

# Noetherian Quasi-Polish Spaces\*

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## Abstract

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In the presence of suitable power spaces, compactness of  $\mathbf{X}$  can be characterized as the singleton  $\{X\}$  being open in the space  $\mathcal{O}(\mathbf{X})$  of open subsets of  $\mathbf{X}$ . Equivalently, this means that universal quantification over a compact space preserves open predicates.

Using the language of represented spaces, one can make sense of notions such as a  $\Sigma_2^0$ -subset of the space of  $\Sigma_2^0$ -subsets of a given space. This suggests higher-order analogues to compactness: We can, e.g., investigate the spaces  $\mathbf{X}$  where  $\{X\}$  is a  $\Delta_2^0$ -subset of the space of  $\Delta_2^0$ -subsets of  $\mathbf{X}$ . Call this notion  $\nabla$ -compactness. As  $\Delta_2^0$  is self-dual, we find that both universal and existential quantifier over  $\nabla$ -compact spaces preserve  $\Delta_2^0$  predicates.

Recall that a space is called Noetherian iff every subset is compact. Within the setting of Quasi-Polish spaces, we can fully characterize the  $\nabla$ -compact spaces: A Quasi-Polish space is Noetherian iff it is  $\nabla$ -compact. Note that the restriction to Quasi-Polish spaces is sufficiently general to include plenty of examples.

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## 1 Introduction

### 1.1 Noetherian spaces

► **Definition 1.** A topological space  $\mathbf{X}$  is called *Noetherian*, iff every strictly ascending chain of open sets is finite.

Noetherian spaces were first studied in algebraic geometry. Here, the prime motivation is that the Zariski topology on the spectrum of a Noetherian commutative ring is Noetherian (which earns the Noetherian spaces their name).

The relevance of Noetherian spaces for computer science was noted by GOUBAULT-LARRECQ [14], based on their relationship to well quasiorders. Via well-structured transition systems [13], well quasiorders are used in verification to prove decidability of termination and related properties. Unfortunately, well quasiorders lack some desirable closure properties (the standard counterexample is due to RADO [33]), which led to the introduction of better quasiorders by NASH-WILLIAMS [25], which is a more restrictive notion avoiding the shortcomings of well quasiorders.

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Noetherian spaces generalize well-quasi orders: The Alexandrov topology on a quasi-order is Noetherian iff the quasi-order is a well-quasi order. As shown by GOUBAULT-LARRECQ [17], results on the preservation of well-quasi orders under various constructions (such as Higman’s Lemma or Kruskal’s Tree Theorem [16]) extend to Noetherian spaces; furthermore, Noetherian spaces exhibit some additional closure properties, e.g. the Hoare space of a Noetherian space is Noetherian again [14]. The usefulness of Noetherian spaces for verification is detailed by GOUBAULT-LARRECQ in [15].

## 1.2 Quasi-Polish spaces

A countably-based topological space is called quasi-Polish if its topology can be derived from a Smyth-complete quasi-metric. Quasi-Polish spaces were introduced by DE BRECHT in [6] as a joint generalization of Polish spaces and  $\omega$ -continuous domains in order to satisfy the desire for a unified setting for descriptive set theory in those areas (expressed e.g. by SELIVANOV [36]).

## 1.3 Synthetic DST

Synthetic descriptive set theory as proposed by the authors in [32] reinterprets descriptive set theory in a category-theoretic context. In particular, it provides notions of lifted counterparts to topological concepts such as open sets (e.g.  $\Sigma$ -classes from descriptive set theory), compactness, and so on.

## 1.4 Our contributions

In the present paper, we will study Noetherian quasi-Polish spaces. As our main result, we show that in the setting of quasi-Polish spaces, being Noetherian is the  $\Delta_2^0$ -analogue to compactness. We present the result in two different incarnations:

Theorem 10 states the result in the language of traditional topology, i.e. a quasi-Polish space is Noetherian iff every  $\Delta_2^0$ -cover has a finite subcover. While we will prove Theorem 10 via the Baire category theorem, it is also a simple consequence of results on the Skula topology for sober spaces.

Our second version (Theorem 39) can be seen as lifting the characterization of compactness by the projections being closed maps. To be able to even formulate this result, we employ the language of synthetic topology. This requires us to define a computable version of being Noetherian (Definition 35). A significant improvement of usefulness of Theorem 39 compared to Theorem 10 is a particular consequence: Universal and existential quantification over Noetherian spaces preserve  $\Delta_2^0$ -predicates – and this characterizes Noetherian spaces (Proposition 43).

## 1.5 Structure of the article

In Section 2 we recall some results on Noetherian spaces and on quasi-Polish spaces, and then prove some observations on Noetherian quasi-Polish spaces. In particular, Theorem 10 shows that for quasi-Polish spaces, being Noetherian is equivalent to any  $\Delta_2^0$ -cover admitting a finite subcover. This section requires only some basic background from topology.

Section 3 introduces the additional background material we need for the remainder of the paper, in particular from computable analysis and synthetic topology.

In Section 4 we investigate how Noetherian spaces ought to be defined in synthetic topology (ESCARDÓ [10]), specifically in the setting of the category of represented spaces (PAULY [29]).

Our main result will be presented in Section 5: The Noetherian spaces can be characterized amongst the quasi-Polish spaces as those allowing quantifier elimination over  $\Delta_2^0$ -statements (Theorem 39 and Corollary 44). The core idea is that just as compact spaces are characterized by  $\{X\}$  being an open subset of the space  $\mathcal{O}(\mathbf{X})$  of open subsets, the Noetherian spaces are (amongst the quasi-Polish) characterized by  $\{X\}$  being a  $\Delta_2^0$ -subset of the space of  $\Delta_2^0$ -subsets.

An extended version is available on the arXiv as [8].

## 2 Initial observations on Noetherian quasi-Polish spaces

### 2.1 Background on Quasi-Polish spaces

Recall that a quasi-metric on  $\mathbf{X}$  is a function  $d : \mathbf{X} \times \mathbf{X} \rightarrow [0, \infty)$  such that  $x = y \Leftrightarrow d(x, y) = d(y, x) = 0$  and  $d(x, z) \leq d(x, y) + d(y, z)$ , i.e. a quasi-metric has all properties of a metric except symmetry. A sequence  $(x_n)_{n \in \mathbb{N}}$  in a quasi-metric space is Cauchy, if  $\forall k \in \mathbb{N} \exists N \in \mathbb{N} \forall m \geq n d(x_n, x_m) < 2^{-k}$ . After SMYTH [37], we call a quasi-metric space Smyth-complete, if every Cauchy sequence converges w.r.t. the metric  $d'$  obtained as  $d'(x, y) := \max\{d(x, y), d(y, x)\}$ .

A quasi-metric induces a topology via the basis  $(B(x, 2^{-k}) := \{y \in \mathbf{X} \mid d(x, y) < 2^{-k}\})_{x \in \mathbf{X}, k \in \mathbb{N}}$ . A topological space is called quasi-Polish, if it is countably-based and the topology can be obtained from a Smyth-complete quasi-metric. For details we refer to [6], and only recall some select results to be used later on here.

The  $\Sigma_2^0$ -subsets of a quasi-Polish space are those of the form  $\bigcup_{n \in \mathbb{N}} U_n \setminus V_n$ , where  $U_n$  and  $V_n$  are open. Complements of  $\Sigma_2^0$ -sets are called  $\Pi_2^0$ -subset. A set is  $\Delta_2^0$  iff it is both  $\Sigma_2^0$  and  $\Pi_2^0$ . It follows that the  $\Sigma_2^0$ -sets are exactly the countable unions of  $\Delta_2^0$ -sets.

► **Proposition 2** (DE BRECHT [6]). A subspace of a quasi-Polish space is a quasi-Polish space iff it is a  $\Pi_2^0$ -subspace.

► **Proposition 3** (DE BRECHT [6]). A space is quasi-Polish iff it is homoeomorphic to a  $\Pi_2^0$ -subspace of the Scott domain  $\mathcal{P}(\omega)$ .

► **Theorem 4** (HECKMANN [19], BECKER & GRIGORIEFF [1, Theorem 3.14]). Let  $\mathbf{X}$  be quasi-Polish. If  $\mathbf{X} = \bigcup_{i \in \mathbb{N}} A_i$  with each  $A_i$  being  $\Sigma_2^0$ , then there is some  $i_0$  such that  $A_{i_0}$  has non-empty interior.

