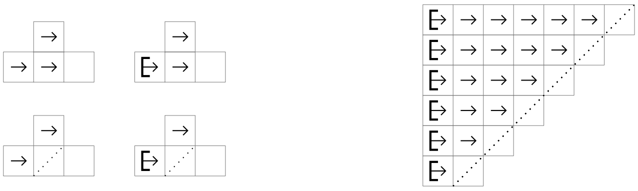
Now, let us focus on what happens in the pre-image of proper factors of intervals. The basic idea is that any proper factor must be issued from a "bigger" factor in the pre-image. To prove this, we first need to introduce a specific notion of size.

▶ **Definition 8** (Size). Given a factor $f \in N^*$ of an interval, the *size* $s(f)$ corresponds to the number of symbols except counting only one half for symbol [⋘.

For example: $s($⧅$) = 1$, $s($[⧅⊣$) = 2$, $s($[⋘←←$) = 2.5$ and $s($[→/←$) = 3$.

▶ **Lemma 9.** *Any pre-image of a proper factor of an interval contains a factor of strictly greater size.*

Proof. The first remark is that the pre-image can indeed be an interval. The proof is done by refining the previous proof. Let us consider the case when the proper factor is a prefix of an interval (the suffix case is similar). This implies that $c_n \in I$. Looking at the transition table, we have that b_n must be in I and b_{n+1} is either in I or in R. Then we have $s(b) > s(c)$, unless the left border in b is b_2; but this is only possible when $c_1 =$ [⋘, and then we have $s(b) = n$ and $s(c) = n - 1 + \frac{1}{2} = n - \frac{1}{2}$. ◀



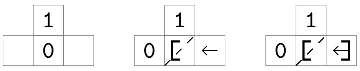
(a) Rules outputting $\boxed{\rightarrow}$ (b) Form of region in the space-time diagram.

Figure 3 Regions of $\boxed{\rightarrow}$ (here, $\boxed{\cdot^{\cdot^{\cdot}}}$ denotes either $\boxed{/}$, $\boxed{//}$ or $\boxed{///}$.)

At last, one can easily see that some words do not appear in any image:

▶ **Lemma 10.** *$\boxed{1}\ \boxed{0}$ and $\boxed{0}\ \boxed{1}$ are not possible in the image.*

Proof. Looking at Table 2, we can see that the only way to output a $\boxed{1}$ are the cases:



Since outputting a $\boxed{0}$ requires that there is no $\boxed{0}$ in the neighbourhood and that neither $\boxed{\leftarrow}$ nor $\boxed{\leftarrow\!]}$ can generate $\boxed{0}$, no $\boxed{0}$ can occur next to a $\boxed{1}$. ◀

With these first non-deterministic rules, any periodic configuration can be divided into intervals or short-lived proper factors of them (since any letter is itself a proper factor) separated by uniform spaces of 0 or 1. The only exception is when the configuration contains only one periodic proper factor of an interval, which can only be of the form ${}^{\omega}\boxed{\rightarrow}^{\omega}$ or ${}^{\omega}\boxed{\leftarrow}^{\omega}$. As we want to keep only full intervals, the option selected is to add one layer to ensure the two following properties: infinite configurations consisting only of interior will eventually disappear ; proper factors of intervals cannot be created. This is done in the following section.

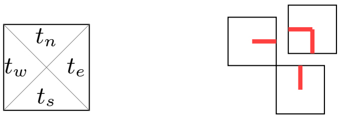
3.2 Ensuring intervals with an aperiodic tiling

To ensure the presence of at least one interval, we want to detect inside uniform $\boxed{\rightarrow}$ or $\boxed{\leftarrow}$ regions the periodicity without perturbing the normal behaviour of such regions. To do this, we shall add a layer containing an aperiodic tiling over each region independently. In this section, we shall only detail the case $\boxed{\rightarrow}$ since $\boxed{\leftarrow}$ is similar.

The first remark is that state $\boxed{\rightarrow}$ can only be generated by the rules depicted in Figure 3a leading to a region in a bottom triangular form (see Figure 3b).

