Decidable Logics with Associative Binary Modalities*

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Abstract

A new family of modal logics with an associative binary modality, called *counting logics* is proposed. These propositional logics allow to express finite cardinalities of sets and more generally to count the number of subsets satisfying some properties. We show that these logics can be seen both as specializations of the Boolean logic of bunched implications and as generalizations of the propositional dependence logic. Moreover, whereas most logics with an associative binary modality are undecidable, we prove that some counting logics are decidable, in particular the basic counting logic bCL. We conjecture that this interesting result is due to the valuation constraints in counting logics' semantics and prove that the logic corresponding to bCL without these constraints is undecidable. Finally, we give lower and upper bounds for the complexity of bCL's validity problem.

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1 Introduction

Although most modal logics have only unary modalities, the concept of modality can easily be extended to any arity. In the present work, we are interested in binary modalities with a relational semantics: for an existential[1] binary modality $*$ there is a ternary relation R such that any formula of the form $\varphi * \psi$ is true at a state w if and only if there are two states x, y such that $R(w, x, y)$, φ is true at x and ψ is true at y. In many logics with such a binary modality, like separation logics [14], logics of bunched implications [13], interval logics [16] and ambient logics [5], the ternary relation R is interpreted as a decomposition of the current state into its constituents: $R(w, x, y)$ is read both as "the state w can be decomposed into the states x and y" and as "combining the states x and y produces the state w". In such a reading, it is usually expected that if a state w can be obtained by combining the state x with the combination of the states y and z then w can also be obtained by first combining x with y and then combining the resulting state with z. This requirement makes the binary modality associative. But it has been proved by Kurucz, Németi, Sain and Simon [8] that most logics with an associative binary modality are undecidable. Hence, many of the aforementioned

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[1] Whereas for unary modality, the universal modality $\Box$ is generally considered as the principal modality and the existential modality $\Diamond$ is defined as its dual, it is generally the contrary for binary modalities. The reason is the close relation between existential binary modalities and tensors of substructural logics.

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logics are undecidable, in particular the more general and abstract ones like the Boolean logic of bunched implications BBI [9, 3].

Despite this fact, a few specialized logics with an associative binary modality are decidable, like the propositional separation logic kSL0 [4], the propositional dependence logic [18] and Pandya's interval logic [12]. In the present work, we try to explain the reason why these logics are decidable. We observe that in each of the aforementioned decidable logics, the valuation of atomic formulas at any state w is not free but depends on the valuation at the components of w. Put differently, the properties of any compound state are determined by the properties of its atomic constituents. We conjecture that this feature is the main cause of the decidability of these logics. To justify this claim, we propose a new family of logics with associative binary modalities that have this property. We call them *counting logics*. We prove that some logics in this family are decidable, including the basic counting logic bCL. More precisely, we prove that bCL's validity problem is NEXPTIME-hard and in 4EXPTIME. In order to start to establish the limit of decidability for logics with associative binary modalities, we prove that the variant of bCL with free valuations is undecidable and that counting logics are more specific than the undecidable logic BBI and more general than the decidable propositional dependence logic. This latter result gives new insights into the relation between team logics and logic of separation, which was already noticed in [1].

Finally, beside the theoretical interest of counting logics, we believe that counting logic could have practical applications. Indeed, in many situations, the properties of compound elements are determined by the properties of their constituents. It could be the case, for instance, for resources and for sets of agents. Moreover, as their name suggests, counting logics allow to count: they can express that an element is composed of a given finite number of atomic constituents and that it can be decomposed into a given finite number of components satisfying a given property.

2 Language and Semantics

Given a countable infinite set Φ_0 of propositional variables, the basic language Φ of counting logics is defined inductively by the following grammar:

$$\varphi, \psi \; \coloneqq \; p \;\mid\; \neg\varphi \;\mid\; \top \;\mid\; (\varphi \wedge \psi) \;\mid\; \boldsymbol{I} \;\mid\; (\varphi * \psi)$$

where p is any propositional variable. The missing operators of classical propositional logic can be defined in the usual way. These classical operators are called *additive*. They are complemented with the *multiplicative* conjunction $*$ and its neutral element $\boldsymbol{I}$. Hence the language of counting logics is closely related to the languages of substructural logics like the Boolean logic of bunched implications BBI [13]. Intuitively, the formula $(\varphi * \psi)$ means that the current state can be decomposed into two complementary states, one satisfying φ, the other satisfying ψ. As usual, the parentheses may be omitted for clarity, the negation having the highest priority and the implication the lowest. For any formula φ, $\mathrm{SF}(\varphi)$, $\mathrm{PV}(\varphi)$ and $|\varphi|$ denotes, respectively, the set of subformulas of φ, the set of propositional variables occurring in φ and the cardinality of $\mathrm{SF}(\varphi)$. Finally, the following abbreviations are defined:

- The dual $\circledast$ of $*$ is defined by $\varphi \circledast \psi \doteq \neg(\neg\varphi * \neg\psi)$.
- The minimal cardinality predicate $\geq n$ is defined inductively by $\geq 0 \doteq \top$ and for all $n > 1$, $\geq n \doteq \neg\boldsymbol{I} * \geq(n-1)$.
- The cardinality predicate $=n$ is defined for all $n \in \mathbb{N}$ by $=n \doteq \geq n \wedge \neg\geq(n+1)$.
- The universal modality $\Box$ is defined by $\Box\varphi \doteq \bot \circledast \varphi$.

The extension Φ^{-*} of Φ with a multiplicative conditional will also be considered. The additional symbol $-\!\!*$, called the *magic wand*, is the residual of the multiplicative conjunction. Intuitively, the formula $(\varphi -\!\!* \psi)$ means that any state obtained by composing the current state with any state satisfying φ satisfies ψ. In fact, Φ^{-*} is exactly the language of the Boolean logic of bunched implications BBI.

A *subset space model* is a tuple $\mathcal{M} = (X, \mathcal{O}, V)$ where X is an arbitrary[2] set called the *universe*, $\mathcal{O}$ is a non-empty set of subsets of X and V is valuation function assigning a subset[3] of X to each propositional variable in Φ_0. We write V^{-1} for the function from X to the powerset of Φ_0, defined by $V^{-1}(x) \doteq \{p \in \Phi_0 \mid x \in V(p)\}$.

Formulas of Φ are interpreted at subsets in subset space models. We write $\mathcal{M}, S \vDash_C \varphi$ if the formula $\varphi \in \Phi$ is satisfied at the subset $S \in \mathcal{O}$ in the subset space model $\mathcal{M} = (X, \mathcal{O}, V)$. The relation $\vDash_C$ is defined inductively by:

$$
\begin{array}{ll}
\mathcal{M}, S \vDash_C p & \text{iff } S \cap V(p) \neq \emptyset \\
\mathcal{M}, S \vDash_C \top & \text{always} \\
\mathcal{M}, S \vDash_C \neg\varphi & \text{iff } \mathcal{M}, S \nvDash \varphi \\
\mathcal{M}, S \vDash_C \varphi \wedge \psi & \text{iff } \mathcal{M}, S \vDash_C \varphi \text{ and } \mathcal{M}, S \vDash_C \psi \\
\mathcal{M}, S \vDash_C \boldsymbol{I} & \text{iff } S = \emptyset \\
\mathcal{M}, S \vDash_C \varphi * \psi & \text{iff there is } S_1, S_2 \in \mathcal{O} \text{ such that} \\
& \quad S = S_1 \uplus S_2, \mathcal{M}, S_1 \vDash_C \varphi \text{ and } \mathcal{M}, S_2 \vDash_C \psi
\end{array}
$$

where $\uplus$ is the disjoint union of sets, i.e., the partial binary operator over sets such that $A = B \uplus C$ iff $B \cap C = \emptyset$ and $A = B \cup C$. This interpretation is extended to the language Φ^{-*} by the following additional rule:

$$
\begin{array}{ll}
\mathcal{M}, S \vDash_C \varphi -\!\!* \psi & \text{iff for all } S_1, S_2 \in \mathcal{O}, \text{ if } S_2 = S_1 \uplus S \text{ and } \mathcal{M}, S_1 \vDash_C \varphi \\
& \qquad \text{then } \mathcal{M}, S_2 \vDash_C \psi
\end{array}
$$

A formula $\varphi \in \Phi$ is *valid* in a subset space model $\mathcal{M} = (X, \mathcal{O}, V)$, denoted by $\mathcal{M} \vDash_C \varphi$, iff $\mathcal{M}, S \vDash_C \varphi$ for all $S \in \mathcal{O}$. It is valid in a class $\mathcal{C}$ of subset space models, denoted by $\mathcal{C} \vDash_C \varphi$, iff for all models $\mathcal{M}$ in $\mathcal{C}$, $\mathcal{M} \vDash_C \varphi$.

A subset space model $\mathcal{M} = (X, \mathcal{O}, V)$ may have some of the following properties:

Closure under inclusion $\mathcal{M}$ is closed under inclusion if for all $S \in \mathcal{O}$ and $S_1 \subseteq S$, $S_1 \in \mathcal{O}$.

Finiteness $\mathcal{M}$ is finite if X is finite.

Finite completeness $\mathcal{M}$ is finitely complete if $\mathcal{O}$ is the set of all finite subsets of X.

Atomicity $\mathcal{M}$ is atomic if for all $x \in X$ there is a unique $p \in \Phi_0$ such that $x \in V(p)$.