Recall that a non-empty closed set is called *irreducible*, if it is not the union of two proper closed subsets. A topological space is called *sober*, if each irreducible closed set is the closure of a unique singleton.

► **Proposition 5** (DE BRECHT [6]). A countably-based locally compact sober space is quasi-Polish. Conversely, each quasi-Polish space is sober.

### 2.2 Background on Noetherian spaces

► **Theorem 6** (Folklore, cf. GOUBAULT-LARRECQ [17]). The following are equivalent for a topological space  $\mathbf{X}$ :

1.  $\mathbf{X}$  is Noetherian, i.e. every strictly ascending chain of open sets is finite (Definition 1).
2. Every strictly descending chain of closed sets is finite.
3. Every open set is compact.
4. Every subset is compact.

As being Noetherian is preserved by soberification<sup>1</sup>, we do not lose much by restricting our attention to sober Noetherian spaces.

► **Lemma 7** (Folklore, cf. GOUBAULT-LARRECQ [17]). *Every closed subset of a sober Noetherian space is the closure of a finite set.*

### 2.3 Some new observations

► **Theorem 8.** *The following are equivalent for a sober Noetherian space  $\mathbf{X}$ :*

1.  $\mathbf{X}$  is countable.
2.  $\mathbf{X}$  is countably-based.
3.  $\mathbf{X}$  is quasi-Polish.

► **Corollary 9.** *A subspace of a quasi-Polish Noetherian space is sober iff it is a  $\Pi_2^0$ -subspace.*

**Proof.** Combine Theorem 8 with Proposition 2. ◀

The following theorem already showcases the link between being Noetherian and a  $\Delta_2^0$ -analogue to compactness. Its proof is split into Lemmata 11,12 and Observation 13. The result can alternatively be obtained as corollary of a result by HOFFMANN [20] on the relationship between Noetherian spaces and the Skula topology<sup>2</sup>.

► **Theorem 10.** *The following are equivalent for a quasi-Polish space  $\mathbf{X}$ :*

1.  $\mathbf{X}$  is Noetherian.
2. Every  $\Delta_2^0$ -cover of  $\mathbf{X}$  has a finite subcover.
3. Every  $\Sigma_2^0$ -cover of  $\mathbf{X}$  has a finite subcover.

► **Lemma 11.** *If a topological space  $\mathbf{X}$  is not Noetherian, then it admits a countably-infinite  $\Delta_2^0$ -partition.*

**Proof.** If  $\mathbf{X}$  is not Noetherian, then there must be an infinite strictly ascending chain  $(U_i)_{i \in \mathbb{N}}$  of open sets. Then  $\{U_{i+1} \setminus U_i \mid i \in \mathbb{N}\} \cup \{U_0, (\bigcup_{i \in \mathbb{N}} U_i)^C\}$  constitutes a  $\Delta_2^0$ -partition with countably-infinitely many non-trivial pieces. ◀

► **Lemma 12.** *Any  $\Delta_2^0$ -cover of a Noetherian quasi-Polish space has a finite subcover.*

**Proof.** Since  $\mathbf{X}$  is countable we can assume the covering is countable. By the Baire category theorem for quasi-Polish spaces (Theorem 4), there is a  $\Delta_2^0$ -set  $A_0$  in the covering such that its interior,  $U_0$  is non-empty.

For  $n \geq 0$ , if  $\mathbf{X} \neq U_n$ , then we repeat the same argument with respect to  $\mathbf{X} \setminus U_n$  to get a  $\Delta_2^0$ -set  $A_{n+1}$  in the covering with non-empty interior relative to  $\mathbf{X} \setminus U_n$ . Define  $U_{n+1}$  to be the union of  $U_n$  and the relative interior of  $A_{n+1}$ . Then  $U_{n+1}$  is an open subset of  $\mathbf{X}$  which strictly contains  $U_n$ . Since  $\mathbf{X}$  is Noetherian, eventually  $\mathbf{X} = U_n$ , and  $A_0, \dots, A_n$  will yield a finite subcovering of  $\mathbf{X}$ . ◀

► **Observation 13.** Any  $\Sigma_2^0$ -cover of a quasi-Polish space can be refined into a  $\Delta_2^0$ -cover, and any  $\Delta_2^0$ -cover is a  $\Sigma_2^0$ -cover.

**Proof.** In a quasi-Polish space, every  $\Sigma_2^0$ -set is a (countable) union of  $\Delta_2^0$ -sets. ◀

<sup>1</sup> Sobrification only adds points, not open sets, and being Noetherian is only about open sets.

<sup>2</sup> Thanks to an anonymous referee for pointing this out to us. See also [17, Exercise 9.7.16].

► **Corollary 14.** Let  $\mathbf{X}$  be a Noetherian quasi-Polish space, and let  $\mathbf{X}^\delta$  be the topology induced by the  $\Delta_2^0$ -subsets of  $\mathbf{X}$ . Then  $\mathbf{X}^\delta$  is a compact Hausdorff space.

**Proof.** That  $\mathbf{X}^\delta$  is compact follows from Lemma 12. To see that it is Hausdorff, we just note that in any  $T_0$ -space, two distinct points can be separated by a disjoint pair of an open and a closed set – hence by  $\Delta_2^0$ -sets. ◀

We can obtain the following special case of GOUBAULT-LARRECQ's Lemma 7 as a corollary of Lemma 12:

► **Corollary 15.** Every closed subset of a quasi-Polish Noetherian space is the closure of a finite set.

**Proof.** Given some closed subset  $A \subseteq \mathbf{X}$ , consider the  $\Delta_2^0$ -cover  $\mathbf{X} = A^C \cup \bigcup_{x \in A} \text{cl}\{x\}$ . By Lemma 12 there is some finite subcover  $\mathbf{X} = A^C \cup \bigcup_{x \in F} \text{cl}\{x\}$ , but then it follows that  $A = \text{cl}F$ . ◀

Neither being sober nor being quasi-Polish is preserved by continuous images in general. However, being Noetherian is not only preserved itself, but in its presence, so are the other properties:

► **Proposition 16.** Let  $\mathbf{X}$  be a Noetherian sober (quasi-Polish) space and  $\sigma : \mathbf{X} \rightarrow \mathbf{Y}$  a continuous surjection. Then  $\mathbf{Y}$  is Noetherian sober (quasi-Polish) space, too.

**Proof.** Let  $C \subseteq \mathbf{Y}$  be irreducible closed. Then  $\sigma^{-1}(C)$  is closed, so by Lemma 7 (or Corollary 15) there is finite  $F \subseteq \mathbf{X}$  such that  $\text{cl}(F) = \sigma^{-1}(C)$ . Continuity implies  $\text{cl}(\sigma(F)) \supseteq \sigma(\text{cl}(F)) = C$ , hence  $\text{cl}(\sigma(F)) = C$ . Since  $\sigma(F)$  is finite and  $C$  is irreducible,  $C$  must be equal to the closure of some element of  $\sigma(F)$ . Therefore,  $\mathbf{Y}$  is sober.

By Theorem 8, for Noetherian sober spaces being quasi-Polish is equivalent to being countable, which is clearly preserved by (continuous) surjections. ◀

### 3 Background

#### 3.1 Computable analysis

In the remainder of this article, we wish to explore the uniform or effective aspects of the theory of Noetherian Quasi-Polish spaces. The basic framework for this is provided by *computable analysis* [40]. Here the core idea is to introduce notions of continuity and in particular continuity on a wide range of spaces by translating them from those on Baire space via the so-called representations. Our notation and presentation follows closely that of [29], which in turn is heavily influenced by ESCARDÓ's *synthetic topology* [10], and by work by SCHRÖDER [34].

► **Definition 17.** A represented space is a pair  $\mathbf{X} = (X, \delta_X)$  where  $X$  is a set and  $\delta_X : \subseteq \mathbb{N}^\mathbb{N} \rightarrow X$  is a partial surjection. A function between represented spaces is a function between the underlying sets.