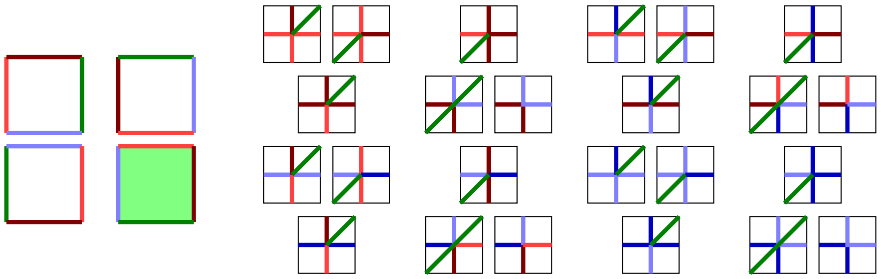
Then, we want to fill this region with a south-west deterministic tiling. Formally, a *Wang tile* t is a square tile with coloured edges, as represented in Figure 4a. It is given by a quadruplet (t_e, t_w, t_n, t_s) of symbols, called *colours*. A tile-set τ is a finite set of Wang tiles. A tiling of the plane by τ is a map t from the discrete plane $\mathbb{Z}^2$ to τ such that two tiles that share a common edge agree on its colour: For all integers i, j we have $t(i,j)_e = t(i+1,j)_w$ and $t(i,j)_n = t(i,j+1)_s$. A tile set is said to be *south-west deterministic* if there is at most one possible tile for every possible (t_s, t_w) (see Figure 4b). A tile-set τ is *aperiodic* if there exists at least a valid tiling of the plane by τ but no periodic tiling.

In this paper, we shall use Kari's tile-set [7] (or more precisely, its horizontal flip). This tile-set is based on Robinson's aperiodic tiling [9] and was introduced to study nilpotency on cellular automata. It can be described with two layers (corresponding to a Cartesian product). The first one is just a regular 4 coloured grid. The second one is usually depicted using lines. The tiles are depicted in Figure 5.



(a) A Wang tile (b) Catenation of tiles

Figure 4 Wang tiles.



(a) Background layer (b) Signals layer

Only signal tiles whose both dark left and bottom lines (, , , , , , ,) can appear above the background 🟩.

Figure 5 A simplified representation of the south-west deterministic aperiodic tile-set.

▶ **Theorem 11** (J. Kari, 1992 [7]). *The tile set in Figure 5 is aperiodic and south-west deterministic.*

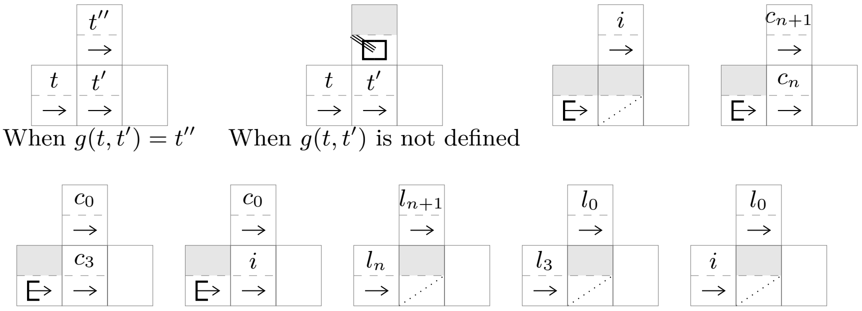
This tiling can locally be implemented as a one-way cellular automaton g over τ by associating, for $t, t' \in \tau$, the only tile $g(t, t')$ fitting (by south-west determinism). One can thus use this to construct the layer in the inside of the region by enhancing the rules as depicted in Figure 6. In case no such tile exists, the transition is defined as ⧄. It can be noted that the additional rule is compatible with Table 2 and even more, it suppresses both cases of non-determinism in case (a).

The previous idea is only valid for the inside and it leaves the case of borders open. For those, we need another property which is the existence of a "regular" north-east quarter of the plane. This is the case for the previous tiling, in particular we shall use the partial tiling depicted in Figure 7. Due to the self-similarity of the tiling, one can see the highlighted column (resp. line) consists of a periodic sequence of size 4 $\{c_0, c_1, c_2, c_3\}$ (resp. $\{l_0, l_1, l_2, l_3\}$) with the possible exception of the common initial tile i. We can use this property to define the last remaining cases of Figure 3a to ensure the filling of the region exists and is as depicted in Figure 6.

