Models of Type Theory Based on Moore Paths

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Abstract

This paper introduces a new family of models of intensional Martin-Löf type theory. We use constructive ordered algebra in toposes. Identity types in the models are given by a notion of Moore path. By considering a particular gros topos, we show that there is such a model that is *non-truncated*, i.e. contains non-trivial structure at all dimensions. In other words, in this model a type in a nested sequence of identity types can contain more than one element, no matter how great the degree of nesting. Although inspired by existing non-truncated models of type theory based on simplicial and on cubical sets, the notion of model presented here is notable for avoiding any form of Kan filling condition in the semantics of types.

1998 ACM Subject Classification F.4.1 Mathematical Logic, F.3.2 Semantics of Programming Languages

Keywords and phrases dependent type theory, homotopy theory, Moore path, topos

Digital Object Identifier 10.4230/LIPIcs.FSCD.2017.28

1 Introduction

Homotopy Type Theory [30] has re-invigorated the study of the intensional version of Martin-Löf type theory [21]. On the one hand, the language of type theory helps to express synthetic constructions and arguments in homotopy theory and higher-dimensional category theory. On the other hand, the geometric and algebraic insights of those highly developed branches of mathematics shed new light on logical and type-theoretic notions. One might say that the familiar *propositions-as-types* analogy has been extended to *propositions-as-types-as-spaces*. In particular, under this analogy there is a *path-oriented* view of intensional (i.e. proof-relevant) equality: proofs of equality of two elements of a type $x, y : A$ correspond to elements of a Martin-Löf identity type $\mathtt{Id}_A\, x\, y$ and behave analogously to paths between two points x, y in a space A. The complicated internal structure of intensional identity types relates to the homotopy classes of path spaces. To make the analogy precise and to exploit it, it helps to have a wide range of models of intensional type theory that embody this path-oriented view of equality in some way. This paper introduces a new family of such models, constructed from *Moore paths* [24] in toposes.

Let $\mathbb{R}_+ = \{r \in \mathbb{R} \mid r \geq 0\}$ be the real half-line with the usual topology. Classically, a Moore path between points x and y in a topological space X is a pair $p = (f, r)$ where $r \in \mathbb{R}_+$ and $f : \mathbb{R}_+ \to X$ is a continuous function with $f\,0 = x$ and $f\,r' = y$ for all $r' \geq r$. We will write $x \sim y$ for the set of Moore paths from x to y, with X understood from the context. Clearly there is a Moore path from x to y iff there is a conventional path, that is, a continuous function $f : [0, 1] \to X$ with $f\,0 = x$ and $f\,1 = y$. The advantage of Moore paths is that they admit degeneracy and composition operations that are unitary and associative

2nd International Conference on Formal Structures for Computation and Deduction (FSCD 2017).
Editor: Dale Miller; Article No. 28; pp. 28:1–28:16
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

up to equality; whereas for conventional paths these identities only hold up to homotopy. Specifically, given $p = (f, r) \in x \sim y$ and $q = (g, s) \in y \sim z$ we get

$$\mathtt{idp}\, x :\equiv (\lambda\_ \to x, 0) \in x \sim x \qquad\qquad q \bullet p :\equiv (g \bullet^r f, r + s) \in x \sim z \tag{1}$$

where $(g \bullet^r f)\, t$ is equal to $f\, t$ if $t \leq r$ and otherwise is equal to $g(t - r)$. These definitions satisfy $p \bullet \mathtt{idp}\, x = p = \mathtt{idp}\, y \bullet p$ and $r \bullet (q \bullet p) = (r \bullet q) \bullet p$.

We abstract from $\mathbb{R}$ some simple order-algebraic structure [4, chapter VI] sufficient for definition (1) to work in a constructive algebraic setting, rather than a classical topological one; initially the structure of an *ordered abelian group* in some topos suffices and then we extend that to an *ordered commutative ring* to make the models satisfy function extensionality. We use this structure in a topos to develop a model of intensional Martin-Löf type theory with: identity types given by Moore paths, Σ-types, Π-types satisfying function extensionality and inductive types (we just consider disjoint unions and W-types). The existence and properties of type-theoretic universes in these models is left for future work. By considering a particular *gros* topos [10] we get a *non-truncated* instance of our model construction, in other words one where iterated identity types $\mathtt{Id}_A, \mathtt{Id}_{\mathtt{Id}_A}, \mathtt{Id}_{\mathtt{Id}_{\mathtt{Id}_A}}, \ldots$ can be non-trivial to any depth of iteration.

The observation that the strictly associative and unitary nature of composition of Moore paths aids in the interpretation of Martin-Löf identity types is not new; see for example [32, sections 5.1 and 5.5]. However, existing non-truncated models of type theory typically make use of some form of Kan filling condition [15] to cut down to a class of *fibrant* families of types with respect to which path types behave as identity types. Perhaps the main contribution of this paper is to show that one can avoid any form of Kan filling and still get a non-truncated model of intensional Martin-Löf type theory. Instead we use a simple notion of fibrant family phrased just in terms of the usual operation of transporting elements along equality proofs. As a consequence every type, regarded as a family over the terminal type, is fibrant in our setting. In particular, this means that the interval is a first-class type in our models, something which is not true for existing path-oriented models, such as the classical simplicial [17] and constructive cubical sets [3, 6] models; and constructing universes in our setting will not need proofs of their fibrancy (something which is quite tricky for cubical sets). These simplifications, together with an algebraic calculus of "bounded abstractions" for Moore paths that we present in this paper suggest that there may be a useful extension of Martin-Löf type theory with path types which is analogous to, but simpler than, Cubical Type Theory [6]. This is something we plan to look at in future.