► **Definition 18.** For  $f : \subseteq \mathbf{X} \rightarrow \mathbf{Y}$  and  $F : \subseteq \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N}^\mathbb{N}$ , we call  $F$  a realizer of  $f$  (notation  $F \vdash f$ ), iff  $\delta_Y(F(p)) = f(\delta_X(p))$  for all  $p \in \text{dom}(f\delta_X)$ . A map between represented spaces is called computable (continuous), iff it has a computable (continuous) realizer.

Represented spaces  $\mathbf{X}$ ,  $\mathbf{X}'$  are *computably isomorphic*, written  $\mathbf{X} \cong \mathbf{X}'$ , if there is a bijection  $f : \mathbf{X} \rightarrow \mathbf{X}'$  such that both  $f$  and  $f'$  are computable.

Two represented spaces of particular importance are the integers  $\mathbb{N}$  and Sierpiński space  $\mathbb{S}$ . The represented space  $\mathbb{N}$  has as underlying set  $\mathbb{N}$  and the representation  $\delta_{\mathbb{N}} : \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}$  defined by  $\delta_{\mathbb{N}}(p) = p(0)$ . The Sierpiński space  $\mathbb{S}$  has the underlying set  $\{\top, \perp\}$  and the representation  $\delta_{\mathbb{S}}$  with  $\delta_{\mathbb{S}}(0^\omega) = \perp$  and  $\delta_{\mathbb{S}}(p) = \top$  for  $p \neq 0^\omega$ .

Represented spaces have binary products, defined in the obvious way: The underlying set of  $\mathbf{X} \times \mathbf{Y}$  is  $X \times Y$ , with the representation  $\delta_{\mathbf{X} \times \mathbf{Y}}(\langle p, q \rangle) = (\delta_{\mathbf{X}}(p), \delta_{\mathbf{Y}}(q))$ . Here  $\langle \cdot, \cdot \rangle : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$  is the pairing function defined via  $\langle p, q \rangle(2n) = p(n)$  and  $\langle p, q \rangle(2n+1) = q(n)$ .

A central reason why the category of represented spaces is such a convenient setting lies in the fact that it is cartesian closed: We have available a function space construction  $\mathcal{C}(\cdot, \cdot)$ , where the represented space  $\mathcal{C}(\mathbf{X}, \mathbf{Y})$  has as underlying set the continuous functions from  $\mathbf{X}$  to  $\mathbf{Y}$ , represented in such a way that the evaluation map  $(f, x) : \mathcal{C}(\mathbf{X}, \mathbf{Y}) \times \mathbf{X} \rightarrow \mathbf{Y}$  becomes computable. This can be achieved, e.g., by letting  $nq$  represent  $f$ , if the  $n$ -th Turing machine equipped with oracle  $q$  computes a realizer of  $f$ . This also makes currying, uncurrying and composition all computable maps.

Having available to us the space  $\mathbb{S}$  and the function space construction, we can introduce the spaces  $\mathcal{O}(\mathbf{X})$  and  $\mathcal{A}(\mathbf{X})$  of open and closed subsets respectively of a given represented space  $\mathbf{X}$ . For this, we identify an open subset  $U$  of  $\mathbf{X}$  with its (continuous) characteristic function  $\chi_U : \mathbf{X} \rightarrow \mathbb{S}$ , and a closed subset with the characteristic function of the complement. As countable join (or) and binary meet (and) on  $\mathbb{S}$  are computable, we can conclude that open sets are uniformly closed under countable unions, binary intersections and preimages under continuous functions by merely using elementary arguments about function spaces. The space  $\mathcal{A}(\mathbf{X})$  corresponds to the upper Fell topology [12] on the hyperspace of closed sets.

Note that neither negation  $\neg : \mathbb{S} \rightarrow \mathbb{S}$  (i.e. mapping  $\top$  to  $\perp$  and  $\perp$  to  $\top$ ) nor countable meet (and)  $\wedge : \mathcal{C}(\mathbb{N}, \mathbb{S}) \rightarrow \mathbb{S}$  (i.e. mapping the constant sequence  $(\top)_{n \in \mathbb{N}}$  to  $\top$  and every other sequence to  $\perp$ ) are continuous or computable operations. They will play the role of fundamental counterexamples in the following. Both operations are equivalent to the *limited principle of omniscience* (LPO) in the sense of Weihrauch reducibility [39].

We need two further hyperspaces, which both will be introduced as subspaces of  $\mathcal{O}(\mathcal{O}(\mathbf{X}))$ . The space  $\mathcal{K}(\mathbf{X})$  of saturated compact sets identifies  $A \subseteq \mathbf{X}$  with  $\{U \in \mathcal{O}(\mathbf{X}) \mid A \subseteq U\} \in \mathcal{O}(\mathcal{O}(\mathbf{X}))$ . Recall that a set is saturated, iff it is equal to the intersection of all open sets containing it (this makes the identification work). The saturation of  $A$  is denoted by  $\uparrow A := \bigcap \{U \in \mathcal{O}(\mathbf{X}) \mid A \subseteq U\}$ . Compactness of  $A$  corresponds to  $\{U \in \mathcal{O}(\mathbf{X}) \mid A \subseteq U\}$  being open itself. The dual notion to compactness is *overtness*<sup>3</sup>. We obtain the space  $\mathcal{V}(\mathbf{X})$  of overt sets by identifying a closed set  $A$  with  $\{U \in \mathcal{O}(\mathbf{X}) \mid A \cap U \neq \emptyset\} \in \mathcal{O}(\mathcal{O}(\mathbf{X}))$ . The space  $\mathcal{V}(\mathbf{X})$  corresponds to the lower Fell (equivalently, the lower Vietoris) topology.

Aligned with the definition of the compact and overt subsets of a space, we can also define when a space itself is compact (respectively overt):

► **Definition 19.** A represented space  $\mathbf{X}$  is (computably) compact, iff  $\text{isFull} : \mathcal{O}(\mathbf{X}) \rightarrow \mathbb{S}$  mapping  $X$  to  $\top$  and any other open set to  $\perp$  is continuous (computable). Dually, it is (computably) overt, iff  $\text{isNonEmpty} : \mathcal{O}(\mathbf{X}) \rightarrow \mathbb{S}$  mapping  $\emptyset$  to  $\perp$  and any non-empty open set to  $\top$  is continuous (computable).

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<sup>3</sup> This notion is much less known than compactness, as it is classically trivial. It is crucial in a uniform perspective, though. The term *overt* was coined by TAYLOR [38], based on the observation that these sets share several closure properties with the open sets.

The relevance of  $\mathcal{K}(\mathbf{X})$  and  $\mathcal{V}(\mathbf{X})$  is found in particular in the following characterizations, which show that compactness just makes universal quantification preserve open predicates, and dually, overtess makes existential quantification preserve open predicates. We shall see later that being Noetherian has the same role for  $\Delta_2^0$ -predicates.

► **Proposition 20** ([29, Proposition 40]). The map  $\exists : \mathcal{O}(\mathbf{X} \times \mathbf{Y}) \times \mathcal{V}(\mathbf{X}) \rightarrow \mathcal{O}(\mathbf{Y})$  defined by  $\exists(R, A) = \{y \in Y \mid \exists x \in A (x, y) \in R\}$  is computable. Moreover, whenever  $\exists : \mathcal{O}(\mathbf{X} \times \mathbf{Y}) \times \mathcal{S}(\mathbf{X}) \rightarrow \mathcal{O}(\mathbf{Y})$  is computable for some hyperspace  $\mathcal{S}(\mathbf{X})$  and some space  $\mathbf{Y}$  containing a computable element  $y_0$ , then  $\bar{\cdot} : \mathcal{S}(\mathbf{X}) \rightarrow \mathcal{V}(\mathbf{X})$  is computable, where  $\bar{\cdot}$  denotes topological closure.

► **Proposition 21** ([29, Proposition 42]). The map  $\forall : \mathcal{O}(\mathbf{X} \times \mathbf{Y}) \times \mathcal{K}(\mathbf{X}) \rightarrow \mathcal{O}(\mathbf{Y})$  defined by  $\forall(R, A) = \{y \in Y \mid \forall x \in A (x, y) \in R\}$  is computable. Moreover, whenever  $\forall : \mathcal{O}(\mathbf{X} \times \mathbf{Y}) \times \mathcal{S}(\mathbf{X}) \rightarrow \mathcal{O}(\mathbf{Y})$  is computable for some hyperspace  $\mathcal{S}(\mathbf{X})$  and some space  $\mathbf{Y}$  containing a computable element  $y_0$ , then  $\uparrow \text{id} : \mathcal{S}(\mathbf{X}) \rightarrow \mathcal{K}(\mathbf{X})$  is computable.