Let us now prove that the additional layer does indeed do what we want: forbid uniform configuration of $\boxed{\rightarrow}$ or $\boxed{\leftarrow}$, and forbid proper prefix of intervals. This is done in the two following lemmas.

▶ **Lemma 12.** *On a periodic configuration of period N, after less than $|\tau|^N$ steps of the automaton, the projected configurations ${}^{\omega}\boxed{\rightarrow}^{\omega}$ and ${}^{\omega}\boxed{\leftarrow}^{\omega}$ cannot appear.*

Proof. By construction, such configurations contain a layer with the aperiodic tiling for every pre-image. By contradiction, if the configuration appears after $|\tau|^N$ steps, we can extract $|\tau|^N$ configurations for the tiling layer. In this case, at least two of them are identical and can be glued into a valid periodic tiling leading to a contradiction. ◀



Tiles i, c_i and l_i are the specific tiles on the border in Figure 7. Tile i is the bottom-left corner. Sequence c_0, c_1, c_2, c_3 (resp. l_0, l_1, l_2, l_3) is the periodic sequence in the leftmost column (resp. bottom line).

Figure 6 Additional aperiodic layer.

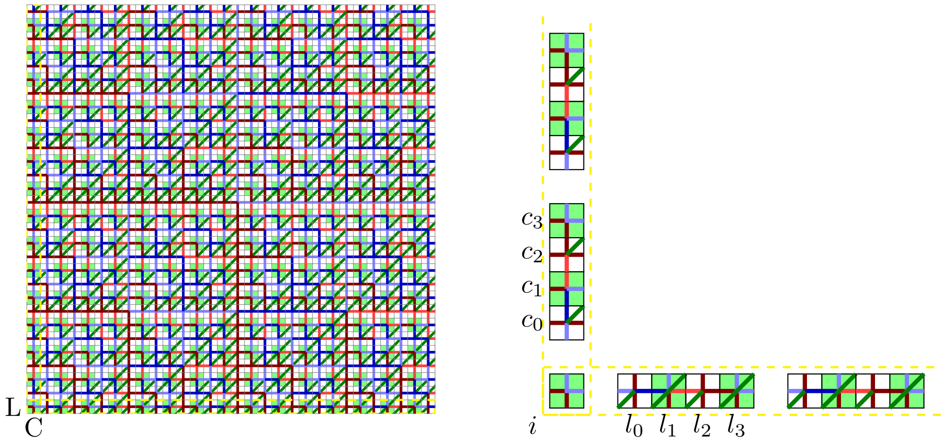


Figure 7 Valid quarter of the plan tiling with regular diagonals.

Let us call $T^K_{\mathtt{max}}(n)$ the bound in the previous lemma. This bound (which is independent of the tiling) is exponential in N. For the specific tiling used here, a better study could almost surely prove a quadratic bound.

More than just eliminating uniform configurations we do not want, this layer avoids the creation of proper prefix of intervals.

▶ **Lemma 13.** *On a periodic configuration of period N, after at most $2N + \max(T^K_{max}(N), 2N)$ steps, the configuration does not contain any proper factor of interval.*

Proof. By contradiction, assume there exists such a proper factor. We can look at its sequence of pre-images.

If this sequence does not contain a full interval, by Lemma 9, it must increase in size (and thus in length) and thus reaches one uniform configuration ${}^{\omega}\boxed{\rightarrow}^{\omega}$ or ${}^{\omega}\boxed{\leftarrow}^{\omega}$ in less than $2N$ steps which contradicts Lemma 12.

Let us consider now the step when the pre-image b is an interval and the image c is a strict factor. The key point is that this case can only happen when there is an error in the aperiodic layer outputting a symbol ▨, otherwise, the image of an interval is an interval. Without loss of generality, we can assume that this error occurs over the symbol $\boxed{\rightarrow}$.