Informal type theory

The new models of type theory we present are given in terms of *categories with families* (CwF) [8, 11]. Specifically, we start with an arbitrary topos $\mathcal{E}$, to which can be associated a CwF, for example as in [18, 2]. We write $\mathcal{E}(\Gamma)$ for the set of families indexed by an object $\Gamma \in \mathcal{E}$ and $\mathcal{E}(\Gamma \vdash A)$ for the set of elements of a family $A \in \mathcal{E}(\Gamma)$. One can make $\mathcal{E}(\Gamma)$ into a category whose morphisms between two families $A, B \in \mathcal{E}(\Gamma)$ are elements in $\mathcal{E}(\Gamma.A \vdash B)$, where $\Gamma.A$ is the comprehension object associated with A. The category $\mathcal{E}(\Gamma)$ is equivalent to the slice category $\mathcal{E}/\Gamma$, the equivalence being given on objects by sending families $A \in \mathcal{E}(\Gamma)$ to corresponding projection morphisms $\Gamma.A \to \Gamma$.

We then construct a new CwF by considering families in $\mathcal{E}$ equipped with certain extra structure (the *transport-along-paths* structure of section 3); the elements of a family in the new CwF are just those of the underlying family in $\mathcal{E}$. One could describe this construction using the language of category theory. Instead, as in [26] we find it clearer to express the construction using an internal language for the CwF associated with $\mathcal{E}$ – a combination of higher-order predicate logic and extensional type theory, with an impredicative universe of propositions given by the subobject classifier in $\mathcal{E}$; see [20]. This use of internal language allows us to give an appealingly simple description of the type constructs in the new CwF. In the text we use this language informally (analogously to the way that [30] develops Homotopy Type Theory); in particular the typing contexts of the judgements in the formal version, such as $[x_0 : A_0, x_1 : A_1(x_0), x_2 : A_2(x_0, x_1)]$, become part of the running text in phrases like "given $x_0 : A_0$, $x_1 : A_1(x_0)$ and $x_2 : A_2(x_0, x_1)$, then…". The arguments we give in sections 2–5 are all constructively valid and in fact do not require the impredicative aspects of topos theory; indeed we have used Agda [1] (with uniqueness of identity proofs and postulates for quotient types) as a tool to develop and to formalise the material presented in those sections. The specific model presented in section 6 uses topological spaces within classical mathematics.

2 Moore Paths in a Topos

Let $\mathcal{E}$ be a topos [19, 14] (which we always assume has a natural number object). A *total order* on an object $\mathtt{R} \in \mathcal{E}$ is given by a subobject $\_ \leqslant \_ \rightarrowtail \mathtt{R} \times \mathtt{R}$ which is not only reflexive, transitive and anti-symmetric, but also satisfies

$$(\forall i, j : \mathtt{R})\ i \leqslant j \ \vee\ j \leqslant i\,. \tag{2}$$

An *ordered abelian group* [4, chapter VI] object in $\mathcal{E}$ is given by $\mathtt{R} \in \mathcal{E}$ plus morphisms $\mathtt{0} : 1 \to \mathtt{R}$, $\mathtt{-} : \mathtt{R} \to \mathtt{R}$ and $\_ + \_ : \mathtt{R} \times \mathtt{R} \to \mathtt{R}$ satisfying the usual abelian group axioms, plus a total order that is respected by addition:

$$(\forall i, j, k : \mathtt{R})\ i \leqslant j \ \Rightarrow\ k + i \leqslant k + j\,. \tag{3}$$

We develop properties of Moore paths in $\mathcal{E}$ with respect to such an object $\mathtt{R}$.

Note that since $\mathcal{E}$ may not be Boolean, we do not necessarily have the trichotomy property $(\forall i, j : \mathtt{R})\ i < j \ \vee\ i = j \ \vee\ j < i$ for the associated relation $i < j \ :\equiv \neg(j \leqslant i)$. So when defining functions by cases using (2) we have to verify that the clauses for $i \leqslant j$ and for $j \leqslant i$ agree on the overlap, where $i = j$ by antisymmetry. For example, the *half-line*

$$\mathtt{R_+} :\equiv \{i : \mathtt{R} \mid \mathtt{0} \leqslant i\} \tag{4}$$

associated with $\mathtt{R}$ has binary operations $\mathtt{R_+} \times \mathtt{R_+} \to \mathtt{R_+}$ of *truncated subtraction* $\_ \dot{-} \_$ and *minimum* $\mathtt{min}$ well-defined as follows (where, as usual, we write $i + (\mathtt{-}j)$ as $i - j$):

$$(\forall i, j : \mathtt{R_+}) \begin{array}{lllll} i \leqslant j & \Rightarrow & i \dot{-} j = \mathtt{0} & \wedge & \mathtt{min}(i, j) = i \\ j \leqslant i & \Rightarrow & i \dot{-} j = i - j & \wedge & \mathtt{min}(i, j) = j \end{array} \tag{5}$$

▶ **Definition 2.1** (Moore path objects)**.** For each object $\Gamma \in \mathcal{E}$, the family $(x \sim y \mid x, y : \Gamma) \in \mathcal{E}(\Gamma \times \Gamma)$ of *Moore path objects* is defined by:

$$x \sim y \ :\equiv\ \{(f, i) : (\mathtt{R_+} \to \Gamma) \times \mathtt{R_+} \mid f\,\mathtt{0} = x \ \wedge\ (\forall j \geqslant i)\ f\,j = y\}\,. \tag{6}$$

We use the following notation:

$$\begin{array}{ll} |\_| : (x \sim y) \to \mathtt{R_+} & \_ \mathbin{\mathtt{@}} \_ : (x \sim y) \to \mathtt{R_+} \to \Gamma \\ |(f, i)| = i & (f, i) \mathbin{\mathtt{@}} j = f\,j \end{array} \tag{7}$$

Following [5, Section 2], we call $|p|$ the *shape* of the path $p : x \sim y$. Thus if $p : x \sim y$, then $p\,0 = x$ and $p \mathbin{@} |p| = y$. *From now on we will just write* $p \mathbin{@} i$ *as* $p\,i$.