### 3.2 Connecting computable analysis and topology

Calling the elements of  $\mathcal{O}(\mathbf{X})$  the *open sets* is justified by noting that they indeed form a topology, namely the final topology  $X$  inherits from the subspace topology of  $\text{dom}(\delta_{\mathbf{X}})$  along  $\delta_{\mathbf{X}}$ . The notion of a continuous map between the represented spaces  $\mathbf{X}, \mathbf{Y}$  however differs from that of a continuous map between the induced topological spaces. For a large class of spaces, the notions do coincide after all, as observed originally by SCHRÖDER [35].

► **Definition 22.** Call  $\mathbf{X}$  admissible, if the map  $x \mapsto \{U \in \mathcal{O}(\mathbf{X}) \mid x \in U\} : \mathbf{X} \rightarrow \mathcal{O}(\mathcal{O}(\mathbf{X}))$  admits a continuous partial inverse.

► **Theorem 23** ([29, Theorem 36]). *A represented space  $\mathbf{X}$  is admissible iff any map  $f : \mathbf{Y} \rightarrow \mathbf{X}$  is continuous as a map between represented spaces iff it is continuous as a map between the induced topological spaces.*

The admissible represented spaces are themselves cartesian closed (in fact, it suffices for  $\mathbf{Y}$  to be admissible in order to make  $\mathcal{C}(\mathbf{X}, \mathbf{Y})$  admissible). They can be seen as a joint subcategory of the sequential topological spaces and the represented spaces, and thus form the natural setting for computable topology. They have been characterized by SCHRÖDER as the QCB<sub>0</sub>-spaces [35], the T<sub>0</sub> quotients of countably based spaces.

WEIHRAUCH [40, 41] introduced the standard representation of a countably based T<sub>0</sub> space: Given some enumeration  $(U_n)_{n \in \mathbb{N}}$  of a basis of a topological space  $\mathbf{X}$ , one can introduce the representation  $\delta_{\mathbf{B}}$  where  $\delta_{\mathbf{B}}(p) = x$  iff  $\{n \in \mathbb{N} \mid \exists i p(i) = n + 1\} = \{n \in \mathbb{N} \mid x \in U_n\}$ . This yields an admissible representation, which in turn induces the original topology on  $\mathbf{X}$ .

Amongst the countably based spaces, the quasi-Polish spaces are distinguished by a completeness properties. We will make use of the following characterization:

► **Theorem 24** (dB [6]). *A topological space  $\mathbf{X}$  is quasi-Polish, iff its topology is induced by an open admissible total representation  $\delta_{\mathbf{X}} : \mathbb{N}^{\mathbb{N}} \rightarrow \mathbf{X}$ .*

### 3.3 Synthetic descriptive set theory

The central addition of synthetic descriptive set theory (as proposed by the authors in [30, 32]) is the notion of a computable endofunctor:

► **Definition 25.** An endofunctor  $d$  on the category of represented spaces is called *computable*, if for any represented spaces  $\mathbf{X}, \mathbf{Y}$  the induced morphism  $d : \mathcal{C}(\mathbf{X}, \mathbf{Y}) \rightarrow \mathcal{C}(d\mathbf{X}, d\mathbf{Y})$  is computable.

To keep things simple, we will restrict our attention here to endofunctors that do not change the underlying set of a represented spaces, but may only modify the representation. Such endofunctors can in particular be derived from certain maps on Baire space, called *jump operators* by DE BRECHT in [7]. Here, we instead adopt the terminology *transparent map* introduced in [4]. Further properties of transparent maps were studied in [27].

► **Definition 26.** Call  $T : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$  *transparent* iff for any computable (continuous)  $g : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$  there is a computable (continuous)  $f : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$  with  $T \circ f = g \circ T$ .

If the required witness  $f$  can be effectively obtained from  $g$ , then  $T$  will induce a computable endofunctor  $t$  by setting  $t\mathbf{X}$  to be  $(X, \delta_{\mathbf{X}} \circ T)$ , and extending to functions in the obvious way.

By applying a suitable endofunctor to Sierpiński space, we can define further classes of subsets; in particular those commonly studied in descriptive set theory. This idea and its relationship to universal sets is further explored in [18]. Basically, we introduce the space  $\mathcal{O}^d(\mathbf{X})$  of  $d$ -open subsets of  $\mathbf{X}$  by identifying a subset  $U$  with its continuous characteristic function  $\chi_U : \mathbf{X} \rightarrow d\mathbb{S}$ . If  $d$  preserves countable products, it automatically follows that the  $d$ -open subsets are effectively closed under countable unions, binary intersections and preimages under continuous maps. The complements of the  $d$ -opens are the  $d$ -closed sets, denoted by  $\mathcal{A}^d(\mathbb{S})$ .

We will use the endofunctors to generate lifted versions of compactness and overtness:

► **Definition 27.** A represented space  $\mathbf{X}$  is (computably)  $d$ -compact, iff  $\text{isFull} : \mathcal{O}^d(\mathbf{X}) \rightarrow d\mathbb{S}$  mapping  $X$  to  $\top$  and any other open set to  $\perp$  is continuous (computable). Dually, it is (computably)  $d$ -overt, iff  $\text{isNonEmpty} : \mathcal{O}^d(\mathbf{X}) \rightarrow d\mathbb{S}$  mapping  $\emptyset$  to  $\perp$  and any non-empty open set to  $\top$  is continuous (computable).

A fundamental example of a computable endofunctor linked to notions from descriptive set theory is the limit or jump endofunctor;

► **Definition 28.** Let  $\lim : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$  be defined via  $\lim(p)(n) = \lim_{i \rightarrow \infty} p(\langle n, i \rangle)$ , where  $\langle , \rangle : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  is a standard pairing function. Define the computable endofunctor  $'$  by  $(X, \delta_{\mathbf{X}})' = (X, \delta_{\mathbf{X}} \circ \lim)$  and the straight-forward lift to functions.

The map  $\lim$  and its relation to the Borel hierarchy and Weihrauch reducibility was studied by BRATTKA in [2]. The jump of a represented spaces was studied in [42, 4]. The  $'$ -open sets are just the  $\Sigma_2^0$ -sets, and the further levels of the Borel hierarchy can be obtained by iterating the endofunctor.

### 3.3.1 Computability with finitely many mindchanges

The most important endofunctor for our investigation of Noetherian Quasi-Polish spaces is the finite mindchange endofunctor  $\nabla$ :

► **Definition 29 ([30]).** Define  $\Delta : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$  via  $\Delta(p)(n) = p(n + 1 + \max\{i \mid p(i) = 0\}) - 1$ . Let the finite mindchange endofunctor be defined via  $(X, \delta_X)^{\nabla} = (X, \delta_X \circ \Delta)$  and  $(f : \mathbf{X} \rightarrow \mathbf{Y})^{\nabla} = f : \mathbf{X}^{\nabla} \rightarrow \mathbf{Y}^{\nabla}$ .

We find that  $\nabla$  is a monad, and moreover, that  $f : \mathbf{X} \rightarrow \mathbf{Y}^\nabla$  is computable (continuous) iff  $f : \mathbf{X}^\nabla \rightarrow \mathbf{Y}^\nabla$  is. The computable maps from  $\mathbf{X}$  to  $\mathbf{Y}^\nabla$  can equivalently be understood as those maps from  $\mathbf{X}$  to  $\mathbf{Y}$  that are computable with finitely many mindchanges.

A machine model for computation with finitely many mindchanges is obtained by adding the option of resetting the output tape to the initial state. To ensure that the output is well-defined, such a reset can only be used finitely many times. Essentially, each  $n + 1$  in the input of  $\Delta$  corresponds to a *write n* command, whereas each 0 represents the *reset-command*.