A class $\mathcal{C}$ of subset space models is said to have any of these properties if all the models in $\mathcal{C}$ have the property.

A *counting logic* is any logic which is obtained by interpreting the language Φ (or any extension of it) in a class of subset space models that are closed under inclusion. The basic counting logic bCL is the logic obtained by interpreting Φ in the class $\mathcal{C}_{\text{all}}$ of all subset space models that are closed under inclusion. The name of these logics is justified by the following proposition.

▶ **Proposition 1.** *For any subset space model $\mathcal{M} = (X, \mathcal{O}, V)$ that is closed under inclusion, any $S \in \mathcal{O}$ and any $n \in \mathbb{N}$, $\mathcal{M}, S \vDash {\geq}n$ iff $|S| \geq n$.*

[2] Since formulas are evaluated at subsets, X may be empty if $\mathcal{O} = \{\emptyset\}$.
[3] This subset does not need to belong to $\mathcal{O}$.

Proof. The proof is by induction on n. The base case and the left-to-right direction of the inductive case are trivial and do not use the closure under inclusion property of $\mathcal{M}$. For the right-to-left direction, suppose $|S| \geq n > 0$. There is $x \in S$ and since $\mathcal{M}$ is closed under inclusion, $\{x\}$ and $S \setminus \{x\}$ belong to $\mathcal{O}$. The conclusion is straightforward. ◀

Actually, this proposition can be generalized to any formula. Given any formula φ, construct inductively the sequence $(\sqsupseteq_k^\varphi)_{k \in \mathbb{N}}$ such that $\sqsupseteq_0^\varphi \doteq \top$ and $\sqsupseteq_{k+1}^\varphi \doteq \varphi \ast \sqsupseteq_k^\varphi$ for all $k \in \mathbb{N}$. The previous proof can easily be adapted to show that for any subset space model $\mathcal{M} = (X, \mathcal{O}, V)$ that is closed under inclusion and any subset $S \in \mathcal{O}$, $\mathcal{M}, S \vDash_C \sqsupseteq_k^\varphi$ if and only if there is at least k disjoint subsets of S satisfying φ in $\mathcal{M}$.

It can easily be checked that the multiplicative conjunction is associative and commutative. Moreover it is a binary modality in the sense that, for any subformulas φ, ψ and χ and any class $\mathcal{C}$ of subset space models we have: $\mathcal{C} \vDash_C (\varphi \circledast (\psi \to \chi)) \to (\varphi \circledast \psi) \to (\varphi \circledast \chi)$ and if $\mathcal{C} \vDash_C \varphi$ then $\mathcal{C} \vDash_C \psi \circledast \varphi$. Nevertheless, no non-trivial counting logics are normal modal logics because the inference rule of uniform substitution is not admissible, as shown by the following proposition.

▶ **Proposition 2.** *Let $\mathcal{C}$ be any class of subset space models that is closed under inclusion and such that there is a model $\mathcal{M} = (X, \mathcal{O}, V)$ in $\mathcal{C}$ with $|\mathcal{O}| > 1$. The inference rule of uniform substitution is not admissible in the counting logic* L *obtained by interpreting the language Φ in $\mathcal{C}$.*

Proof. We prove that the formula $(p \ast \neg \boldsymbol{I}) \to p$ is valid in L while $(\boldsymbol{I} \ast \neg \boldsymbol{I}) \wedge \neg \boldsymbol{I}$ is satisfiable in L. Suppose $\mathcal{M}, S \vDash p \ast \neg \boldsymbol{I}$ for some model $\mathcal{M} = (X, \mathcal{O}, V)$ in $\mathcal{C}$ and some $S \in \mathcal{O}$. There is a subset $S_1 \subset S$ such that $S_1 \cap V(p) \neq \emptyset$. Therefore, $S \cap V(p) \neq \emptyset$ and $\mathcal{M}, S \vDash p$. Now, let $\mathcal{M} = (X, \mathcal{O}, V)$ be a model in $\mathcal{C}$ such that $|\mathcal{O}| > 1$. There is a subset $S \in \mathcal{O}$ such that $S \neq \emptyset$, hence $\mathcal{M}, S \vDash \neg \boldsymbol{I}$. Moreover, since $\mathcal{M}$ is closed under inclusion, $\emptyset \in \mathcal{O}$. Since $S = \emptyset \uplus S$, $\mathcal{M}, S \vDash \boldsymbol{I} \ast \neg \boldsymbol{I}$. ◀

The non-admissibility of the uniform substitution is clearly due to the constraint on the "valuation" of subsets in subset space models. Broadly, the valuation of a subset is not free but is determined by the valuation of its constituents. Observing that most decidable logics with an associative binary modalities have such kind of constraints, like the propositional separation logic kSL0 [14, 4] and the propositional dependence logic [18], we conjecture that this property is a key feature for a logic with an associative binary modality to be decidable.

3 Undecidability of the free disjoint union logic

We justify our claim that the constraint on the valuation of subsets in the semantics of counting logic is essential for the decidability of the basic counting logic. For that purpose, a new logic is defined that corresponds to the basic counting logic with free valuations. The logic is called the free disjoint union logic FDUL, and is proved to be undecidable.

A disjoint union model is a triple $\mathcal{M} = (\Omega, W, V)$ where Ω is a set, W a non-empty set of subsets of Ω that is closed under inclusion and V a valuation function assigning a subset of W to each propositional variable. Observe that the only difference with subset space models is that the valuation assigns subsets of W instead of subsets of Ω. The new satisfiability relation $\vDash_F$ between disjoint union models $\mathcal{M} = (\Omega, W, V)$, subsets $S \in W$ and formulas in Φ is defined by similar rules than for $\vDash_C$, except for the propositional variables in which case $\mathcal{M}, S \vDash_F p$ iff $S \in V(p)$. The free disjoint union logic FDUL is the logic obtained by interpreting Φ in the class of all disjoint union models. We first prove that FDUL does not have the finite model property with respect to these semantics.

▶ **Lemma 3.** *If* $\mathcal{M}, S \vDash_F p \wedge \Box(q \to {=}1) \wedge \Box(p \to (p * q))$ *then there is an infinite sequence* $x_1 x_2 \ldots$ *of pairwise distinct elements of* S *such that for all* $k \geq 1$, $\{x_k\} \in V(q)$ *and* $S \setminus \{x_1, \ldots, x_k\} \in V(p)$.

Proof. The infinite sequence $x_1 x_2 \ldots$ is constructed by induction on k. For $k = 1$, since $\mathcal{M}, S \vDash_F (p * q) \wedge \Box(q \to {=}1)$, there is $x_1 \in S$ such that $\{x_1\} \in V(q)$ and $S \setminus \{x_1\} \in V(p)$. For $k > 1$, let $S_k \doteq S \setminus \{x_1, \ldots, x_{k-1}\}$. By induction hypothesis, $S_k \in V(p)$. Therefore $\mathcal{M}, S_k \vDash_F (p * q) \wedge \Box(q \to {=}1)$ and the construction is similar as in the base case. ◀

To prove the undecidability of FDUL's satisfiability problem, the algebraic method proposed in [8] is difficult to apply because the underlying binary operator for the binary modality is partial. We use instead the following tiling problem [6]. A tiles set is a triple $(\mathcal{T}, \mathit{horz}, \mathit{vert})$ where $\mathcal{T}$ is a finite set whose elements are called tiles, and horz and vert are binary relations over $\mathcal{T}$. A tiling of $\mathbb{N} \times \mathbb{N}$ by the tiles set $(\mathcal{T}, \mathit{vert}, \mathit{horz})$ is a function $t : \mathbb{N} \times \mathbb{N} \longrightarrow \mathcal{T}$ such that for all $(x, y) \in \mathbb{N} \times \mathbb{N}$, $(t(x, y), t(x + 1, y)) \in \mathit{horz}$ and $(t(x, y), t(x, y + 1)) \in \mathit{vert}$. The $\mathbb{N} \times \mathbb{N}$ tiling problem consists in deciding whether there is a tiling of $\mathbb{N} \times \mathbb{N}$ by a given tiles set. This problem has been proved to be undecidable in [2].