Given $\gamma : \Gamma \to \Delta$ in $\mathcal{E}$, there is a congruence operation mapping paths in Γ to those in Δ:

$$\gamma \text{ ' } : (x \sim y) \to (\gamma\, x \sim \gamma\, y) \qquad\qquad \gamma \text{ ' } (f, i) :\equiv (\gamma \circ f, i) \tag{8}$$

▶ **Lemma 2.2** (degenerate and composite paths). *For each* $\Gamma \in \mathcal{E}$*, given* $x, y, z : \Gamma$*,* $p : x \sim y$ *and* $q : y \sim z$*, there are* degenerate paths $\mathtt{idp}\, x : x \sim x$ *and* composite paths $q \bullet p : x \sim z$ *well-defined by the requirements*

$$|\mathtt{idp}\, x| = 0 \quad \wedge \quad (\forall i : \mathtt{R}_{+})\, (\mathtt{idp}\, x)\, i = x$$

$$|q \bullet p| = |p| + |q| \quad \wedge \quad (\forall i : \mathtt{R}_{+}) \begin{array}{lcl} i \leqslant |p| & \Rightarrow & (q \bullet p)\, i = p\, i \\ |p| \leqslant i & \Rightarrow & (q \bullet p)\, i = q\,(i - |p|) \end{array}$$

and satisfying $p \bullet (\mathtt{idp}\, x) = p = (\mathtt{idp}\, y) \bullet p$*,* $(r \bullet q) \bullet p = r \bullet (q \bullet p)$*,* $\gamma \text{ ' } (\mathtt{idp}\, x) = \mathtt{idp}(\gamma\, x)$ *and* $\gamma \text{ ' } (q \bullet p) = (\gamma \text{ ' } q) \bullet (\gamma \text{ ' } p)$*.*

Proof. One just has to check that the usual definitions for classical Moore paths are constructive and only make use of the ordered abelian group properties of the reals. ◀

▶ **Lemma 2.3** (path reversal). *For each* $\Gamma \in \mathcal{E}$*, given* $x, y : \Gamma$ *and* $p : x \sim y$*, there is a* reversed path $\mathtt{rev}\, p : y \sim x$ *well-defined by the requirements* $|\mathtt{rev}\, p| = |p|$ *and* $(\forall i : \mathtt{R}_{+})\, (\mathtt{rev}\, p)\, i = p\,(|p| \dot{-} i)$ *(where* $\dot{-}$ *is as in* (5)*) and satisfying* $\mathtt{rev}(\mathtt{idp}\, x) = \mathtt{idp}\, x$*,* $\mathtt{rev}(q \bullet p) = (\mathtt{rev}\, p) \bullet (\mathtt{rev}\, q)$*,* $\mathtt{rev}(\mathtt{rev}\, p) = p$ *and* $\gamma \text{ ' } (\mathtt{rev}\, p) = \mathtt{rev}(\gamma \text{ ' } p)$*.*

Proof. As for the previous lemma, this follows straightforwardly from the axioms of ordered abelian groups within constructive logic. ◀

Although the definition of $\mathtt{rev}\, p$ is standard, the above equational properties are not often mentioned in the literature. However, they are crucial for the construction of identity types in section 3 to work. What usually gets a mention is the fact that up to homotopy $\mathtt{rev}\, p$ is a two-sided inverse for p with respect to $\bullet$. Paths $(\mathtt{rev}\, p) \bullet p \sim \mathtt{idp}\, x$ and $p \bullet (\mathtt{rev}\, p) \sim \mathtt{idp}\, y$ can be constructed using the path contraction operation given below; we do not bother to do that, because they are also a consequence of the *path induction* [30, 1.12.1] property of identity types that follows from Theorem 3.9.

▶ **Definition 2.4** (bounded abstractions). The following binding syntax is very convenient for describing Moore paths. For each $\Gamma \in \mathcal{E}$, if $\lambda i \to \varphi(i)$ describes a function in $\mathtt{R}_{+} \to \Gamma$, then for each $j : \mathtt{R}_{+}$ using the $\mathtt{min}$ function (5) we get a path $\langle i \leqslant j \rangle \varphi(i) : \varphi(0) \sim \varphi(j)$ in Γ by defining:

$$\langle i \leqslant j \rangle \varphi(i) :\equiv (\lambda i \to \varphi(\mathtt{min}(i, j)) \;, j) \tag{9}$$

(i is bound in the above expression). Note that

$$\gamma \text{ ' } \langle i \leqslant j \rangle \varphi(i) = \langle i \leqslant j \rangle \gamma(\varphi(i))\,, \tag{10}$$

$$\langle i \leqslant 0 \rangle \varphi(i) = \mathtt{idp}(\varphi(0))\,, \tag{11}$$

$$\langle i \leqslant |p| \rangle (p\, i) = p\,. \tag{12}$$

▶ **Lemma 2.5** (path contraction). *Given* $\Gamma \in \mathcal{E}$*, for any path* $p : x \sim y$ *in* Γ *and* $i : \mathtt{R}_{+}$*, there is a path* $p_{\leqslant i} : x \sim p\, i$ *satisfying* $p_{\leqslant 0} = \mathtt{idp}\, x$ *and* $(\forall i \geqslant |p|)\; p_{\leqslant i} = p$*.*

Proof. Using the bounded abstraction notation and the `min` function (5), we define

$$p_{\leqslant i} :\equiv \langle j \leqslant \mathtt{min}(|\,p\,|\,, i)\rangle(p\,j)\,. \tag{13}$$

Since $p\,(\mathtt{min}(|\,p\,|\,, i)) = p\,i$, this does give a path $x \sim p\,i$; and it has the required properties by (11) and (12). ◀

Using $p_{\leqslant i}$ we get for each $x : \Gamma$ that $\sum_{y:\Gamma} x \sim y$ is path-contractible [30, section 3.11] with centre $(x, \mathtt{idp}\,x)$, since for each $y : \Gamma$ and $p : x \sim y$ we have a path $\langle i \leqslant |\,p\,|\rangle(p\,i, p_{\leqslant i})$ in $(x, \mathtt{idp}\,x) \sim (y, p)$. This is part of the more general fact that Moore paths model identity types, which we show in the next section.