In all cases, the error over $\boxed{\rightarrow}$ occurs in a triangle of such states. However, the transitions are made so that any point of the triangle behaves as the tiling depicted in Figure 7.

Thus there cannot be errors in the tiling. This implies that c is an interval, which is a contradiction. ◀

Since no proper prefix of interval may be created after a certain time denoted as $T_R(N)$, we can thus give a precise characterization of configurations which can appear after some initial transient behaviour.

▶ **Proposition 14.** *After $T_R(N)$ steps, the configuration consists only of intervals possibly separated by uniform portions of* 0 *or* 1*.*

Proof. This result follows directly from the two previous Lemmas and Lemma 10. ◀

Moreover, in this case, we have some additional very useful property.

▶ **Lemma 15.** *After $T_R(N)$ steps, no right border is created and the number of intervals in the configuration is decreasing.*

Proof. It is sufficient to remark, looking at Figure 2, that only transitions outputting ⧄ can create a right border and that it cannot happen in the conditions depicted by the previous lemma. Either because of the form of configurations (for the gray cases) or because there cannot be an error (by the proof of Lemma 13). ◀

3.3 Comparing intervals

The last point of the construction is to effectively synchronize. The idea is to keep the largest interval by comparing any two adjacent intervals and suppressing the smaller one. Here, we shall at last use the property of primality which will ensure, in some sense, that such a largest interval exists.

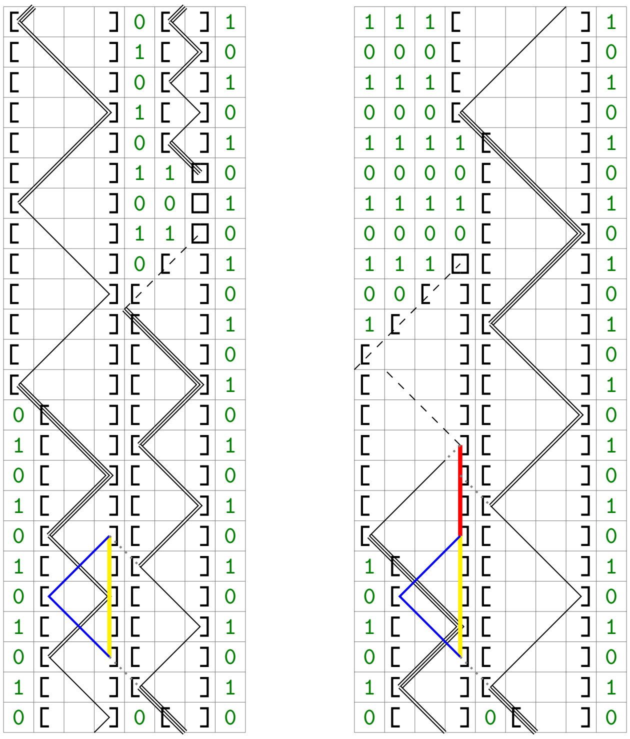
For this last proof, we add a layer C which contains a new full speed signal (╲ , ╱) and two vertical markers (│ , │). This layer works in the following way (see Figure 8 for a global view): Each time it encounters a new right neighbour (that is, when an interval on its left has grown and reaches it), a comparison is started. This comparison launches the new signal inside the left interval and adds a static marker to indicate a undergoing comparison. The new signal makes a round trip around the left interval whereas the signal (╲ , ╱) makes a similar trip in the right interval. The result of the comparison is determined according to which of these signals comes back first on the marker. If the new signal arrives first (meaning the left interval is strictly smaller than the right one), then a mark is left to call for the destruction of the left interval by ensuring that the next time a signal ╱ , ⫽ or ⫻ arriving is transformed into ˥ (lifting the last non-deterministic case). In the other case, nothing is done since the right interval will erase itself if it cannot grow.

One notable property is that the signal takes some time to go back and forth so the comparison is not immediate and synchronous.

Let us now prove the constructed cellular automaton achieves the desired synchronizing behaviour. For a fixed period N, the number of possible configurations is finite and thus any configuration eventually reaches a cycle. In this cycle, the number of intervals is constant. Let us define as s_∞ the maximal size of an interval inside this cycle.