Given a tiles set $(\mathcal{T}, \mathit{horz}, \mathit{vert})$, we construct a formula $\Psi \in \Phi$ such that Ψ is FDUL-satisfiable if and only if there is a tiling of $\mathbb{N} \times \mathbb{N}$ by $(\mathcal{T}, \mathit{horz}, \mathit{vert})$. First, we associate to each tile $i \in \mathcal{T}$ three propositional variables p_i, r_i and u_i. Intuitively, p_i, r_i and u_i state that the tile i is at the current position, on the right and above, respectively. Additionally, we use the propositional variables c, v and h, whose intuitive meaning will become clear shortly. All the aforementioned propositional variables must be pairwise distinct. The formula Ψ is defined as the conjunction of the following subformulas.

$$c \wedge \Box(v \vee h \to {=}1 \wedge (\neg v \vee \neg h)) \wedge \Box(c \to (v * c) \wedge (h * c)) \tag{1}$$

$$\Box\left(c \to \bigvee_{i \in \mathcal{T}} p_i\right) \tag{2}$$

$$\bigwedge_{i \in \mathcal{T}} \Box\left(p_i \to \bigwedge_{j \neq i} \neg p_j \wedge \bigvee_{j \in \mathit{horz}(i)} r_j \wedge \bigvee_{j \in \mathit{vert}(i)} u_j\right) \tag{3}$$

$$\bigwedge_{i \in \mathcal{T}} \Box\left(u_i \to (\neg v \circledast (c \wedge p_i)) \wedge \bigwedge_{j \neq i} \neg u_j\right) \tag{4}$$

$$\bigwedge_{i \in \mathcal{T}} \Box\left(r_i \to (\neg h \circledast (c \wedge p_i)) \wedge \bigwedge_{j \neq i} \neg r_j\right) \tag{5}$$

▶ **Lemma 4.** *There is a tiling of* $\mathbb{N} \times \mathbb{N}$ *by* $(\mathcal{T}, \mathit{horz}, \mathit{vert})$ *if and only if* Ψ *is FDUL-satisfiable.*

Proof. For the left-to-right direction, suppose there is a tiling t of $\mathbb{N} \times \mathbb{N}$ by $(\mathcal{T}, \mathit{horz}, \mathit{vert})$. Then for each cofinite subset S of $\mathbb{N}$, let x_S be the number of even numbers not in S and y_S the number of odd numbers not in S. The model $\mathcal{M} = (\Omega, W, V)$ is constructed such that $\Omega = \mathbb{N}$, $W = \mathcal{P}(\mathbb{N})$, $V(c)$ is the set of all cofinite subsets, $V(h)$ is the set of all singletons whose element is even, $V(v)$ is the set of all singletons whose element is odd, and for all $i \in \mathcal{T}$, a set $S \subseteq \mathbb{N}$ belongs to $V(p_i)$ (resp. r_i and u_i) iff S is cofinite and $t(x_S, y_S) = i$ (resp. $t(x_S + 1, y_S) = i$ and $t(x_S, y_S + 1) = i$). It can easily be checked that $\mathcal{M}, \mathbb{N} \vDash_F \Psi$. For instance for (4), suppose that $\mathcal{M}, S \vDash_F u_i$ and $S = S_1 \uplus S_2$. By definition, S is cofinite. Moreover, if $\mathcal{M}, S_1 \vDash_F v$ then S_1 is a singleton whose element is odd, S_2 is cofinite and $(x_{S_2}, y_{S_2}) = (x_S, y_S + 1)$, hence $\mathcal{M}, S_2 \vDash_F c \wedge p_i$.

For the right-to-left direction, suppose $\mathcal{M}, S \vDash_F \Psi$ for some $\mathcal{M} = (\Omega, W, V)$ and $S \in W$. Since S satisfies (1), by Lemma 3, there are two sequences $x_1x_2\ldots$ and $y_1y_2\ldots$ of distinct elements of S such that for all $k \geq 1$, $\{x_k\} \in V(h)$, $\{y_k\} \in V(v)$ and $S \setminus \{x_1, \ldots, x_k\} \in V(c)$. For all $(x, y) \in \mathbb{N} \times \mathbb{N}$, let $\mathcal{S}(x, y) \doteq S \setminus (\{x_k \mid k \leq x\} \cup \{y_k \mid k \leq y\})$. Using (4), it can easily be proved by induction on y that for all $(x, y) \in \mathbb{N} \times \mathbb{N}$, $\mathcal{S}(x, y) \in V(c)$. By (2) and (3), the function f from $V(c)$ to $\mathcal{T}$ can be defined such that $f(S') = i$ iff $S' \in V(p_i)$. It can easily be checked that the composition of f and $\mathcal{S}$ is a tiling of $\mathbb{N} \times \mathbb{N}$ by $(\mathcal{T}, \mathit{horz}, \mathit{vert})$. ◀

As a consequence, we have the following proposition.

▶ **Proposition 5.** *The satisfiability problem of* FDUL *is undecidable.*

4 Finite Model Property

We prove that the basic counting logic bCL has the finite model property and that its validity problem is decidable. We believe that the most important result is the finite model property since decidability can be deduced from it (see Section 7). Moreover, neither the free disjoint union logic FDUL from Section 3 nor the basic counting logic with magic wand $\mathsf{bCL}^{-\!*}$ has this property (with respect to the proposed semantics).

▶ **Lemma 6.** *For any subset space model $\mathcal{M} = (X, \mathcal{O}, V)$ and any subset $S \in \mathcal{O}$, if $\mathcal{M}, S \vDash_C \top -\!\!* \neg(\neg \boldsymbol{I} -\!\!* \bot)$ then X is infinite.*

Proof. Suppose that $\mathcal{M}, S \vDash_C \top -\!\!* \neg(\neg \boldsymbol{I} -\!\!* \bot)$, $\mathcal{M} = (X, \mathcal{O}, V)$ and X is finite. The cardinality n of S belongs to $\mathbb{N}$. It can easily be proved by induction on k that for all $k \geq n$, there is a subset $S_k \in \mathcal{O}$ with cardinality k such that $S \subseteq S_k$. ◀

Notice that the previous lemma is weaker than Lemma 3 since it does not prove that $\mathsf{bCL}^{-\!*}$ does not have the finite-subset model property: we do not know whether there is a formula of $\Phi^{-\!*}$ which is $\mathsf{bCL}^{-\!*}$-satisfiable only at infinite subsets.

The results of this section are obtained by a faithful translation from bCL to the monadic second-order logic without functions MSO. The language Φ_{MSO} of this logic is defined from a set of term variables and a set of predicate variables by the following grammar:

$$\varphi, \psi \;\coloneqq\; P(x) \;\mid\; x = y \;\mid\; \neg\varphi \;\mid\; \top \;\mid\; (\varphi \wedge \psi) \;\mid\; (\forall x.\varphi) \;\mid\; (\forall P.\varphi)$$

where x and y are term variables and P a predicate variable. The missing boolean operator and the existential quantifiers are defined as usual. The formulas of Φ_{MSO} are evaluated by the relation $\vDash_{\mathsf{MSO}}$ at triples $(\mathfrak{D}, \mathfrak{t}, \mathfrak{p})$ where $\mathfrak{D}$ is a non-empty set, $\mathfrak{t}$ is a function assigning an element of $\mathfrak{D}$ to each term variable and $\mathfrak{p}$ is a function assigning a subset of $\mathfrak{D}$ to each predicate variable. The definition of $\vDash_{\mathsf{MSO}}$ is standard.

We now define the translation function τ from the language Φ of bCL to the language Φ_{MSO} of MSO. First, to each propositional variable $p \in \Phi_0$ is associated the predicate variable Q_p. The translation τ is also parametrized by a predicate variable and is defined inductively as follows:

$$\tau(P, p) \doteq (\exists x.P(x) \wedge Q_p(x)) \qquad \tau(P, \top) \doteq \top$$
$$\tau(P, \boldsymbol{I}) \doteq (\forall x.\neg P(x)) \qquad \tau(P, \neg\varphi) \doteq \neg\tau(P, \varphi)$$
$$\tau(P, \varphi \wedge \psi) \doteq \tau(P, \varphi) \wedge \tau(P, \psi)$$
$$\tau(P, \varphi * \psi) \doteq (\exists P_1 \exists P_2.\, (\forall x.P(x) \leftrightarrow (P_1(x) \vee P_2(x))) \wedge (\forall x.\neg P_1(x) \vee \neg P_2(x)) \wedge \tau(P_1, \varphi) \wedge \tau(P_2, \psi))$$

where the symbols P_1 and P_2 are chosen to be distinct from P.

▶ **Lemma 7.** *The formula φ is* bCL*-satisfiable if and only if $\tau(P,\varphi)$ is* MSO*-satisfiable.*

Proof. For the left-to-right direction, suppose $\mathcal{M}, S \vDash_C \varphi$ and let $\mathfrak{D} \doteq S$. It can easily be proved by structural induction on ψ that for any subformula ψ of φ, any subset $S' \subseteq S$, any term interpretation $\mathfrak{t}$ and any predicate interpretation $\mathfrak{p}$ such that $\mathfrak{p}(P) = S'$ and $\mathfrak{p}(Q_p) = V(p)$ for all $p \in \Phi_0$, $\mathcal{M}, S' \vDash_C \psi$ if and only if $\mathfrak{D}, \mathfrak{t}, \mathfrak{p} \vDash_{\mathsf{MSO}} \tau(P,\psi)$.

For the right-to-left direction, suppose $\mathfrak{D}, \mathfrak{t}, \mathfrak{p} \vDash_{\mathsf{MSO}} \tau(P,\varphi)$ and let $\mathcal{M} \doteq (X, \mathcal{O}, V)$ such that $X = \mathfrak{p}(P)$, $\mathcal{O} = \mathcal{P}(\mathfrak{p}(P))$ and $V(p) = \mathfrak{p}(P) \cap \mathfrak{p}(Q_p)$ for all $p \in \Phi_0$. Clearly, $\mathcal{M}$ is a subset space model closed under inclusion. It can easily be proved by structural induction on ψ that for any subformula ψ of φ and any predicate interpretation $\mathfrak{p}'$ such that $\mathfrak{p}'(P) \subseteq \mathfrak{p}(P)$ and $\mathfrak{p}'(Q_p) = \mathfrak{p}(Q_p)$ for all $p \in \Phi_0$, $\mathfrak{D}, \mathfrak{t}, \mathfrak{p}' \vDash_{\mathsf{MSO}} \tau(P,\psi)$ if and only if $\mathcal{M}, \mathfrak{p}'(P) \vDash_C \psi$. ◀

Löwenheim [10] proved that MSO is decidable and has the finite model property. Hence, we have the following proposition.