In the context of computable analysis, computation with finitely many mindchanges was studied by a number of authors [42, 9, 3, 5, 26]. For our purposes, an equivalent model based on non-deterministic computation turns out to be more useful. We say that a function from  $\mathbf{X}$  to  $\mathbf{Y}$  is non-deterministically computable with advice space  $\mathbb{N}$ , if on input  $p$  (a name for some  $x \in \mathbf{X}$ ) the machine can guess some  $n \in \mathbb{N}$  and then either continue for  $\omega$  many steps and output a valid name for  $f(x)$ , or at some finite time reject the guess. We demand that for any  $p$  there is some  $n \in \mathbb{N}$  that is not rejected. The equivalence of the two models is shown in [3].

The interpretation of  $\nabla$  in descriptive set theory is related to the  $\Delta_2^0$ -sets. In particular, the  $\nabla$ -open sets are the  $\Delta_2^0$ -sets, the continuous functions from  $\mathbf{X}$  to  $\mathbf{Y}^\nabla$  are the piecewise continuous functions for Polish  $\mathbf{X}$ , and the lifted version of admissibility under  $\nabla$  corresponds to the Jayne-Rogers theorem (cf. [21, 23, 22]). This was explored in detail by the authors in [31].

## 4 $\nabla$ -computably Noetherian spaces

In this section, we want to investigate the notion of being Noetherian in the setting of synthetic topology. We will see that the naive approach fails, but then provide a well-behaved definition. That it is adequate will be substantiated by providing a computable counterpart to the relationship between Noetherian spaces and well-quasiorders. First, however, we will explore a prototypical example.

### 4.1 A case study on computably Noetherian spaces

Let  $\mathbb{N}_<$  be the natural numbers with the topology  $\mathcal{T}_< := \{L_n := \{i \in \mathbb{N} \mid i \geq n\} \mid n \in \mathbb{N}\} \cup \{\emptyset\}$ . Then let  $\bar{\mathbb{N}}_<$  be the result of adjoining  $\infty$ , which is contained in all non-empty open sets. In  $\bar{\mathbb{N}}_<$  we find a very simple yet non-trivial example of a quasi-Polish Noetherian space.

Similarly, let  $\mathbb{N}_>$  be the natural numbers with the topology  $\mathcal{T}_> := \{U_n := \{i \in \mathbb{N} \mid i < n\} \mid n \in \mathbb{N}\} \cup \{\mathbb{N}\}$ . By  $\bar{\mathbb{N}}_>$  we denote the space resulting from adjoining an element  $\infty$ , which is only contained in one open set. In terms of representations, we can conceive of an element in  $\mathbb{N}_<$  as being given as the limit of an increasing sequence, and of an element in  $\mathbb{N}_>$  as the limit of a decreasing sequence.

Looking at the way how we defined  $\mathcal{T}_<$ , we see that we have a countable basis, and given indices of open sets, can e.g. decide subset inclusion. The indexing is fully effective, in the sense that this is a computable basis as follows:

► **Definition 30** ([18, Definition 9]). An effective countable base for  $\mathbf{X}$  is a computable sequence  $(U_i)_{i \in \mathbb{N}} \in \mathcal{C}(\mathbb{N}, \mathcal{O}(\mathbf{X}))$  such that the multivalued partial map  $\text{Base} : \subseteq \mathbf{X} \times \mathcal{O}(\mathbf{X}) \rightrightarrows \mathbb{N}$  is computable. Here  $\text{dom}(\text{Base}) = \{(x, U) \mid x \in U\}$  and  $n \in \text{Base}(x, U)$  iff  $x \in U_n \subseteq U$ .

Even though all open sets are basis elements, we should still distinguish computability on the open sets themselves, and computability on the indices. For example, the map  $\bigcup : \mathcal{O}(\mathbf{X})^\mathbb{N} \rightarrow \mathcal{O}(\mathbf{X})$ , i.e. the countable union of open sets, should always be a computable

operation. This, however, cannot be done on the indices. More generally, in the synthetic topology framework the space of open subsets of a given space automatically comes with its own natural topology. This topology is obtained by demanding that given a point and an open set, we can recognize (semidecide) membership. In the case of  $\mathbb{N}_<$ , we can establish a quite convenient characterization of its open subsets:

► **Proposition 31.** The map  $n \mapsto \{i \in \mathbb{N} \mid i \geq n\} : \overline{\mathbb{N}}_> \rightarrow \mathcal{O}(\mathbb{N}_<)$  is a computable isomorphism.

**Proof.**

1. The map is computable.

Given  $m \in \mathbb{N}_<$  and  $n \in \overline{\mathbb{N}}_>$ , we can semidecide  $m \geq n$  (just wait until the increasing and the decreasing approximations pass each other).

2. The map is surjective.

At the moment some number  $m$  is recognized to be an element of some open set  $U \in \mathcal{O}(\mathbb{N}_<)$ , we have only learned some lower bound on  $m$  so far. Thus, any number greater than  $m$  is contained in  $U$ , too. Hence all open subsets of  $\mathbb{N}_<$  are final segments.

3. The inverse of the map is computable.

Given  $U \in \mathcal{O}(\mathbb{N}_<)$ , we can simultaneously begin testing  $i \in U$ ? for all  $i \in \mathbb{N}$ . Any positive test provides an upper bound for the  $n$  such that  $U = \{i \in \mathbb{N} \mid i \geq n\}$ . ◀

The space of (saturated) compact subsets likewise comes with its own topology, in this case obtained by demanding that given a compact  $K$  and an open  $U$ , we can recognize if  $K \subseteq U$ . Similarly to the preceding proposition, we can also characterize the compact subsets of  $\mathbb{N}_>$ :

► **Proposition 32.** The map  $n \mapsto \{i \in \mathbb{N} \mid i \geq n\} : \overline{\mathbb{N}}_< \rightarrow \mathcal{K}(\mathbb{N}_<)$  is a computable isomorphism.

**Proof.**

1. The map is computable.

We need to show that given  $n \in \overline{\mathbb{N}}_<$  and  $U \in \mathcal{O}(\mathbb{N}_>)$  we can recognize that  $\{i \in \mathbb{N} \mid i \geq n\} \subseteq U$ . By Proposition 31, we can assume that  $U$  is of the form  $U = \{i \in \mathbb{N} \mid i \geq m\}$  with  $m \in \overline{\mathbb{N}}_>$ . Now for such  $n, m$ , we can indeed semidecide  $m \leq n$  – again, just wait until the approximating sequences reach the same value.

2. The map is surjective.

While any subset of  $\mathbb{N}_<$  is compact, only the saturated compact sets appear in  $\mathcal{K}(\mathbb{N}_>)$ , and these are the given ones.

3. The inverse map is computable.

Given a compact set  $K \in \mathcal{K}(\mathbb{N}_<)$ , we simultaneously test if it is covered by open sets of the form  $\{i \mid i \geq m\}$ . Any such  $m$  we find provides a lower bound for the  $n$  for which  $K = \{i \mid i \geq n\}$  holds. ◀

So we see that while the **spaces**  $\mathcal{O}(\mathbb{N}_<)$  and  $\mathcal{K}(\mathbb{N}_<)$  contain the same points, their topologies differ – and are, in fact, incomparable. There are two potential ways to capture the idea that *opens are compact* in a synthetic way:

We could work with open and compact sets when in a Noetherian space, i.e. with the space  $\mathcal{O}(\mathbb{N}_<) \wedge \mathcal{K}(\mathbb{N}_<)$  carrying the join of the topologies. As  $\mathbb{N}_< \wedge \mathbb{N}_> \cong \mathbb{N}$ , in this special case we would end up in the same situation as using computability on base indices straightforwardly. In general though it is not even obvious if  $\cap : (\mathcal{O}(\mathbf{X}) \wedge \mathcal{K}(\mathbf{X})) \times (\mathcal{O}(\mathbf{X}) \wedge \mathcal{K}(\mathbf{X})) \rightarrow (\mathcal{O}(\mathbf{X}) \wedge \mathcal{K}(\mathbf{X}))$  should be computable.

The second approach relies on the observation that  $\mathbb{N}_<$  and  $\mathbb{N}_>$  do not differ by *that much*. We can consider *computability with finitely many mindchanges* – and the distinction between  $\mathbb{N}_<$ ,  $\mathbb{N}_>$  and  $\mathbb{N}$  disappears, as we find  $\mathbb{N}_<^\nabla \cong \mathbb{N}_>^\nabla \cong \mathbb{N}^\nabla$ . As the next subsection shows, computability with finitely many mindchanges seems adequate to give *opens are compact* a computable interpretation.