3 Transport along Paths

In this section we continue with the assumptions of the previous one: $\mathcal{E}$ is a topos (with associated CwF) containing an ordered abelian group object $\mathtt{R}$. We want objects of Moore paths with respect to $\mathtt{R}$ (Definition 2.1) to give a model of identity types, as well as other type formers. Recall that in Martin-Löf type theory elements of identity types $\mathtt{Id}_\Gamma\, x\, y$ give rise to transport functions $A\,x \to A\,y$ between members of a family of types $(A\,x \mid x : \Gamma)$ over a type Γ; see for example [30, section 2.3]. Therefore, for each object $\Gamma \in \mathcal{E}$, we should restrict attention to families $A \in \mathcal{E}(\Gamma)$ that come equipped at least with some sort of transport operation taking a path $p : x \sim y$ in Γ and an element $a : A\,x$ to an element $p_*a : A\,y$. This leads to the following definition.

▶ **Definition 3.1** (fibrations). Given an object $\Gamma \in \mathcal{E}$, a *transport-along-paths* (tap) structure for a family $A \in \mathcal{E}(\Gamma)$ is a $(\Gamma \times \Gamma)$-indexed family of morphisms $((\_)_* : x \sim y \to (A\,x \to A\,y) \mid x, y : \Gamma)$ satisfying for all $x : \Gamma$ and $a : A\,x$

$$(\mathtt{idp}\,x)_*a = a\,. \tag{14}$$

We write $\mathbf{Fib}(\Gamma)$ for the families over Γ equipped with a tap structure and call them *fibrations*. They are stable under re-indexing: given $\gamma : \Delta \to \Gamma$ in $\mathcal{E}$ and $A \in \mathbf{Fib}(\Gamma)$, then $A\,\gamma :\equiv (A(\gamma\,x) \mid x : \Delta) \in \mathcal{E}(\Delta)$ has a tap structure via the congruence operation (8), taking the transport of $a : (A\,\gamma)\,x = A(\gamma\,x)$ along $p : x \sim y$ to be $(\gamma\,\text{'}\,p)_*a : A(\gamma\,y)$. This re-indexing of tap structure respects composition and identities in $\mathcal{E}$. So $\mathbf{Fib}$ inherits the structure of a CwF from that of $\mathcal{E}$, with the set of elements of a fibration being the elements of the underlying family in $\mathcal{E}$, that is $\mathbf{Fib}(\Gamma \vdash A) = \mathcal{E}(\Gamma \vdash A)$.

▶ Remark 3.2. In the above definition we do not require the transport action $\_{}_*\_$ to be strictly associative, because that property is not needed to prove that fibrations give a model of type theory. However, we always have associativity up to homotopy, i.e. there is a path $(q \bullet p)_*a \sim q_*(p_*a)$. This is a consequence of the fact that, with respect to the above notion of fibration, Moore paths model identity types (see Theorem 3.9 below) and hence support path induction [30, section 2.9]). This allows one to construct a path $(q \bullet p)_*a \sim q_*(p_*a)$ for any p from the case when $p = \mathtt{idp}\,x$, where one has the degenerate path for $(q \bullet (\mathtt{idp}\,x))_*a = q_*a = q_*((\mathtt{idp}\,x)_*a)$.

We show that the CwF $\mathbf{Fib}$ inherits some type structure from that of $\mathcal{E}$. To begin with, note that every object $\Gamma \in \mathcal{E}$, regarded as a family over the terminal object 1, can be made into a fibration via the trivial transport action $p_*a = a$. So in particular the half-line $\mathtt{R}_+$ is a fibration over 1. Also, it is easy to see that the tap structure on families $A, B \in \mathbf{Fib}(\Gamma)$ lifts to one on the family of disjoint unions $(A\,x + B\,x \mid x : \Gamma)$ (stably under re-indexing) so that the CwF **Fib** supports disjoint union types. To describe Σ- and Π-types in **Fib** we have to lift paths in Γ to paths in comprehension objects $\Gamma.A$ of fibrations $A \in \mathbf{Fib}(\Gamma)$:

▶ **Lemma 3.3** (path lifting). *Given $\Gamma \in \mathcal{E}$ and $A \in \mathbf{Fib}(\Gamma)$, for each path $p : x \sim y$ in Γ and each $a : A\,x$, there is a path $\mathtt{lift}(p, a) : (x, a) \sim (y, p_* a)$ in $\Gamma.A$ satisfying*

$$\mathtt{lift}(\mathtt{idp}\,x, a) = \mathtt{idp}(x, a) \tag{15}$$

and stable under re-indexing along morphisms $\Delta \to \Gamma$ in $\mathcal{E}$.

Proof. We can use path contraction (Lemma 2.5) to express path lifting using the bounded abstraction notation from Definition 2.4:

$$\mathtt{lift}(p, a) :\equiv \langle i \leqslant |p|\rangle(p\,i, (p_{\leqslant i})_* a)\,. \tag{16}$$

Since the path contraction operation satisfies $p_{\leqslant 0} = \mathtt{idp}\,x$ and $p_{\leqslant |p|} = p$, this does indeed give a path in $(x, a) \sim (y, p_* a)$. The desired properties of `lift` follow from corresponding properties of path contraction. ◀

▶ **Theorem 3.4** (Σ- and Π-types). *Given $\Gamma \in \mathcal{E}$, $A \in \mathbf{Fib}(\Gamma)$ and $B \in \mathbf{Fib}(\Gamma.A)$, the families $\Sigma\,A\,B :\equiv (\sum_{a:A\,x} B(x, a) \mid x : \Gamma)$ and $\Pi\,A\,B :\equiv (\prod_{a:A\,x} B(x, a) \mid x : \Gamma)$ in $\mathcal{E}(\Gamma)$ have tap structures that are stable under re-indexing along morphisms $\Delta \to \Gamma$ in $\mathcal{E}$. Hence the CwF* **Fib** *supports Σ-and Π-types [11, Definitions 3.15 and 3.18].*