▶ **Proposition 8.** *bCL has the finite model property and its validity problem is decidable.*

5 Comparison with BBI

The Boolean logic of bunched implications BBI [13] has been devised to reason about resources. This logic is usually considered as the foundation of separation logics [14] and other logics with a separative multiplicative conjunction [5]. The language of BBI is Φ^{-*} and we prove in this section that bCL^{-*} can be seen as BBI interpreted in a restricted class of BBI models. By the results in [8], the magic wand-free fragment of BBI is already undecidable, hence the present section gives new insight into the decidability of separation logics.

A *commutative partial monoid* is a triple $(M, \circ, \mathbf{e})$ where M is a set, $\circ$ a partial function from $M \times M$ to M and $\mathbf{e} \in M$, satisfying for all $a, b, c \in M$:

$$\mathbf{e} \circ a{\downarrow} \text{ and } \mathbf{e} \circ a = a \qquad \text{(identity)}$$
$$\text{if } a \circ b{\downarrow} \text{ then } b \circ a{\downarrow} \text{ and } a \circ b = b \circ a \qquad \text{(commutativity)}$$
$$\text{if } a \circ (b \circ c){\downarrow} \text{ then } (a \circ b) \circ c{\downarrow} \text{ and } a \circ (b \circ c) = (a \circ b) \circ c \qquad \text{(associativity)}$$

where $a \circ b{\downarrow}$ denotes that $\circ$ is defined at (a, b). Notice that, for instance, $a \circ (b \circ c){\downarrow}$ implies $b \circ c{\downarrow}$. A BBI model is a tuple $\mathcal{M} = (M, \circ, \mathbf{e}, V)$ where $(M, \circ, \mathbf{e})$ is a commutative partial monoid and V is a valuation function assigning a subset of M to each propositional variable. Formula in Φ^{-*} are interpreted in BBI models by the relation $\vDash_B$ defined by:

$$\mathcal{M}, a \vDash_B p \quad \text{iff } a \in V(p)$$
$$\mathcal{M}, a \vDash_B \boldsymbol{I} \quad \text{iff } a = \mathbf{e}$$
$$\mathcal{M}, a \vDash_B \varphi * \psi \quad \text{iff there is } b, c \in M \text{ s.t. } b \circ c{\downarrow},\ a = b \circ c,\ \mathcal{M}, b \vDash_B \varphi \text{ and } \mathcal{M}, c \vDash_B \psi$$
$$\mathcal{M}, a \vDash_B \varphi \mathrel{-\!\!*} \psi \quad \text{iff for all } b, c \in M \text{ if } a \circ b{\downarrow},\ a \circ b = c \text{ and } \mathcal{M}, b \vDash_B \varphi \text{ then } \mathcal{M}, c \vDash_B \psi$$

the omitted rules for the Boolean constructs being classical. The logic obtained by interpreting the language Φ^{-*} in the class of all BBI models is BBI[4].

We now describe the class of BBI models which corresponds to bCL^{-*}. Let $(M, \circ, \mathbf{e})$ be a commutative partial monoid. The binary relation $\preccurlyeq$ on M is defined such that for all $a, b \in M$, $a \preccurlyeq b$ iff $b = a \circ c$ for some $c \in M$. An element $a \in M$ is an *atom*

[4] More precisely, this logic is usually called $\mathsf{BBI}_{\mathsf{PD}}$, for instance in [9].

if $a \neq \mathbf{e}$ and for all b, if $b \preccurlyeq a$ then $b \in \{\mathbf{e}, a\}$. For any element $b \in M$, we define $\mathrm{At}\,(b) \doteq \{a \in M \mid a$ is an atom and $a \preccurlyeq b\}$. We define the class $\mathcal{M}_{\mathsf{CL}}$ of all the commutative partial monoids $(M, \circ, \mathbf{e})$ satisfying the following properties:

Atomicity for all $b \in M \setminus \{\mathbf{e}\}$, there is an atom $a \in M$ such that $a \preccurlyeq b$;

Disjointness for all $a \in M$, if $a \circ a\downarrow$ then $a = \mathbf{e}$;

Weak cross-split for all $a, b, c, d \in M$, if $a \circ b = c \circ d$ and $a \neq \mathbf{e}$ then there is $s \in M$ such that $s \neq \mathbf{e}$, $s \preccurlyeq a$ and $s \preccurlyeq c$ or $s \preccurlyeq d$;

Bounded completeness every set $S \subseteq M$ that has an upper bound with respect to $\preccurlyeq$ has a least upper bound.

For the last property to make sense it has to be observed that the relation $\preccurlyeq$ is a partial order on any disjoint commutative partial monoid. The class $\mathcal{C}_{\mathsf{CL}}$ of BBI models is defined as all the BBI models $\mathcal{M} = (M, \circ, \mathbf{e}, V)$ such that $(M, \circ, \mathbf{e})$ is in $\mathcal{M}_{\mathsf{CL}}$ and for all $p \in \Phi_0$ and all $b \in M$ that is not an atom, $b \in V(p)$ iff there is an atom $a \in M$ such that $a \preccurlyeq b$ and $a \in V(p)$.

▶ **Lemma 9.** *Every commutative partial monoid $(M, \circ, \mathbf{e})$ in $\mathcal{M}_{\mathsf{CL}}$ is atomistic, i.e., any element $a \in M$ is the least upper bound of $At(a)$.*

Proof. By definition, a is an upper bound of $\mathrm{At}\,(a)$ and since $(M, \circ, \mathbf{e})$ is bounded complete, there is a least upper bound b of $\mathrm{At}\,(a)$. Let us suppose that $b \neq a$. There must exists $c \in M$, different from $\mathbf{e}$, such that $a = b \circ c$. By atomicity, there is an atom $d \in M$ such that $d \preccurlyeq c$. Obviously $d \in \mathrm{At}\,(a)$ and by associativity and commutativity, $d \circ d\downarrow$. By disjointness, $d = \mathbf{e}$ which is not possible. ◀

▶ **Lemma 10.** *For any commutative partial monoid $(M, \circ, \mathbf{e})$ in $\mathcal{M}_{\mathsf{CL}}$ and any $a, b, c \in M$, $a = b \circ c$ iff $At(a) = At(b) \uplus At(c)$.*

Proof. For the left-to-right direction, suppose that $a = b \circ c$. By the disjointness property, $\mathrm{At}\,(b) \cap \mathrm{At}\,(c) = \emptyset$. By transitivity of $\preccurlyeq$, $\mathrm{At}\,(b) \uplus \mathrm{At}\,(c) \subseteq \mathrm{At}\,(a)$. By the weak cross-split property, for any atom $d \in \mathrm{At}\,(a)$, $d \preccurlyeq b$ or $d \preccurlyeq c$.

For the right-to-left direction, suppose $\mathrm{At}\,(a) = \mathrm{At}\,(b) \uplus \mathrm{At}\,(c)$. By Lemma 9 on b, there is $c' \in M$ such that $a = b \circ c'$. For all $d \in \mathrm{At}\,(c')$, $d \in \mathrm{At}\,(a)$ and by disjointness $d \notin \mathrm{At}\,(b)$, hence $\mathrm{At}\,(c') \subseteq \mathrm{At}\,(c)$. For all $d \in \mathrm{At}\,(c)$, $d \preccurlyeq b \circ c'$ and by the weak cross-split property $d \preccurlyeq b$ or $d \preccurlyeq c'$, hence $\mathrm{At}\,(c) \subseteq \mathrm{At}\,(c)$. Therefore $\mathrm{At}\,(c) = \mathrm{At}\,(c')$ and by Lemma 9, $c' = c$. ◀

▶ **Proposition 11.** *The logic obtained by interpreting $\Phi^{-\!*}$ in $\mathcal{C}_{\mathsf{CL}}$ is $\mathsf{bCL}^{-\!*}$.*

Proof. It can easily be checked that any formula that is satisfiable in a subset space model $\mathcal{M} = (X, \mathcal{O}, V)$ that is closed under inclusion, is satisfiable in the BBI model $(\mathcal{O}, \uplus, \emptyset, V')$ where $S \in V'(p)$ iff $\mathcal{M}, S \vDash_C p$, and that this BBI model belongs to $\mathcal{C}_{\mathsf{CL}}$. For the other direction, from any BBI model $\mathcal{M} = (M, \circ, \mathbf{e}, V)$ in $\mathcal{C}_{\mathsf{CL}}$ construct the subset space model $\mathcal{M}' \doteq (X, \mathcal{O}, V')$ where X is the set of all atoms in M, $\mathcal{O}$ is the set of all subset of atoms that has an upper bound by $\preccurlyeq$ and V' is the reduction of V to X. It can easily be checked that $\mathcal{M}'$ is closed under inclusion and satisfies the same formulas than $\mathcal{M}$. ◀

6 Comparison with the Propositional Dependence logic

The team semantics has been devised by Hodges [7] to provide a compositional semantics for the independence-friendly logic. The main idea is to evaluate formulas at *sets* of valuations, called teams, instead of at single valuations. Following this idea, new propositional logics, called propositional team logics, has been devised and studied recently [17, 18, 19]. Observing that, by identifying each element of a universe to a valuation, counting logics can be seen as generalizations of propositional team logics, we prove in this section that the *propositional downward closed team logic* PD [18], also called propositional dependence logic, can be embedded in a large class of counting logics.