## 4.2 The abstract approach

The straightforward approach to formulate a synthetic topology version of Noetherian would be the following:

► **Definition 33** (Hypothetical). Call a space  $\mathbf{X}$  *computably Noetherian*, iff  $\text{id}_{\mathcal{O},\mathcal{K}} : \mathcal{O}(\mathbf{X}) \rightarrow \mathcal{K}(\mathbf{X})$  is well-defined and computable.

This fails entirely, though:

► **Observation 34.** Let  $\mathbf{X}$  be non-empty. Then  $\mathbf{X}$  is not computably Noetherian according to Definition 33.

**Proof.** Note that  $\subseteq : \mathcal{K}(\mathbf{X}) \times \mathcal{O}(\mathbf{X}) \rightarrow \mathbb{S}$  is by definition of  $\mathcal{K}$  a computable map, i.e. inclusion of a compact in an open set is semidecidable. Furthermore,  $\iota : \mathbb{S} \rightarrow \mathcal{O}(\mathbf{X})$  defined via  $\iota(\top) = X$  and  $\iota(\perp) = \emptyset$  is always a computable injection for non-empty  $\mathbf{X}$ . Now if  $\mathbf{X}$  were computably Noetherian, then the map  $t \mapsto \subseteq(\text{id}_{\mathcal{O},\mathcal{K}}(\iota(t)), \emptyset)$  would be computable and identical to  $\neg : \mathbb{S} \rightarrow \mathbb{S}$ , but the latter is non-computable. ◀

We can avoid this problem by relaxing the computability-requirement to computability with finitely many mindchanges. Now we can try again:

► **Definition 35.** Call a space  $\mathbf{X}$   $\nabla$ -*computably Noetherian*, iff  $\text{id}_{\mathcal{O},\mathcal{K}} : \mathcal{O}(\mathbf{X}) \rightarrow (\mathcal{K}(\mathbf{X}))^\nabla$  is well-defined and computable.

Say that an effective countable base is *nice*, if  $\{\langle u, v \rangle \mid (U_{u(1)} \cup \dots \cup U_{u(|u|)}) \subseteq (U_{v(1)} \cup \dots \cup U_{v(|v|)})\}$  is a decidable subset of  $\mathbb{N}^* \times \mathbb{N}^*$ . Clearly any effective countable base is nice relative to some oracle, hence this requirement is unproblematic from the perspective of continuity.

We can now state and prove the following theorem, which can be seen as a uniform counterpart to Theorem 6:

► **Theorem 36.** Let  $\mathbf{X}$  be quasi-Polish, and in particular have a nice effective countable base. Then the following are equivalent:

1.  $\mathbf{X}$  is  $\nabla$ -computably Noetherian
2.  $\text{id}_{\mathcal{O},\mathcal{K}} : \mathcal{O}(\mathbf{X}) \rightarrow (\mathcal{K}(\mathbf{X}))^\nabla$  is well-defined and computable.
3.  $\subseteq : \mathcal{O}(\mathbf{X}) \times \mathcal{O}(\mathbf{X}) \rightarrow \mathbb{S}^\nabla$  is computable.
4.  $\text{Stabilize} : \mathcal{C}(\mathbb{N}, \mathcal{O}(\mathbf{X})) \rightrightarrows \mathbb{N}^\nabla$  is well-defined and computable, where  $N \in \text{Stabilize}((V_i)_{i \in \mathbb{N}})$  iff  $(\bigcup_{i=0}^N V_i) = (\bigcup_{i \in \mathbb{N}} V_i)$ .
5.  $\text{Stabilize} : \mathcal{C}(\mathbb{N}, \mathcal{A}(\mathbf{X})) \rightrightarrows \mathbb{N}^\nabla$  is well-defined and computable, where  $N \in \text{Stabilize}((A_i)_{i \in \mathbb{N}})$  iff  $(\bigcap_{i=0}^N A_i) = (\bigcap_{i \in \mathbb{N}} A_i)$ .
6. The computable map  $u \mapsto (U_{u(1)} \cup \dots \cup U_{u(|u|)}) : \mathbb{N}^* \rightarrow \mathcal{O}(\mathbf{X})$  is a surjection and has a  $\nabla$ -computable right-inverse.

Note that the forward implications hold for arbitrary represented spaces, as long as they make sense.

**Proof.**

1.  $\Leftrightarrow$  2. This is the definition.
  2.  $\Rightarrow$  3. By taking into account the definition of  $\mathcal{K}$ , we have  $\text{id}_{\mathcal{O}, \mathcal{K}} : \mathcal{O}(\mathbf{X}) \rightarrow (\mathcal{C}(\mathcal{O}(\mathbf{X}), \mathbb{S}))^\nabla$ . Moreover,  $\text{id} : \mathcal{C}(\mathbf{Y}, \mathbf{Z})^\nabla \rightarrow \mathcal{C}(\mathbf{Y}, \mathbf{Z}^\nabla)$  is always computable, so currying yields the claim.
  3.  $\Rightarrow$  4. First, we prove that Stabilize is well-defined. Assume that it is not, then there is a family  $(V_i)_{i \in \mathbb{N}}$  of open sets such that  $V := \bigcup_{i \in \mathbb{N}} V_i \neq \bigcup_{i=0}^N V_i$  for all  $N \in \mathbb{N}$ . Consider the computable map  $q \mapsto \subseteq(V, \bigcup_{i \in \mathbb{N}} V_{q(i)}) : \mathbb{N}^\mathbb{N} \rightarrow \mathbb{S}^\nabla$ . If the range of  $q$  is finite, then the output must be  $\perp$ , if the range of  $q$  is  $\mathbb{N}$ , then the output must be  $\top$ . However, these two cases cannot be distinguished in a  $\Delta_2^0$ -way, thus the  $(V_i)_{i \in \mathbb{N}}$  cannot exist, and Stabilize is well-defined.
- To see that we can compute the (multivalued) inverse, we employ the equivalence to  $\nabla$ -computability and non-deterministic computation with advice space  $\mathbb{N}$  from [3]. Given  $(V_i)_{i \in \mathbb{N}}$ , we guess  $N \in \mathbb{N}$  together with an upper bound  $b$  on the number of mindchanges happening in verifying that  $\subseteq(\bigcup_{i \in \mathbb{N}} V_i, \bigcup_{i=0}^N V_i) = \top$ . Any correct guess contains a valid solution, and any wrong guess can be rejected.
4.  $\Leftrightarrow$  5. By de Morgan's law.
  4.  $\Rightarrow$  6. In a quasi-Polish space  $\mathbf{X}$  with effectively countable basis  $(U_i)_{i \in \mathbb{N}}$ , any  $U \in \mathcal{O}(\mathbf{X})$  can be effectively represented by  $p \in \mathbb{N}^\mathbb{N}$  with  $U = \bigcup_{i \in \mathbb{N}} U_{p(i)}$ . Applying stabilize to the family  $(U_{p(i)})_{i \in \mathbb{N}}$  shows subjectivity and computability of the multivalued inverse.
  6.  $\Rightarrow$  2. First we argue that (6.) implies that Stabilize from (4.) is well-defined. For the sake of a contradiction, assume that  $p \in \mathbb{N}^\mathbb{N}$  is such that  $\bigcup_{i \in \mathbb{N}} U_{p(i)} \neq \bigcup_{i \leq n} U_{p(i)}$  for all  $n \in \mathbb{N}$ ; and that  $I$  is a  $\nabla$ -computable realizer of the inverse to  $u \mapsto (U_{u(1)} \cup \dots \cup U_{u(|u|)}) : \mathbb{N}^* \rightarrow \mathcal{O}(\mathbf{X})$ . We show how to construct some  $q \in \mathbb{N}^\mathbb{N}$  such that  $I$  changes its mind infinitely often on  $q$ . We start copying  $p$  to  $q$ . At any stage, the current output of  $I$  cannot be both equal to  $\bigcup_{i \in \mathbb{N}} U_{p(i)}$  and  $\bigcup_{i \leq n} U_{p(i)}$  (for any  $n \in \mathbb{N}$ ). If it is the former, we extend  $q$  by copying its last entry until  $I$  changes its mind. If  $I$  never changed its mind, then  $I$  answered wrong. If it is the latter, we copy  $p$  to  $q$  until  $I$  changes its mind. Again, if  $I$  never changed its mind, then  $I$  answered wrong.