Proof. Given a path $p : x \sim y$ in Γ, we get functions $\Sigma\,A\,B\,x \to \Sigma\,A\,B\,y$ and $\Pi\,A\,B\,x \to \Pi\,A\,B\,y$ using path lifting and (in the second case) path reversal:

$$p_*(a, b) :\equiv (p_* a, \mathtt{lift}(p, a)_* b) \in \Sigma\,A\,B\,y \qquad \text{where } (a, b) : \Sigma\,A\,B\,x \tag{17}$$

$$(p_* f)\,a :\equiv (\mathtt{rev}(\mathtt{lift}(\mathtt{rev}\,p, a)))_* f((\mathtt{rev}\,p)_* a) \in B(y, a) \qquad \begin{array}{l}\text{where } f : \Pi\,A\,B\,x\\ \text{and } a : A\,y\end{array} \tag{18}$$

The properties of path lifting given in Lemma 3.3 suffice to show that (17) inherits the properties of a tap structure from those for A and B and is stable under re-indexing. The same is true for (18) except that one also has to use the properties of path reversal given in Lemma 2.3. ◀

Since we only consider toposes with a natural number object, the CwF associated with $\mathcal{E}$ can interpret types of well-founded trees (W-types) [30, section 5.3]; see [23, Propositions 3.6 and 3.8]. We write $\mathtt{W}_{x:A}\,B\,x$ for the object of well-founded trees determined by a family $B \in \mathcal{E}(A)$, with constructor $\mathtt{sup} : \sum_{y:A}(B\,y \to \mathtt{W}_{x:A}\,B\,x) \to \mathtt{W}_{x:A}\,B\,x$.

▶ **Theorem 3.5** (W-types). *Given $A \in \mathbf{Fib}(\Gamma)$ and $B \in \mathbf{Fib}(\Gamma.A)$, the family*

$$W\,A\,B :\equiv (\mathtt{W}_{a:A\,x}\,B(x, a) \mid x : \Gamma) \in \mathcal{E}(\Gamma)$$

has a tap structure that is stable under re-indexing along morphisms $\Delta \to \Gamma$ in $\mathcal{E}$. Therefore the CwF **Fib** *supports W-types.*

Proof. Given a path $p : x \sim y$ in Γ, we get a function $W\,A\,B\,x \to W\,A\,B\,y$ via the following well-founded recursion equation:

$$p_*\,\mathtt{sup}(a\,,\,f) = \mathtt{sup}\,(p_* a\,,\,\lambda b \to p_* f((\mathtt{rev}(\mathtt{lift}(p, a)))_* b)) \qquad \begin{array}{l}\text{where } a : A\,x \text{ and}\\ f : B(x, a) \to W\,A\,B\,x\end{array} \tag{19}$$

Well-founded inductions using the properties of reversal and lifting given in Lemmas 2.3 and 3.3 suffice to show that this inherits the properties of a tap structure from those for A and B and is stable under re-indexing. ◀

So far we have lifted some type structure from the CwF associated with $\mathcal{E}$ to the CwF **Fib**. Since $\mathcal{E}$ is a topos, it certainly has *extensional* identity types [22], inhabitation of which coincides with judgemental equality, and those could be lifted to **Fib**. However, we wish to show that Moore path objects become *intensional* identity types in **Fib**. We will use Hofmann's version [11] of the structure in a CwF needed to model such types. To do so, let us fix some notation for a CwF $\mathbf{C}$. Re-indexing of a family $A \in \mathbf{C}(\Gamma)$ and an element $\alpha \in \mathbf{C}(\Gamma \vdash A)$ along a morphism $\gamma : \Delta \to \Gamma$ will just be denoted by $A\,\gamma \in \mathbf{C}(\Delta)$ and $\alpha\,\gamma \in \mathbf{C}(\Delta \vdash A\,\gamma)$; and given an element $\beta \in \mathbf{C}(\Delta \vdash A\,\gamma)$, then $\langle \gamma\,,\beta\rangle$ denotes the unique morphism $\Delta \to \Gamma.A$ whose composition with $\mathtt{fst} : \Gamma.A \to \Gamma$ is γ and whose re-indexing of the generic element $\mathtt{snd} \in \mathbf{C}(\Gamma.A \vdash A\,\mathtt{fst})$ is β.

▶ **Definition 3.6.** Following Hofmann [11, Definition 3.19], we say that a CwF $\mathbf{C}$ *supports the interpretation of intensional identity types* if for each object $\Gamma \in \mathbf{C}$ and each family $A \in \mathbf{C}(\Gamma)$ the following data is given and is stable under re-indexing along any $\gamma : \Delta \to \Gamma$:

- a family $\mathtt{Id}_A \in \mathbf{C}(\Gamma.A.A\,\mathtt{fst})$,
- a morphism $\mathtt{Refl}_A : \Gamma.A \to \Gamma.A.A\,\mathtt{fst}.\,\mathtt{Id}_A$ such that $\mathtt{fst} \circ \mathtt{Refl}_A$ equals the diagonal morphism $\langle \mathtt{id}\,,\mathtt{snd}\rangle : \Gamma.A \to \Gamma.A.A\,\mathtt{fst}$,
- a function mapping each $B \in \mathbf{C}(\Gamma.A.A\,\mathtt{fst}.\,\mathtt{Id}_A)$ and $\beta \in \mathbf{C}(\Gamma.A \vdash B\,\mathtt{Refl}_A)$ to an element $\mathtt{J}_A\,B\,\beta \in \mathbf{C}(\Gamma.A.A\,\mathtt{fst}.\,\mathtt{Id}_A \vdash B)$ such that the re-indexing $(\mathtt{J}_A\,B\,\beta)\,\mathtt{Refl}_A$ equals β.