We first recall the syntax and semantics of PD. Given a set Φ_0 of propositional variables, the language Φ_{PD} is defined inductively by:

$$\varphi, \psi \ \coloneqq \ p \mid \neg p \mid \bot \mid (\varphi \wedge \psi) \mid (\varphi \vee \psi) \mid (\varphi \otimes \psi) \mid {=}(q_1, \ldots, q_n, p)$$

where $p, q_1, \ldots, q_n$ are propositional variables. The symbol $\otimes$ is called the *tensor* and formulas of the form ${=}(q_1, \ldots, q_n, p)$ *dependence atoms*. Notice that negation is allowed only in front of propositional variables. A *valuation* is a subset $v \subseteq \Phi_0$ of propositional variables and a *team* T is a subset of valuations. Formulas in Φ_{PD} are evaluated at teams by the relation $\vDash_{\mathsf{PD}}$ defined inductively by:

$$\begin{array}{ll}
T \vDash_{\mathsf{PD}} p & \text{iff for all } v \in T, p \in v \\
T \vDash_{\mathsf{PD}} \neg p & \text{iff for all } v \in T, p \notin v \\
T \vDash_{\mathsf{PD}} \bot & \text{iff } T = \emptyset \\
T \vDash_{\mathsf{PD}} \varphi \wedge \psi & \text{iff } T \vDash_{\mathsf{PD}} \varphi \text{ and } T \vDash_{\mathsf{PD}} \psi \\
T \vDash_{\mathsf{PD}} \varphi \vee \psi & \text{iff } T \vDash_{\mathsf{PD}} \varphi \text{ or } T \vDash_{\mathsf{PD}} \psi \\
T \vDash_{\mathsf{PD}} \varphi \otimes \psi & \text{iff there is } T_1, T_2 \text{ such that } T = T_1 \cup T_2, T_1 \vDash_{\mathsf{PD}} \varphi \text{ and } T_2 \vDash_{\mathsf{PD}} \psi \\
T \vDash_{\mathsf{PD}} {=}(q_1, \ldots, q_n, p) & \text{iff for all } v_1, v_2 \in T, \text{ if } v_1 \cap \{q_1, \ldots, q_n\} = v_2 \cap \{q_1, \ldots, q_n\} \\
& \qquad\qquad\qquad\qquad \text{then } v_1 \cap \{p\} = v_2 \cap \{p\}
\end{array}$$

A formula $\varphi \in \Phi_{\mathsf{PD}}$ is PD-valid, denoted by $\vDash_{\mathsf{PD}} \varphi$, if $T \vDash_{\mathsf{PD}} \varphi$ for all teams T. The propositional dependence logic PD is obtained by interpreting Φ_{PD} in the class of all teams over Φ_0. An important property of this logic is the *downward closure*: if $T \vDash_{\mathsf{PD}} \varphi$ then for any subset $T' \subseteq T$, $T' \vDash_{\mathsf{PD}} \varphi$ (see for instance [18]).

We prove that PD can be embedded in any counting logic obtained by interpreting Φ (or any of its extensions) in any class $\mathfrak{C}$ of subset space models that are closed under inclusion and that satisfies the following *comprehensive condition*: for all $P \subseteq \Phi_0$, there is a subset space model $\mathcal{M} = (X, \mathcal{O}, V)$ in $\mathfrak{C}$ and a subset $S \in \mathcal{O}$ such that for any valuation $v \subseteq P$, there is $x \in S$ such that $v = V^{-1}(x)$. Remark that $\mathfrak{C}_{\mathrm{all}}$ satisfies the comprehensive condition. The translation τ from Φ_{PD} to Φ is defined inductively as follows:

$$\begin{array}{ll}
\tau(p) \doteq \Box(\boldsymbol{I} \vee p) & \tau(\varphi \wedge \psi) \doteq \tau(\varphi) \wedge \tau(\psi) \\
\tau(\neg p) \doteq \neg p & \tau(\varphi \vee \psi) \doteq \tau(\varphi) \vee \tau(\psi) \\
\tau(\bot) \doteq \boldsymbol{I} & \tau(\varphi \otimes \psi) \doteq \tau(\varphi) \ast \tau(\psi)
\end{array}$$

$$\tau({=}(q_1, \ldots, q_n, p)) \doteq \Box\left({=}2 \to \left(\bigwedge_{i \in 1..n} \mathrm{same}(q_i)\right) \to \mathrm{same}(p)\right)$$

where same is defined by $\mathrm{same}(p) \doteq (p \ast p) \vee ((\neg \boldsymbol{I} \wedge \neg p) \ast (\neg \boldsymbol{I} \wedge \neg p))$ for all propositional variables p. This translation is clearly polynomial. To prove that it preserves validity (Proposition 14), the following lemmas are needed. The proof of Lemma 12 is straightforward and Lemma 13 has already been proved, for instance in [17].

▶ **Lemma 12.** *For any subset space model $\mathcal{M} = (X, \mathcal{O}, V)$, any $\{x, y\} \in \mathcal{O}$ and any propositional variable $p \in \Phi_0$, $\mathcal{M}, \{x, y\} \vDash_C$ same(p) iff $\{x, y\} \subseteq V(p)$ or $\{x, y\} \cap V(p) = \emptyset$.*

▶ **Lemma 13.** *For any formula $\varphi \in \Phi$, let $T_\varphi \doteq \mathcal{P}(PV(\varphi))$. Then φ is PD-valid iff $T_\varphi \vDash_{PD} \varphi$.*

▶ **Proposition 14.** *For any class $\mathcal{C}$ of subset space models that are closed under inclusion and any formula $\varphi \in \Phi_{PD}$, if $\mathcal{C}$ fulfils the comprehensive condition then $\vDash_{PD} \varphi$ iff $\mathcal{C} \vDash_C \tau(\varphi)$.*

Proof. For the left-to-right direction, suppose that $\vDash_{\mathsf{PD}} \varphi$. Let $\mathcal{M} = (X, \mathcal{O}, V)$ be a subset space model in $\mathcal{C}$. For all $S \in \mathcal{O}$, let $T(S) \doteq \{v \subseteq \mathrm{PV}(\varphi) \mid \text{there is } x \in S,\ V^{-1}(x) = v\}$. We prove by structural induction on ψ that for all $\psi \in \mathrm{SF}(\varphi)$ and all $S \in \mathcal{O}$, if $T(S) \vDash_{\mathsf{PD}} \psi$ then $\mathcal{M}, S \vDash_C \tau(\psi)$. We detail only some cases, the missing ones being either similar or straightforward. For propositional variables, suppose that $\mathcal{M}, S \nvDash_C \Box(\boldsymbol{I} \vee p)$. Then $\mathcal{M}, S \vDash_C \top * (\neg \boldsymbol{I} \wedge \neg p)$. Hence there is a subset $S_2 \subseteq S$ such that $\mathcal{M}, S_2 \vDash_C \neg \boldsymbol{I} \wedge \neg p$. Therefore there is $x \in S$ such that $x \notin V(p)$. By definition, there is $v \in T(S)$ such that $p \notin v$. We have proved that $T(S) \nvDash_{\mathsf{PD}} p$. For tensor products, let us suppose that $T(S) \vDash_{\mathsf{PD}} \psi_1 \otimes \psi_2$. There are T_1, T_2 such that $T(S) = T_1 \cup T_2$, $T_1 \vDash_{\mathsf{PD}} \psi_1$ and $T_2 \vDash_{\mathsf{PD}} \psi_2$. By the downward closure property, $T_2 \setminus T_1 \vDash_{\mathsf{PD}} \psi_2$ too. Let $S_1 \doteq \{x \in S \mid V^{-1}(x) \in T_1\}$ and $S_2 \doteq \{x \in S \mid V^{-1}(x) \in T_2 \setminus T_1\}$. Clearly $S = S_1 \uplus S_2$ and by the induction hypothesis, $\mathcal{M}, S \vDash_C \tau(\psi_1) * \tau(\psi_2)$. For dependence atoms, suppose that $\mathcal{M}, S \nvDash_C \tau\,(=\!(q_1, \ldots, q_n, p))$. There is $\{x, y\} \subseteq S$ such that $x \neq y$ and $\mathcal{M}, \{x, y\} \vDash_C \left(\bigwedge_{i \in 1..n} \mathrm{same}(q_i)\right) \wedge \neg \mathrm{same}(p)$. By Lemma 12, $V^{-1}(x) \cap \{q_1, \ldots, q_n\} = V^{-1}(y) \cap \{q_1, \ldots, q_n\}$ and $V^{-1}(x) \cap \{p\} \neq V^{-1}(y) \cap \{p\}$. Therefore $T(S) \nvDash_{\mathsf{PD}} =\!(q_1, \ldots, q_n, p)$.