Next we conclude that  $u \mapsto (U_{u(1)} \cup \dots \cup U_{u(|u|)}) : \mathbb{N}^* \rightarrow \mathcal{K}(\mathbf{X})$  is computable, provided that  $(U_n)_{n \in \mathbb{N}}$  is a nice basis. For this, note that given  $u \in \mathbb{N}^*$  and  $p \in \mathbb{N}^\mathbb{N}$ , we can semidecide whether  $(U_{u(1)} \cup \dots \cup U_{u(|u|)}) \subseteq \bigcup_{n \in \mathbb{N}} U_{p(n)}$  by well-definedness of Stabilize from (4.).

Finally, concatenating  $u \mapsto (U_{u(1)} \cup \dots \cup U_{u(|u|)}) : \mathbb{N}^* \rightarrow \mathcal{K}(\mathbf{X})$  with the  $\nabla$ -computable inverse to  $u \mapsto (U_{u(1)} \cup \dots \cup U_{u(|u|)}) : \mathbb{N}^* \rightarrow \mathcal{O}(\mathbf{X})$  yields  $\text{id}_{\mathcal{O}, \mathcal{K}} : \mathcal{O}(\mathbf{X}) \rightarrow (\mathcal{K}(\mathbf{X}))^\nabla$  as a well-defined and computable map.  $\blacktriangleleft$

► **Corollary 37.** Every Noetherian Quasi-Polish space is  $\nabla$ -computably Noetherian relative to some oracle.

**Proof.** It is clear that every Quasi-Polish space has a nice countably basis relative to some oracle, as we can just code all relevant information on the intersection of basis elements into the oracle. Then we use the 6th characterization from Theorem 36, and notice that the space being Noetherian suffices to prove the surjectivity of the map, and that access to the oracle making the countable basis nice suffices for the computability of the inverse.  $\blacktriangleleft$

We point out some further simple observations on  $\nabla$ -computably Noetherian spaces: All finite spaces containing only computable points are  $\nabla$ -computably Noetherian.  $\nabla$ -computably Noetherian spaces are closed under finite products and finite coproducts, and computable images of  $\nabla$ -computably Noetherian spaces are  $\nabla$ -computably Noetherian. With some more

effort, one can also show that any computable well-quasiorder induces a  $\nabla$ -computably Noetherian space (see [8]).

## 5 Noetherian spaces as $\nabla$ -compact spaces

For some hyperspace  $P(\mathbf{X})$  of subsets of a represented space  $\mathbf{X}$ , and a space  $B$  of truth values  $\perp, \top$ , we define the map  $\text{isFull} : P(X) \rightarrow B$  by  $\text{isFull}(X) = \top$  and  $\text{isFull}(A) = \perp$  for  $A \neq X$ . We recall from [29] that a represented space is (computably) compact iff  $\text{isFull} : \mathcal{O}(\mathbf{X}) \rightarrow \mathbb{S}$  is continuous (computable).

The space  $\mathbb{S}^\nabla \cong \mathbf{2}^\nabla$  can be considered as the space of  $\Delta_2^0$ -truth values. In particular, we can identify  $\Delta_2^0$ -subsets of  $\mathbf{X}$  with their continuous characteristic functions into  $\mathbf{2}^\nabla$ , just as the open subsets are identifiable with their continuous characteristic functions into  $\mathbb{S}$ . By replacing both occurrences of  $\mathbb{S}$  in the definition of compactness (one is hidden inside  $\mathcal{O}$ ) by  $\mathbb{S}^\nabla$ , we arrive at:

► **Definition 38.** A represented space  $\mathbf{X}$  is called  $\nabla$ -compact, iff  $\text{isFull} : \Delta_2^0(\mathbf{X}) \rightarrow \mathbb{S}^\nabla$  is computable.

► **Theorem 39.** A Quasi-Polish space is  $\nabla$ -compact iff it is  $\nabla$ -computably Noetherian (relative to some oracle).

The proof is provided in the following lemmata and propositions.

Recall that constructible subsets of a topological space are finite boolean combinations of open subsets. For a represented space  $\mathbf{X}$ , there is an obvious represented space  $\mathfrak{C}(\mathbf{X})$  of constructible subsets of  $\mathbf{X}$ : A set  $A \in \mathfrak{C}(\mathbf{X})$  is given by a (Goedel-number of a) boolean expression  $\phi$  in  $n$  variables, and an  $n$ -tuple of open sets  $U_1, \dots, U_n$  such that  $A = \phi(U_1, \dots, U_n)$ . Straight-forward calculation shows that we can always assume that  $\phi(x_1, \dots, x_{2n}) = (x_1 \setminus x_2) \cup \dots \cup (x_{2n-1} \setminus x_{2n})$  without limitation of generality.

► **Lemma 40** ((<sup>4</sup>)). Let  $\mathbf{X}$  be a  $\nabla$ -computably Noetherian Quasi-Polish space. Then  $\text{id} : \Delta_2^0(\mathbf{X}) \rightarrow \mathfrak{C}(\mathbf{X})^\nabla$  is well-defined and computable.

**Proof.** As  $\mathbf{X}$  is Quasi-Polish, we can take it to be represented by an effectively open representation  $\delta_{\mathbf{X}} : \mathbb{N}^\mathbb{N} \rightarrow \mathbf{X}$ .

We can consider our input  $A \in \Delta_2^0(\mathbf{X})$  to be given by a realizer  $f : \mathbb{N}^\mathbb{N} \rightarrow \{0, 1\}$  of a finite mindchange computation. We consider the positions where a mindchange happens, i.e. those  $w \in \mathbb{N}^*$  which if read by  $f$  will cause a mindchange to happen before reading any more of the input. W.l.o.g. we may assume that the realizer makes at most one mindchange at a given position  $w \in \mathbb{N}^*$ , and the realizer initially outputs 0 before reading any of the input.

Let  $W \subseteq \mathbb{N}^*$  be the set of mindchange positions. To simplify the following, we will view  $\varepsilon$  (the empty string in  $\mathbb{N}^*$ ) as being an element of  $W$  (this assumption can be justified formally by viewing the initial output of 0 as being a mindchange from “undefined” to 0). Note that  $W$  is decidable by simply observing the computation of  $f$ . If we denote the prefix relation on  $\mathbb{N}^*$  by  $\sqsubseteq$ , we see that there are no infinite strictly ascending sequences in  $W$  with respect to  $\sqsubseteq$ , since any such sequence would correspond to an input that induces infinitely many mindchanges. It follows that  $(W, \preceq)$  is a computable total well-order with maximal element  $\varepsilon$ , where  $\preceq$  is the (restriction of the) Kleene-Brouwer order and defined as  $v \preceq w$  if and only

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<sup>4</sup> This is based on an adaption of the proof of the computable Hausdorff-Kuratowski theorem in [28].

if (i)  $w \sqsubseteq v$ , or (ii)  $v(n) < w(n)$ , where  $n$  is the least position where  $v$  and  $w$  are both defined and disagree.

We first note that  $\min : \subseteq \Delta_2^0(W) \rightarrow W$  is  $\nabla$ -computable, where  $\min$  is the function mapping each non-empty  $S \in \Delta_2^0(W)$  to the  $\preceq$ -minimal element of  $S$ . A realizer for  $\min$  on input  $S$  can test in parallel whether each element of  $W$  is in  $S$ , and output as a guess the  $\preceq$ -minimal element which it currently believes to be in  $S$ . Since  $\preceq$  is a well-order and it only takes finitely many mindchanges to determine whether or not a given element is in  $S$ , this computation is guaranteed to converge to the correct answer.