Given an object Γ of the topos $\mathcal{E}$ and a family $A \in \mathcal{E}(\Gamma)$, we can use the family of Moore path objects $\_ \sim \_$ (Definition 2.1) to define a family $\mathtt{Id}_A \in \mathcal{E}(\Gamma.A.A\,\mathtt{fst})$ as follows:

$$\mathtt{Id}_A :\equiv (a_1 \sim a_2 \mid ((x, a_1), a_2) : \Gamma.A.A\,\mathtt{fst})\,. \tag{20}$$

We will show that this together with suitable $\mathtt{Refl}$ and $\mathtt{J}$ operations give an instance of Definition 3.6 for the CwF **Fib**. In particular $\mathtt{Id}_A$ has a tap structure when A does. To see this we first need to analyse paths in $\Gamma.A$ in terms of paths in Γ and in the fibres $A\,x$ (for $x : \Gamma$).

▶ **Lemma 3.7.** *Given $\Gamma \in \mathcal{E}$ and $A \in \mathbf{Fib}(\Gamma)$, for each path $p : (x, a) \sim (y, b)$ in $\Gamma.A$, there is a path $\mathtt{snd}(p) : (\mathtt{fst}\,\text{'}\,p)_* a \sim b$ in $A\,y$ satisfying*

$$\mathtt{snd}(\mathtt{idp}(x, a)) = \mathtt{idp}\,a \tag{21}$$

and stable under re-indexing along any $\gamma : \Delta \to \Gamma$ in $\mathcal{E}$.

Proof. We use a reversed version of the path-contraction operation from Lemma 2.5: for each path $q : x \sim y$ and each $i : \mathtt{R_+}$, we define $q_{\geqslant i} : q\,i \sim y$ by

$$q_{\geqslant i} :\equiv \mathtt{rev}\left((\mathtt{rev}\,q)_{\leqslant(|q| \dot{-} i)}\right) \tag{22}$$

where $\dot{-}$ is the truncated subtraction operation from (5). Given $p : (x, a) \sim (y, b)$, then $\mathtt{fst}\,\text{'}\,p : x \sim y$ and hence for each $i : \mathtt{R_+}$ we have $(\mathtt{fst}\,\text{'}\,p)_{\geqslant i} : (\mathtt{fst}\,\text{'}\,p)\,i \sim y$; and since $\mathtt{snd}(p\,i) : A(\mathtt{fst}(p\,i)) = A((\mathtt{fst}\,\text{'}\,p)\,i)$, we get $((\mathtt{fst}\,\text{'}\,p)_{\geqslant i})_*\,\mathtt{snd}(p\,i) : A\,y$. So we can define

$$\mathtt{snd}(p) = \langle i \leqslant |p|\rangle\,(((\mathtt{fst}\,\text{'}\,p)_{\geqslant i})_*\,\mathtt{snd}(p\,i)) \tag{23}$$

to get a path in $(\mathtt{fst}\,\text{'}\,p)_* a \sim b$ in $A\,y$. Property (21) and stability under re-indexing follow from (10) and (11). ◀

▶ **Lemma 3.8.** *Given* $\Gamma \in \mathcal{E}$ *and a fibration* $A \in \mathbf{Fib}(\Gamma)$*, the family* $\mathtt{Id}_A \in \mathcal{E}(\Gamma.A.A\,\mathtt{fst})$ *defined in* (20) *has a tap structure that is stable under re-indexing along morphisms* $\Delta \to \Gamma$ *in* $\mathcal{E}$*.*

Proof. First note that re-indexing the fibration $A \in \mathbf{Fib}(\Gamma)$ along $\mathtt{fst} : \Gamma.A \to \Gamma$ we get $A\,\mathtt{fst} \in \mathbf{Fib}(\Gamma.A)$. Given a path $p : ((x,a_1),a_2) \sim ((y,b_1),b_2)$ in $\Gamma.A.A\,\mathtt{fst}$, we can apply Lemma 3.7 to the paths $\mathtt{fst}\ \text{'}\ p : (x,a_1) \sim (y,b_1)$ and $\langle \mathtt{fst}\circ\mathtt{fst}\,,\mathtt{snd}\rangle\ \text{'}\ p : (x,a_2) \sim (y,b_2)$ in $\Gamma.A$ to get paths $p_1 :\equiv \mathtt{snd}(\mathtt{fst}\ \text{'}\ p) : (\mathtt{fst}\ \text{'}\ (\mathtt{fst}\ \text{'}\ p))_* a_1 \sim b_1$ and $p_2 :\equiv \mathtt{snd}(\langle \mathtt{fst}\circ\mathtt{fst}\,,\mathtt{snd}\rangle\ \text{'}\ p) : (\mathtt{fst}\ \text{'}\ (\mathtt{fst}\ \text{'}\ p))_* a_2 \sim b_2$ in $A\,y$ (using the fact that $\mathtt{fst}\circ\langle \mathtt{fst}\circ\mathtt{fst}\,,\mathtt{snd}\rangle = \mathtt{fst}\circ\mathtt{fst}$). Thus for each path $q : a_1 \sim a_2$ in $A\,x$, we can compose together the paths

$$b_1 \xrightarrow{\mathtt{rev}(p_1)} (\mathtt{fst}\ \text{'}\ (\mathtt{fst}\ \text{'}\ p))_* a_1 \xrightarrow{((\mathtt{fst}\text{'}(\mathtt{fst}\text{'}p))_*)\text{'}q} (\mathtt{fst}\ \text{'}\ (\mathtt{fst}\ \text{'}\ p))_* a_2 \xrightarrow{p_2} b_2$$

in $A\,y$ to get a path $b_1 \sim b_2$. So we get a function $p_* : \mathtt{Id}_A((x,a_1),a_2) \to \mathtt{Id}_A((y,b_1),b_2)$ defined by:

$$p_* q = \mathtt{snd}(\langle \mathtt{fst}\circ\mathtt{fst}\,,\mathtt{snd}\rangle\ \text{'}\ p) \bullet (((\mathtt{fst}\ \text{'}\ (\mathtt{fst}\ \text{'}\ p))_*)\ \text{'}\ q) \bullet \mathtt{rev}(\mathtt{snd}(\mathtt{fst}\ \text{'}\ p))\,. \tag{24}$$