For the right-to-left direction, suppose that $\mathcal{C} \vDash_C \tau(\varphi)$. By the comprehensive condition, there is $\mathcal{M} = (X, \mathcal{O}, V)$ in $\mathcal{C}$ and $S \in \mathcal{O}$ such that $T_\varphi \subseteq T(S)$ where T_φ is defined as in Lemma 13 and $T(S)$ as in the previous direction. By Lemma 13 and the downward closure property, it is sufficient to prove that for all $\psi \in \mathrm{SF}(\varphi)$ and all $S' \subseteq S$, if $\mathcal{M}, S' \vDash_C \tau(\psi)$ then $T(S') \vDash_{\mathsf{PD}} \psi$. The proof is by structural induction on ψ. We detail only some cases, the missing ones being either similar or straightforward. For propositional variables, suppose that $T(S') \nvDash_{\mathsf{PD}} p$. There is $v \in T(S')$ such that $p \notin v$. By definition, there is $x \in S'$ such that $x \notin V(p)$. Therefore $\mathcal{M}, \{x\} \vDash \neg \boldsymbol{I} \wedge \neg p$ and $\mathcal{M}, S' \nvDash_C \Box(\boldsymbol{I} \vee p)$. For tensor products, suppose that $\mathcal{M}, S' \vDash_C \tau(\psi_1) * \tau(\psi_2)$. There are $S_1, S_2 \in \mathcal{O}$ such that $S' = S_1 \uplus S_2$, $\mathcal{M}, S_1 \vDash_C \tau(\psi_1)$ and $\mathcal{M}, S_2 \vDash_C \tau(\psi_2)$. Obviously, $T(S_1) \cup T(S_2) = T(S_1 \cup S_2)$. Therefore, by induction hypothesis, $T(S') \vDash_{\mathsf{PD}} \psi_1 \otimes \psi_2$. For dependence atoms, suppose that $T(S') \nvDash_{\mathsf{PD}} =\!(q_1, \ldots, q_n, p)$. There are $x, y \in S'$ such that $V^{-1}(x) \cap \{q_1, \ldots, q_n\} = V^{-1}(y) \cap \{q_1, \ldots, q_n\}$ and $V^{-1}(x) \cap \{p\} \neq V^{-1}(y) \cap \{p\}$. By Lemma 12, $\mathcal{M}, \{x, y\} \vDash_C \left(\bigwedge_{i \in 1..n} \mathrm{same}(q_i)\right) \wedge \neg \mathrm{same}(p)$. Therefore $\mathcal{M}, S' \nvDash_C \tau\,(=\!(q_1, \ldots, q_n, p))$. ◀

Since the validity problem of PD has been proved in [17] to be NEXPTIME-complete, and the translation is polynomial, we have the following corollary. It implies that bCL's validity problem is NEXPTIME-hard, because $\mathcal{C}_{\mathrm{all}}$ satisfies the comprehensive condition.

▶ **Corollary 15.** *The validity problem of any counting logic obtained by interpreting Φ (or any of its extension) in a class of subset space model that is closed under inclusion and that satisfies the comprehensive condition is NEXPTIME-hard.*

7 Complexity upper bounds

We proved in Section 4 that the basic counting logic bCL is decidable, and in the previous section that its validity problem is NEXPTIME-hard. We now establish a 4EXPTIME upper bound. For that matter, we consider two new counting logics. The finitely complete counting logic $\mathsf{CL}^{-*}_{\mathrm{comp}}$ is the logic obtained by interpreting the language Φ^{-*} in the class $\mathcal{C}_{\mathrm{comp}}$ of all finitely complete subset models. The finitely complete atomic counting logic $\mathsf{CL}^{-*}_{\mathrm{atom}}$ is the logic obtained by interpreting the language $\Phi^{-\!*}$ in the class $\mathcal{C}_{\text{atom}}$ of all subset models that are finitely complete and atomic. We first prove that $\mathsf{CL}^{-\!*}_{\text{atom}}$'s satisfiability problem is in 3EXPTIME by a reduction to the satisfiability of Presburger arithmetic. Then we provide a faithful but exponential translation from $\mathsf{CL}^{-\!*}_{\text{comp}}$ to $\mathsf{CL}^{-\!*}_{\text{atom}}$. Finally we prove that bCL is the magic wand-free fragment of $\mathsf{CL}^{-\!*}_{\text{comp}}$.

7.1 Complexity of the satisfiability problem of $\mathsf{CL}^{-\!*}_{\text{atom}}$

We provide a polynomial reduction of $\mathsf{CL}^{-\!*}_{\text{atom}}$'s satisfiability problem to the satisfiability problem of Presburger arithmetic Presburger arithmetic is the first-order theory of the natural numbers $\mathbb{N}$ with addition $+$. The set of quantified variables of Presburger arithmetic is denoted by $\mathbb{V}$. Formulas of this theory are called *constraints*. A Presburger assignment[5] is a function $\mathcal{A}$ from $\mathbb{V}$ to $\mathbb{N}$. We write $\mathcal{A} \vDash_{\mathsf{PA}} C$ to denote that the constraint C is satisfied by the assignment $\mathcal{A}$.

Given a formula $\varphi_0 \in \Phi^{-\!*}$, we construct a constraint that is satisfiable in Presburger arithmetic if and only if φ_0 is $\mathsf{CL}^{-\!*}_{\text{atom}}$-satisfiable. For that purpose, we use *vectors* which are ordered finite sets, i.e., finite sequences without repetitions (each component of a vector occurs exactly once). Vectors are distinguished by an arrow accent. We write v_i to denote the i^{th} component of $\overrightarrow{v}$. We may abusively consider a vector to be the set of its components and write for instance $u \in \overrightarrow{v}$. In this section, it is assumed that all vectors have length $n \doteq |\text{PV}(\varphi_0)| + 1$. From the formula φ_0, a vector $\overrightarrow{p}$ of propositional variables is defined such that $\{p_1, \ldots, p_{n-1}\} = \text{PV}(\varphi_0)$ and $p_n \notin \text{PV}(\varphi_0)$. Moreover, two vectors $\overrightarrow{r}$ and $\overrightarrow{s}$ of Presburger arithmetic's variables are selected such that $\overrightarrow{r} \cap \overrightarrow{s} = \emptyset$.

A model $\mathcal{M} = (X, \mathcal{O}, V)$ is *flat* on a subset $P \subseteq \Phi_0$ of propositional variables if $V(p) = \emptyset$ for all $p \in P$.

▶ **Lemma 16.** *If φ_0 is $\mathsf{CL}^{-\!*}_{atom}$-satisfiable then it is satisfiable in a model from $\mathcal{C}_{atom}$ that is flat on $\Phi_0 \setminus \overrightarrow{p}$.*

Proof. Suppose that $\mathcal{M}, S \vDash_C \varphi$. Construct $\mathcal{M}' \doteq (X, \mathcal{O}, V')$ such that $V'(p) = V(p)$ for all $p \in \text{PV}(\varphi_0)$, $V'(p_n) = \bigcup_{p \notin \text{PV}(\varphi_0)} V(p)$ and $V'(p) = \emptyset$ for all $p \notin \overrightarrow{p}$. It can easily be proved by structural induction on ψ that for all $\psi \in \text{SF}(\varphi_0)$ and all $S' \in \mathcal{O}$, $\mathcal{M}', S' \vDash_C \psi$ iff $\mathcal{M}, S' \vDash_C \psi$. ◀

A pair $(\mathcal{M}, S)$ composed of a subset space model $\mathcal{M} = (X, \mathcal{O}, V)$ in $\mathcal{C}_{\text{atom}}$ and a subset $S \in \mathcal{O}$ *is in correspondence with* a Presburger arithmetic assignment $\mathcal{A}$ by a vector $\overrightarrow{x}$ of Presburger arithmetic variables, denoted by $\mathcal{M}, S \xrightarrow{\overrightarrow{x}} \mathcal{A}$, if all the following conditions hold for all $i \in 1\,..\,n$:

- $\mathcal{M}$ is flat on $\Phi_0 \setminus \overrightarrow{p}$;
- if $\mathcal{A}(r_i) = 0$ then $|V(p_i)| = \mathcal{A}(s_i)$;
- if $\mathcal{A}(r_i) > 0$ then $|V(p_i)|$ is infinite;
- $|V(p_i) \cap S| = \mathcal{A}(x_i)$.

Clearly, for any model $\mathcal{M} = (X, \mathcal{O}, V)$ in $\mathcal{C}_{\text{atom}}$ that is flat on $\Phi_0 \setminus \overrightarrow{p}$, any subset $S \in \mathcal{O}$ and any vector $\overrightarrow{x}$ of Presburger arithmetic disjoint from $\overrightarrow{r}$ and $\overrightarrow{s}$ (i.e., $\overrightarrow{x} \cap \overrightarrow{r} = \overrightarrow{x} \cap \overrightarrow{s} = \emptyset$), there is a Presburger arithmetic assignment $\mathcal{A}$ such that $\mathcal{M}, S \xrightarrow{\overrightarrow{x}} \mathcal{A}$.

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[5] For the sake of simplicity, we only consider here the standard interpretation of Presburger arithmetic over natural numbers.

As a first component of the translation of φ_0, the constraint $\Psi(\overrightarrow{x})$ is defined for all vectors $\overrightarrow{x}$ of Presburger arithmetic variables that is disjoint from $\overrightarrow{r}$ and $\overrightarrow{s}$, by:

$$\Psi\left(\overrightarrow{x}\right) \doteq \bigwedge_{i \in 1..n} \left(r_i = 0 \rightarrow x_i \leq s_i\right)$$

As stated by the following lemma, this formula ensures that there is a pair $(\mathcal{M}, S)$ in correspondence with any assignment satisfying it.