For each  $w \in W$ , define  $U_w := \bigcup_{v \in W, v \preceq w} \delta_X[v\mathbb{N}^\mathbb{N}]$ , which is an effectively open subset of  $X$  and a uniform definition because  $\preceq$  is decidable. Next, let  $\mathbf{1} = \{*\}$  be the totally represented space with a single point, and define  $h: W \rightarrow (\Delta_2^0(W) + \mathbf{1})$  as  $h(w) = *$  if  $U_w = X$  and  $h(w) = \{v \in W \mid U_v \subsetneq U_w\}$ , otherwise. The computability of the mapping  $w \mapsto U_w$  and the assumption that  $X$  is  $\nabla$ -computably Noetherian implies that it is  $\nabla$ -decidable whether  $U_w = X$ , and also that the characteristic function of the set  $\{v \in W \mid U_v \subsetneq U_w\}$  is  $\nabla$ -computable given  $w \in W$ . It follows that  $h$  is well-defined and  $\nabla$ -computable.

We construct a finite sequence  $v_0 \prec \dots \prec v_k$  in  $W$  by defining  $v_0 = \min(W)$  and  $v_{n+1} = \min(h(v_n))$  whenever  $h(v_n) \neq *$ . This sequence is necessarily finite because the  $U_{v_n}$  form a strictly increasing sequence of open sets and  $X$  is Noetherian. Note that the last element  $v_k$  in the sequence satisfies  $h(v_k) = *$ . It follows that the sequence  $\langle v_0, \dots, v_k \rangle \in W^*$  can be  $\nabla$ -computed from the realizer  $f$  because it only involves a finite composition of  $\nabla$ -computable functions, and it can be  $\nabla$ -decided when the sequence terminates.

Define  $\eta: W \rightarrow \{0, 1\}$  to be the computable function mapping each  $w \in W$  to the output of the realizer  $f$  after the mindchange upon reading  $w$  (thus  $\eta(\varepsilon) = 0$ ). For  $n \leq k$  define  $V_n := U_{v_n} \setminus \bigcup_{m < n} U_{v_m}$ . We claim that  $A = \bigcup\{V_n \mid 0 \leq n \leq k \ \& \ \eta(v_n) = 1\}$ , from which it will follow that we can  $\nabla$ -compute a name for  $A \in \mathfrak{C}(X)$  from the realizer  $f$ .

Fix  $x \in X$ , and let  $w \in W$  be  $\preceq$ -minimal such that  $x \in \delta_X[w\mathbb{N}^\mathbb{N}]$ . It follows that  $x \in A$  if and only if  $\eta(w) = 1$ , because  $w$  is a prefix of some name  $p$  for  $x$ , and the  $\preceq$ -minimality of  $w$  implies that the realizer  $f$  does not make any additional mindchanges on input  $p$  after reading  $w$ . Next, let  $n \in \{0, \dots, k\}$  be the least number satisfying  $x \in V_n$ . It is clear that  $w \preceq v_n$ . Conversely, if  $n = 0$  then  $v_n = v_0 \preceq w$  by the  $\preceq$ -minimality of  $v_0$ . If  $n > 0$ , then  $w \not\preceq v_{n-1}$  hence  $x$  is a witness to  $U_{v_{n-1}} \subsetneq U_w$ , which implies  $v_n = h(v_{n-1}) \preceq w$ . Thus  $w = v_n$ , and it follows that  $x \in A$  if and only if  $x \in \bigcup\{V_n \mid 0 \leq n \leq k \ \& \ \eta(v_n) = 1\}$ , which completes the proof.  $\blacktriangleleft$

► **Proposition 41.** Let  $X$  be  $\nabla$ -computably Noetherian. Then  $\text{isFull} : \mathfrak{C}(X) \rightarrow \mathbf{2}^\nabla$  is computable.

**Proof.** It is well-known that the sets in  $\mathfrak{C}(X)$  have a normal form  $A = (U_0 \setminus V_0) \cup \dots \cup (U_n \setminus V_n)$ , and this is obtainable uniformly. Now  $A = X$  iff  $\forall I \subseteq \{0, \dots, n\} \left( \bigcap_{j \notin I} V_j \right) \subseteq \left( \bigcup_{i \in I} U_i \right)$ . To see this, first note that the special case  $I = \{0, \dots, n\}$  yields  $X = \bigcup_{i \in I} U_i$ . Now consider for each  $x \in X$  the statement for  $I = \{i \mid x \notin V_i\}$ .

In a  $\nabla$ -computably Noetherian space, we can compute  $\left( \bigcap_{j \notin I} V_j \right)$  as a compact set, and decide its inclusion in  $\left( \bigcup_{i \in I} U_i \right)$  with finitely many mindchanges. Doing this for the finitely many choices of  $I$  is unproblematic, thus yielding the claim.  $\blacktriangleleft$

► **Proposition 42.** Let  $X$  admit a partition  $(A_n)_{n \in \mathbb{N}}$  into non-empty  $\Delta_2^0$ -sets. Then  $X$  is not  $\nabla$ -compact.

**Proof.** Given some  $(t_i)_{i \in \mathbb{N}} \in (\mathbf{2}^\nabla)^\mathbb{N}$ , we can compute the set  $A := \{x \in \mathbf{X} \mid \exists n \in \mathbb{N} x \in A_n \wedge t_n = 1\} \in \Delta_2^0(\mathbf{X})$ . If  $\mathbf{X}$  were  $\nabla$ -compact, then applying  $\text{isFull} : \Delta_2^0(\mathbf{X}) \rightarrow \mathbb{S}^\nabla \cong \mathbf{2}^\nabla$  to  $A$  would yield a computable realizer of  $\bigwedge : (\mathbf{2}^\nabla)^\mathbb{N} \rightarrow \mathbf{2}^\nabla$ .  $\blacktriangleleft$

**Proof of Theorem 39.** By combining Lemma 40 and Proposition 41, we see that for a  $\nabla$ -computably Noetherian quasi-Polish space  $\mathbf{X}$  the map  $\text{isFull} : \Delta_2^0(\mathbf{X}) \rightarrow \mathbb{S}^\nabla$  is computable, i.e. it is  $\nabla$ -compact. Conversely, if  $\mathbf{X}$  is not Noetherian, then by Lemma 11 there is a countably-infinite  $\Delta_2^0$ -partition of  $\mathbf{X}$ , so by Proposition 42, it cannot be  $\nabla$ -compact.  $\blacktriangleleft$

The significance of  $\nabla$ -compactness and Theorem 39 lies in the following proposition that supplies the desired quantifier-elimination result. The proof is a straight-forward adaption of the corresponding result for compact spaces and open predicates from [29] (recalled here as Propositions 20,21), which in turn has [11] and [24] as intellectual predecessors. Note that as  $\neg : \mathbb{S}^\nabla \rightarrow \mathbb{S}^\nabla$  is computable, it follows that  $\nabla$ -compactness and  $\nabla$ -overtness coincide:

► **Proposition 43.** The following are equivalent for a represented space  $\mathbf{X}$ :

1.  $\mathbf{X}$  is  $\nabla$ -compact.
2. For any represented space  $\mathbf{Y}$ , the map  $\forall : \Delta_2^0(\mathbf{X} \times \mathbf{Y}) \rightarrow \Delta_2^0(\mathbf{Y})$  mapping  $R$  to  $\{y \in \mathbf{Y} \mid \forall x \in \mathbf{X} (x, y) \in R\}$  is computable.
3. For any represented space  $\mathbf{Y}$ , the map  $\exists : \Delta_2^0(\mathbf{X} \times \mathbf{Y}) \rightarrow \Delta_2^0(\mathbf{Y})$  mapping  $R$  to  $\{y \in \mathbf{Y} \mid \exists x \in \mathbf{X} (x, y) \in R\}$  is computable.

► **Corollary 44.** A formula built from  $\Delta_2^0$ -predicates, boolean operations and universal and existential quantification over Noetherian quasi-Polish spaces defines itself a  $\Delta_2^0$ -predicate.

► **Corollary 45.** Let  $\mathbf{X} = \mathbf{X}_0 \times \dots \times \mathbf{X}_n$  be a Noetherian Quasi-Polish space. If a subset  $U \subseteq \mathbf{X}_0$  is definable using a finite expression involving open predicates in  $\mathbf{X}$ , boolean operations, and existential and universal quantification, then  $U$  is definable using a finite expression involving open predicates in  $\mathbf{X}_0$  and boolean operations.

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## 16:16 Noetherian Quasi-Polish Spaces

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