The properties of Moore path reversal (Lemma 2.3) together with Lemma 3.7 suffice to show that this definition inherits the property of a tap structure (14) from the one for A and that it is stable under re-indexing. ◀

▶ **Theorem 3.9** (identity types)**.** *The CwF* **Fib** *of Definition 3.1 supports the interpretation of intensional identity types (Definition 3.6), given by Moore path objects as in* (20)*.*

Proof. In view of Lemma 3.8, it just remains to define the $\mathtt{Refl}$ and $\mathtt{J}$ operations as in Definition 3.6. Given $\Gamma \in \mathcal{E}$ and $A \in \mathbf{Fib}(\Gamma)$, we get $\mathtt{Refl}_A : \Gamma.A \to \Gamma.A.A\,\mathtt{fst}.\,\mathtt{Id}_A$ by defining

$$\mathtt{Refl}_A(x,a) :\equiv (((x,a),a), \mathtt{idp}\,a)\,. \tag{25}$$

Note that $(\mathtt{fst}\circ\mathtt{Refl}_A)(x,a) = ((x,a),a) = \langle \mathtt{id}\,,\mathtt{snd}\rangle(x,a)$, as required.

Given $B \in \mathbf{Fib}(\Gamma.A.A\,\mathtt{fst}.\,\mathtt{Id}_A)$ and $\beta \in \mathcal{E}(\Gamma.A \vdash B\ \mathtt{Refl}_A)$, for each path $p : a_1 \sim a_2$ in $A\,x$ using Lemma 2.5 we have the following path in $\Gamma.A.A\,\mathtt{fst}.\,\mathtt{Id}_A$:

$$\langle i \leqslant |p|\rangle(((x,a_1),p\,i),p_{\leqslant i}) : (((x,a_1),a_1),\mathtt{idp}\,a_1) \sim (((x,a_1),a_2),p)\,.$$

Since B has a tap structure we can transport $\beta(x,a_1) : B(((x,a_1),a_1),\mathtt{idp}\,a_1)$ along this path to get an element of $B(((x,a_1),a_2),p)$. So we can define $\mathtt{J}_A\,B\,\beta \in \mathcal{E}(\Gamma.A.A\,\mathtt{fst}.\,\mathtt{Id}_A \vdash B)$ by:

$$\mathtt{J}_A\,B\,\beta\,(((x,a_1),a_2),p) :\equiv (\langle i \leqslant |p|\rangle(((x,a_1),p\,i),p_{\leqslant i}))_*\,\beta(x,a_1)\,. \tag{26}$$

$\mathtt{J}$ has the required computation property, because

$$\begin{aligned}
((\mathtt{J}_A\,B\,\beta)\,\mathtt{Refl}_A)(x,a) &= \mathtt{J}_A\,B\,\beta\,(((x,a),a),\mathtt{idp}\,a) && \text{by (25)}\\
&= (\langle i \leqslant 0\rangle(((x,a),a)\,\mathtt{idp}\,a))_*\,\beta(x,a) && \text{by (26) and (13)}\\
&= \mathtt{idp}(((x,a),a)\,\mathtt{idp}\,a)_*\beta(x,a) && \text{by (11)}\\
&= \beta(x,a) && \text{by (14)}
\end{aligned}$$

Stability of $\mathtt{Refl}$ under re-indexing follows from the fact that the congruence operations γ ' preserve degenerate paths; and stability of $\mathtt{J}$ uses (10) and the fact that the path contraction operation $(\_)_{\leqslant i}$ is preserved by the congruence operations γ ' . ◀

4 Function Extensionality

In this section we investigate whether functions in the CwF **Fib** behave extensionally with respect to identity types given by Moore paths (Theorem 3.9). So far we have needed only rather weak assumptions about the object $\mathtt{R}$, namely that it is an ordered abelian group. For function extensionality to hold, we need to assume more. To see why, consider the constant function $\mathtt{k}_0 = \lambda i \to 0 : \mathtt{R}_+ \to \mathtt{R}_+$ and the identity function $\mathtt{id} : \mathtt{R}_+ \to \mathtt{R}_+$.[1] If equality means existence of a Moore path, then these functions are extensionally equal, because we have $\lambda i \to \langle j \leqslant i \rangle j : \prod_{i:\mathtt{R}_+}(\mathtt{k}_0\, i \sim \mathtt{id}\, i)$. So if the principle of function extensionality is to hold with respect to Moore paths, then there will have to be a path $p : \mathtt{k}_0 \sim \mathtt{id}$ in $\mathtt{R}_+ \to \mathtt{R}_+$. What is its shape $|p|$? Since p does not depend upon any assumptions, $|p|$ would have to be a global element $1 \to \mathtt{R}$ in the topos $\mathcal{E}$. Our current assumptions about $\mathtt{R}$ only guarantee the existence of one such global element, namely 0. However, if $|p| = 0$ then $\mathtt{k}_0 = \mathtt{id}$, that is $(\forall i : \mathtt{R}_+)\ 0 = i$, and the model of type theory is degenerate. So we need some extra assumptions about $\mathtt{R}$ if function extensionality is to hold without collapsing everything. We do not attempt to find the minimal such assumptions, but rather identify some well-known part of (constructive, ordered) Algebra [4, chapter VI] that does the job, in the expectation that this makes easier the task of finding specific models with good properties (which we address in section 6).