▶ **Lemma 17.** *For any vector $\overrightarrow{x}$ of Presburger arithmetic variables that is disjoint from $\overrightarrow{r}$ and $\overrightarrow{s}$ and for any Presburger arithmetic assignment $\mathcal{A}$, $\mathcal{A} \vDash_{\mathsf{PA}} \Psi(\overrightarrow{x})$ if and only if there is a subset space model $\mathcal{M} = (X, \mathcal{O}, V)$ in $\mathcal{C}_{atom}$ and a subset $S \in \mathcal{O}$ such that $\mathcal{M}, S \xrightarrow{\overrightarrow{x}} \mathcal{A}$.*

Proof. For the left-to-right direction, suppose $\mathcal{A} \vDash_{\mathsf{PA}} \Psi(\overrightarrow{x})$. Construct $\mathcal{M} = (X, \mathcal{O}, V)$ and S such that $X = \{(i, j) \in \{1\,..\,n\} \times \mathbb{N} \mid \text{if } \mathcal{A}(r_i) = 0 \text{ then } j < \mathcal{A}(s_i)\}$, $\mathcal{O} = \mathcal{P}(X)$, $V(p_i) = \{(i', j) \in X \mid i' = i\}$ for all $i \in 1\,..\,n$, $V(q) = \emptyset$ for all $q \notin \overrightarrow{p}$ and $S = \{(i, j) \in X \mid j < \mathcal{A}(x_i)\}$. It can easily be checked that $\mathcal{M}$ belongs to $\mathcal{C}_{\text{atom}}$ and $\mathcal{M}, S \xrightarrow{\overrightarrow{x}} \mathcal{A}$. The right-to-left direction is straightforward. ◀

The second component of the translation is the function θ assigning a constraint to any pair $(\overrightarrow{x}, \psi)$ composed of a vector $\overrightarrow{x}$ of Presburger arithmetic variables that is disjoint from $\overrightarrow{r}$ and $\overrightarrow{s}$ and a subformula ψ of φ_0. This function is defined inductively as follows:

$$\theta(\overrightarrow{x}, p_i) \doteq x_i > 0 \qquad \theta(\overrightarrow{x}, \top) \doteq \top$$

$$\theta(\overrightarrow{x}, \boldsymbol{I}) \doteq \bigwedge_{i \in 1..n} x_i = 0 \qquad \theta(\overrightarrow{x}, \neg\varphi) \doteq \neg\theta(\overrightarrow{x}, \varphi)$$

$$\theta(\overrightarrow{x}, \varphi \wedge \psi) \doteq \theta(\overrightarrow{x}, \varphi) \wedge \theta(\overrightarrow{x}, \psi)$$

$$\theta(\overrightarrow{x}, \varphi * \psi) \doteq (\exists \overrightarrow{y}, \overrightarrow{z}.\ \bigwedge_{i \in 1..n} x_i = y_i + z_i \ \wedge\ \Psi(\overrightarrow{y}) \wedge \theta(\overrightarrow{y}, \varphi) \wedge \Psi(\overrightarrow{z}) \wedge \theta(\overrightarrow{z}, \psi))$$

$$\theta(\overrightarrow{x}, \varphi \mathrel{-\!\!*} \psi) \doteq (\forall \overrightarrow{y}, \overrightarrow{z}.\ \bigwedge_{i \in 1..n} z_i = x_i + y_i \ \rightarrow\ \Psi(\overrightarrow{y}) \rightarrow \theta(\overrightarrow{y}, \varphi) \rightarrow \Psi(\overrightarrow{z}) \rightarrow \theta(\overrightarrow{z}, \psi))$$

where $\overrightarrow{y}$ and $\overrightarrow{z}$ are chosen such that $\overrightarrow{r}$, $\overrightarrow{s}$, $\overrightarrow{x}$, $\overrightarrow{y}$ and $\overrightarrow{z}$ are pairwise disjoint.

▶ **Lemma 18.** *For any subformula ψ of φ_0, any Presburger arithmetic assignment $\mathcal{A}$, any vector $\overrightarrow{x}$ of Presburger arithmetic variables that is disjoint from $\overrightarrow{r}$ and $\overrightarrow{s}$, any model $\mathcal{M} = (X, \mathcal{O}, V)$ in $\mathcal{C}_{atom}$ and any subset $S \in \mathcal{O}$ such that $\mathcal{M}, S \xrightarrow{\overrightarrow{x}} \mathcal{A}$, $\mathcal{M}, S \vDash_C \psi$ if and only if $\mathcal{A} \vDash_{\mathsf{PA}} \theta(\overrightarrow{x}, \psi)$.*

Proof. The proof is by structural induction on ψ. Only the cases for the magic wand are detailed, the other ones being either similar or straightforward.

For the left-to-right direction, suppose $\mathcal{A} \nvDash_{\mathsf{PA}} \theta(\overrightarrow{x}, \varphi \mathrel{-\!\!*} \psi)$. There is $\mathcal{A}'$ such that $\mathcal{A}' \nvDash_{\mathsf{PA}} \Psi(\overrightarrow{y}) \rightarrow \theta(\overrightarrow{y}, \varphi) \rightarrow \Psi(\overrightarrow{z}) \rightarrow \theta(\overrightarrow{z}, \psi)$ and for all $i \in 1\,..\,n$, $\mathcal{A}'(r_i) = \mathcal{A}(r_i)$, $\mathcal{A}'(s_i) = \mathcal{A}(s_i)$, $\mathcal{A}'(x_i) = \mathcal{A}(x_i)$ and $\mathcal{A}'(z_i) = \mathcal{A}'(x_i) + \mathcal{A}'(y_i)$. Since $\mathcal{A}' \vDash_{\mathsf{PA}} \Psi(\overrightarrow{z})$, for all $i \in 1\,..\,n$, there are at least $\mathcal{A}'(y_i)$ elements $x \in X$ such that $x \in V(p_i) \setminus S$. Since $\mathcal{M}$ is atomic, there is $S_1 \subseteq X \setminus S$ such that $\mathcal{M}, S_1 \xrightarrow{\overrightarrow{y}} \mathcal{A}'$. Moreover, it can easily be checked that $\mathcal{M}, S_2 \xrightarrow{\overrightarrow{z}} \mathcal{A}'$ for $S_2 \doteq S \uplus S_1$. By induction hypothesis, $\mathcal{M}, S_1 \vDash_C \varphi$ and $\mathcal{M}, S_2 \nvDash_C \psi$. Therefore $\mathcal{M}, S \nvDash_C \varphi \mathrel{-\!\!*} \psi$.

For the right-to-left direction, suppose $\mathcal{M}, S \nvDash_C \varphi \mathrel{-\!\!*} \psi$. There are S_1, S_2 such that $S_2 = S \uplus S_1$, $\mathcal{M}, S_1 \vDash_C \varphi$ and $\mathcal{M}, S_2 \nvDash_C \psi$. Define the Presburger arithmetic assignment $\mathcal{A}'$ such that $\mathcal{A}'(r_i) = \mathcal{A}(r_i)$, $\mathcal{A}'(s_i) = \mathcal{A}(s_i)$, $\mathcal{A}'(x_i) = \mathcal{A}(x_i)$, $\mathcal{A}'(y_i) = |V(p_i) \cap S_1|$ and

$\mathcal{A}'(z_i) = |V(p_i) \cap S_2|$, for all $i \in 1..n$. By definition, $\mathcal{M}, S_1 \xrightarrow{\overrightarrow{y}} \mathcal{A}'$, $\mathcal{M}, S_2 \xrightarrow{\overrightarrow{z}} \mathcal{A}'$ and $\mathcal{A}' \vDash_{\mathsf{PA}} \bigwedge_{i \in 1..n} z_i = x_i + y_i$. By Lemma 17, $\mathcal{A}' \vDash_{\mathsf{PA}} \Psi(\overrightarrow{y}) \wedge \Psi(\overrightarrow{z})$. By induction hypothesis, $\mathcal{A}' \vDash_{\mathsf{PA}} \theta(\overrightarrow{y}, \varphi)$ and $\mathcal{A}' \nvDash_{\mathsf{PA}} \theta(\overrightarrow{z}, \psi)$. Therefore, $\mathcal{A} \nvDash_{\mathsf{PA}} \theta(\overrightarrow{x}, \varphi \twoheadrightarrow \psi)$. ◀

We can now prove the complexity upper bound for $\mathsf{CL}^{\twoheadrightarrow}_{\text{atom}}$.