▶ **Lemma 16.** *The case $s_\infty = 1$ is not possible.*

Proof. By contradiction assume that $s_\infty = 1$. Since all intervals are of maximal size 1, there is no L, R or I states in the configuration. Thus, it only consists of states either in V or in D. After some time, those are stable since no interval is created nor destroyed. Let us look



(a) left neighbour is larger (b) left neighbour is strictly smaller

Figure 8 Comparison of intervals.

at some state in D at this point. Looking at the transition rule, the only possibility for this state to stay in D is to alternate the sequence ⧅, ⧄, □ over time. However, the transition ⧅ → ⧄ requires its left state to be in D whereas the one ⧄ → □ requires it to be in V, which is not possible. ◀

▶ **Lemma 17.** *The case $s_\infty \geq 2$ is not possible.*

Proof. By contradiction assume that $s_\infty \geq 2$. There exists a configuration with an interval of size s_∞. This interval must be persistent over time and thus makes a cycle going from ⧅ up to an interval of size s_∞ then erases itself to ⧄, □ and then goes back to ⧅. For the proof, let us look at the moment the interval goes into erasing mode. This can be due to two different cases: either because its growth was denied or because a comparison has produced an erasing signal (see Figure 2). Let us review the two cases.

When the interval disappears because of left blocking. In this case, the first trivial remark is that there must be an interval to the left. As right borders are persistent in this case conditions, there must be an interval directly to the left. Let us now look at the moment t_0 our interval has grown to size s_∞. It has thus launched a comparison.

If the interval to the left is of size $s < s_\infty$, then the comparison would have detected this before instant $t_0 + 2(s-1)$ which is the time for the signal to do a round trip. Moreover, this comparison would have resulted, for the left neighbour, in the appearance of an erasure signal after at most $2(s-1)$ steps. At last, this results in the erasure of the left interval in less than $2(s-1)+2$ steps. Since our interval is alive for at least $6*(s_\infty - 1)$ steps, the left interval would have disappeared before our interval even tries to grow which contradicts it being blocked.

Thus we can assume that the interval to the left is of size s_∞ when comparing. It implies that it will also erase itself later which can only be for the same reason (blocked) as the initial interval.

By iterating the proof, we can see that there is an infinite sequence of intervals which are all of size s_∞. Even if this sequence is found at different times, since right borders do not move, appear, or disappear, this means that the configuration has a right border every s_∞ cells. At last, since the period is a prime number, this implies that there is only one interval (of size s_∞). However, in this case, it is easy to see that it goes into the uniform configuration ${}^{\omega}\boxed{0}^{\omega}$ at the end of the erasure, which contradicts $s_\infty > 0$.

When the interval disappears because it is erased. This proof is very similar to the previous one. This case is only possible if the interval is strictly smaller than its right neighbour. This can indeed happen if the comparison was done just before an increase. In this case, the right neighbour was of size s_∞ (before). Moreover, since the erasure is, as in the previous case, done before the right neighbour grows, it implies that the latter also disappears because of erasure. Thus, we are in the same case, when the configuration is split into intervals regularly spaced which can only mean there is only one interval. This leads to the same contradiction as before. ◀

4 Generalisation and extension

To summarise, this paper shows that synchronisation cannot be achieved in a spatially uniform deterministic context with only local information. However, it gives one sufficient and surprising global additional condition (primality) that can be exploited to achieve synchronisation. Moreover, an anonymous referee has hinted that the proof can be extended to the odd case by forcing the lifetime of an interval to be even. In that case, the contradiction for Lemma 17 is achieved by the fact that at least two consecutive intervals should have a even shift between which would results in their disappearance.

The construction done in the previous section can be extended to achieve one cycle of any size. Once this is done, the same remark as in the end of Section 2 applies: using Cartesian product, it is easy to add any additional distinct number of cycles of any length.

Nevertheless, our construction is quite complex and relies heavily on the fact that the dimension is one. This opens immediately the question of the validity of this result in higher dimension; especially since the extension of the primality condition is not clear. One other extension is to consider robust computation by introducing in the transition rule a small probability of error (see [6]). In this case, it may even be possible to get rid of the primality requirement.

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