▶ **Proposition 19.** *The satisfiability problem of $\mathsf{CL}^{\twoheadrightarrow}_{atom}$ is in 3EXPTIME.*

Proof. Given a formula $\varphi_0 \in \Phi^{\twoheadrightarrow}$, let us choose $\overrightarrow{p}$, $\overrightarrow{r}$ and $\overrightarrow{s}$ as specified previously and $\overrightarrow{x}$ such that $\overrightarrow{x}$, $\overrightarrow{r}$ and $\overrightarrow{s}$ are pairwise disjoint. The constraint $\Psi(\overrightarrow{x}) \wedge \theta(\overrightarrow{x}, \varphi_0)$ can be computed in time polynomial in $|\varphi_0|$. If φ_0 is $\mathsf{CL}^{\twoheadrightarrow}_{\text{atom}}$-satisfiable, by Lemma 16, there is a model $\mathcal{M} = (X, \mathcal{O}, V)$ in $\mathfrak{C}_{\text{atom}}$, a subset $S \in \mathcal{O}$ and a Presburger arithmetic assignment $\mathcal{A}$ such that $\mathcal{M}, S \vDash_C \varphi_0$ and $\mathcal{M}, S \xrightarrow{\overrightarrow{x}} \mathcal{A}$. By Lemmas 17 and 18, $\mathcal{A} \vDash_{\mathsf{PA}} \Psi(\overrightarrow{x}) \wedge \theta(\overrightarrow{x}, \varphi_0)$. Conversely, suppose that $\mathcal{A} \vDash_{\mathsf{PA}} \Psi(\overrightarrow{x}) \wedge \theta(\overrightarrow{x}, \varphi_0)$. By Lemma 17, there is a model $\mathcal{M} = (X, \mathcal{O}, V)$ in $\mathfrak{C}_{\text{atom}}$ and a subset $S \in \mathcal{O}$ such that $\mathcal{M}, S \xrightarrow{\overrightarrow{x}} \mathcal{A}$. By Lemma 18, $\mathcal{M}, S \vDash_C \varphi_0$. Therefore, there is a polynomial reduction from the satisfiability problem of $\mathsf{CL}^{\twoheadrightarrow}_{\text{atom}}$ to the satisfiability problem in Presburger arithmetic. Since the satisfiability problem in Presburger arithmetic has been proved in [11] to be decidable in 3EXPTIME, the satisfiability of $\mathsf{CL}^{\twoheadrightarrow}_{\text{atom}}$ is in 3EXPTIME too. ◀

7.2 Complexity of the satisfiability problem of $\mathsf{CL}^{\twoheadrightarrow}_{\text{comp}}$

We provide an exponential reduction of the satisfiability problem of $\mathsf{CL}^{\twoheadrightarrow}_{\text{comp}}$ to the satisfiability problem of $\mathsf{CL}^{\twoheadrightarrow}_{\text{atom}}$. We first define a *power correspondence* as a triple (Q_1, b, Q_2) such that Q_1 and Q_2 are subsets of Φ_0 and b is a bijection assigning an element of Q_2 to every subset of Q_1. Then, given a power correspondence (Q_1, b, Q_2) and a formula $\varphi \in \Phi^{\twoheadrightarrow}$ such that the set of propositional variables occurring in φ is included in Q_1, the translation $\tau\,(Q_1, b, Q_2)(\varphi)$ of φ is obtained by replacing each occurrence of p in φ with

$$\bigvee_{p \in P \subseteq Q_1} b(P)$$

for all $p \in Q_1$. The resulting formula has size exponential in the size of the original formula. Moreover, given any universe X and any power correspondence (Q_1, b, Q_2), the relation $\frac{Q_1, b, Q_2}{X}$ over valuations on X is defined such that $V_1 \frac{Q_1, b, Q_2}{X} V_2$ iff for all $x \in X$ and all $Q \subseteq Q_1$, $x \in V_2\,(b\,(Q))$ iff $Q = \{p \in Q_1 \mid x \in V_1(p)\}$. We first state the following lemmas.

▶ **Lemma 20.** *For any subset model $\mathcal{M} = (X, \mathcal{O}, V_1)$ in $\mathfrak{C}_{comp}$ and any power correspondence (Q_1, b, Q_2), there is a valuation V_2 on X such that $V_1 \frac{Q_1, b, Q_2}{X} V_2$ and $(X, \mathcal{O}, V_2)$ is a subset model in $\mathfrak{C}_{atom}$.*

Proof. Define V_2 such that, for all $p \in \Phi_0$, $V_2(p) = \left\{x \in X \,\middle|\, b^{-1}(p) = V_1^{-1}(x)\right\}$ if $p \in Q_2$ and $V_2(p) = \emptyset$ otherwise. It can easily be checked that $V_1 \frac{Q_1, b, Q_2}{X} V_2$ and $(X, \mathcal{O}, V_2)$ is a subset model in $\mathfrak{C}_{\text{atom}}$. ◀

▶ **Lemma 21.** *For all subset model $\mathcal{M} = (X, \mathcal{O}, V_2)$ in $\mathfrak{C}_{atom}$ and all power correspondence (Q_1, b, Q_2), there is a valuation V_1 on X such that $V_1 \frac{Q_1, b, Q_2}{X} V_2$ and $(X, \mathcal{O}, V_1)$ is a subset model in $\mathfrak{C}_{comp}$.*

Proof. Define V_1 by

$$V_1(p) \doteq \bigcup_{p \in P \subseteq Q_1} V_2\,(b\,(P))\,, \text{ for all } p \in \Phi_0.$$

It can easily be checked that $V_1 \frac{Q_1, b, Q_2}{X} V_2$ and $(X, \mathcal{O}, V_1)$ is a subset model in $\mathfrak{C}_{\text{comp}}$. ◀

▶ **Lemma 22.** *For any subset models $\mathcal{M}_1 = (X, \mathcal{O}, V_1)$ and $\mathcal{M}_2 = (X, \mathcal{O}, V_2)$, any power correspondence (Q_1, b, Q_2) such that $V_1 \xrightarrow[X]{Q_1,b,Q_2} V_2$, any subset $S \in \mathcal{O}$ and any formula $\varphi \in \Phi^{-\!*}$ such that the set of propositional variables occurring in φ is included in Q_1, $\mathcal{M}_1, S \vDash \varphi$ if and only if $\mathcal{M}_2, S \vDash \tau(Q_1, b, Q_2)(\varphi)$.*

Proof. The proof is by a straightforward induction on $|\varphi|$, left to the reader. ◀

▶ **Proposition 23.** *The satisfiability problem of $\mathsf{CL}^{-\!*}_{comp}$ is in 4EXPTIME.*

Proof. Suppose now that we want to check whether a formula $\varphi_0 \in \Phi^{-\!*}$ is satisfiable in $\mathsf{CL}^{-\!*}_{\mathrm{comp}}$. Since Φ_0 is infinite, we can construct a power correspondence (Q_1, b, Q_2) such that Q_1 is the set of all the propositional variables occurring in φ_0. Then we check whether $\tau(Q_1, b, Q_2)(\varphi)$ is satisfiable in $\mathsf{CL}^{-\!*}_{\mathrm{atom}}$. If it is the case, then there is a model $\mathcal{M}_2$ in $\mathcal{C}_{\mathrm{atom}}$ satisfying $\tau(Q_1, b, Q_2)(\varphi)$. By Lemmas 21 and 22, there is a model $\mathcal{M}_1$ in $\mathcal{C}_{\mathrm{atom}}$ satisfying φ. Therefore, our procedure is complete. Conversely, if there is a model $\mathcal{M}_1$ in $\mathcal{C}_{\mathrm{comp}}$ satisfying φ then, by Lemmas 20 and 22, there is a model $\mathcal{M}_2$ in $\mathcal{C}_{\mathrm{comp}}$ satisfying $\tau(Q_1, b, Q_2)(\varphi)$. Therefore, our procedure is sound. Finally, computing (Q_1, b, Q_2) and $\tau(Q_1, b, Q_2)(\varphi)$ takes deterministic exponential time and the satisfiability problem of $\mathsf{CL}^{-\!*}_{\mathrm{atom}}$ can be decided in triple exponential time in the size of the input formula. Since the size of $\tau(Q_1, b, Q_2)(\varphi)$ is exponential in the size of φ, the satisfiability problem of $\mathsf{CL}^{-\!*}_{\mathrm{comp}}$ is in 4EXPTIME. ◀

▶ **Corollary 24.** *The satisfiability problem of* bCL *is in 4EXPTIME.*

Proof. We prove that bCL is the magic wand-free fragment of $\mathsf{CL}^{-\!*}_{\mathrm{comp}}$. Clearly, any model in $\mathcal{C}_{\mathrm{comp}}$ is in $\mathcal{C}_{\mathrm{all}}$. Suppose that φ is bCL-satisfiable. By Prop. 8, there is a finite model $\mathcal{M} = (X, \mathcal{O}, V)$ such that $\mathcal{M}, S \vDash_C \varphi$ for some $S \in \mathcal{O}$. It can easily be checked that $\mathcal{M}' \doteq (S, \mathcal{P}(S), V\!\downharpoonright_S)$ is in $\mathcal{C}_{\mathrm{comp}}$ and that $\mathcal{M}', S \vDash \varphi$. ◀

8 Conclusion

We have proposed a new family of logics with an associative binary modality, called counting logics. We have shown that these logics can be seen both as specializations of the Boolean logic of bunched implications and as generalizations of the propositional dependence logic. We have also proved that the validity problem of the basic counting logic bCL is decidable, NEXPTIME-hard and in 4EXPTIME. We conjecture that this decidability result is due to particular constraints on the valuation of propositional variable and we have proved that the logic obtained by removing these constraints is undecidable. But the present work is exploratory and many problems remain open. First, the exact complexity of the basic counting logic is still unknown. Another avenue for future work is to further explore the connection between counting logic and propositional team logics, in particular logics that lack the locality property like the propositional independence logic. Finally, the most challenging problem we have left open is the decidability status of $\mathsf{bCL}^{-\!*}$, the basic counting logic with magic wands. We believe that this problem is difficult and is strongly related to the subset finite problem which consists in determining whether there is a $\mathsf{bCL}^{-\!*}$ formula that can only be satisfied at infinite subsets.

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