▶ **Definition 4.1** (order-ringed topos)**.** An *order-ringed topos* $(\mathcal{E}, \mathtt{R})$ is a topos $\mathcal{E}$ (with an associated CwF) together with an *ordered commutative ring object* $\mathtt{R}$ in $\mathcal{E}$. Thus as well as an ordered abelian group structure (see section 2), $\mathtt{R}$ has a unit $\mathtt{1} : 1 \to \mathtt{R}$ and a multiplication $\_\cdot\_ : \mathtt{R} \times \mathtt{R} \to \mathtt{R}$ satisfying the usual axioms for a commutative ring; furthermore multiplication respects the order relation in the sense that

$$(\forall i, j : \mathtt{R})\ 0 \leqslant i \ \wedge\ 0 \leqslant j \ \Rightarrow\ 0 \leqslant i \cdot j\,. \tag{27}$$

From now on we assume that we are given such an order-ringed topos $(\mathcal{E}, \mathtt{R})$.

Given an object $\Gamma \in \mathcal{E}$ and a family $A \in \mathcal{E}(\Gamma)$, if $p : f \sim g$ is a path between dependent functions $f, g : \prod_{x:\Gamma} A\,x$, then for each $x : \Gamma$ we can apply the path congruence operation (8) to $\lambda f \to f\,x : (\prod_{x:\Gamma} A\,x) \to A\,x$ and p to obtain a path $(\lambda f \to f\,x)$ **'** $p : f\,x \sim g\,x$ in $A\,x$. This gives us the following function (which coincides with the canonical function obtained by path induction as in [30, section 2.9]):

$$\begin{aligned} &\mathtt{happly} : (f \sim g) \to \textstyle\prod_{x:\Gamma}(f\,x \sim g\,x) \\ &\mathtt{happly}\; p\,x :\equiv (\lambda f \to f\,x) \text{ ' } p \end{aligned} \qquad \text{(where } f, g : \textstyle\prod_{x:\Gamma} A\,x) \tag{28}$$

▶ **Theorem 4.2** (function extensionality modulo $\sim$)**.** *Given an order-ringed topos* $(\mathcal{E}, \mathtt{R})$*, an object* $\Gamma \in \mathcal{E}$ *and a family* $A \in \mathcal{E}(\Gamma)$*, for all* $f, g : \prod_{x:\Gamma} A\,x$ *there is a function* $\mathtt{funext} : (\prod_{x:\Gamma}(f\,x \sim g\,x)) \to (f \sim g)$*. Furthermore this function is quasi-inverse [30, Definition 2.4.6] to* $\mathtt{happly}$*, that is, that for all* $e : \prod_{x:\Gamma}(f\,x \sim g\,x)$ *and* $p : f \sim g$ *there are paths* $\varepsilon\, e : \mathtt{happly}(\mathtt{funext}\, e) \sim e$ *in* $\prod_{x:\Gamma}(f\,x \sim g\,x)$ *and* $\eta\, p : p \sim \mathtt{funext}(\mathtt{happly}\, p)$ *in* $f \sim g$*.*

Proof. If $e : \prod_{x:\Gamma}(f\,x \sim g\,x)$, then for all $x : \Gamma$ we have $e\,x\,(0 \cdot |e\,x|) = e\,x\,0 = f\,x$ and $e\,x\,(\mathtt{1} \cdot |e\,x|) = e\,x\,|e\,x| = g\,x$. So there is a path $\mathtt{funext}\, e : f \sim g$ in $\prod_{x:\Gamma} A\,x$ given by:

$$\mathtt{funext}\, e = \langle i \leqslant \mathtt{1} \rangle \lambda x \to e\,x\,(i \cdot |e\,x|)\,. \tag{29}$$

[1] Function extensionality for **Fib** only concerns functions between fibrant families; but recall that in **Fib** all objects, and in particular $\mathtt{R}_+$, are fibrant as trivial families over the terminal object.

Note that by (10) we have for each $x : \Gamma$

$$\begin{aligned}\mathtt{happly}(\mathtt{funext}\, e)\, x &= (\lambda f \to f\, x) \text{ ' } \langle i \leqslant \mathtt{1}\rangle \lambda x \to e\, x\, (i \cdot |\, e\, x\,|) \\ &= \langle i \leqslant \mathtt{1}\rangle (e\, x\, (i \cdot |\, e\, x\,|))\end{aligned}$$

whereas $e\, x = \langle i \leqslant |\, e\, x\,|\rangle (e\, x\, i)$ by (12). So to get a path from $\mathtt{happly}(\mathtt{funext}\, e)\, x$ to $e\, x$ we need to interpolate shapes from $\mathtt{1}$ to $|\, e\, x\,|$ while at the same time interpolating the argument of $e\, x\, \_$ from $i \cdot |\, e\, x\,|$ to $i \cdot \mathtt{1} = i$. So for each $j : \mathtt{R}$ with $\mathtt{0} \leqslant j \leqslant \mathtt{1}$, consider

$$u_{j,x} :\equiv \mathtt{1} - j + j \cdot |\, e\, x\,| \quad \text{and} \quad v_{j,x} :\equiv (\mathtt{1} - j) \cdot |\, e\, x\,| + j\,.$$

Calculating with the ring axioms we find that $u_{j,x} \cdot v_{j,x} = |\, e\, x\,| + j \cdot (\mathtt{1} - j) \cdot (|\, e\, x\,| - \mathtt{1})^2$. Since $\mathtt{0} \leqslant j$ and $\mathtt{0} \leqslant \mathtt{1} - j$ by assumption and since squares are always positive in an ordered ring [4, VI.19], we have that $|\, e\, x\,| \leqslant u_{j,x} \cdot v_{j,x}$; hence $e\, x\, (u_{j,x} \cdot v_{j,x}) = g\, x$. So for all $j : \mathtt{R}$ with $\mathtt{0} \leqslant j \leqslant \mathtt{1}$ we have a path $\langle i \leqslant u_{j,x}\rangle (e\, x\, (i \cdot v_{j,x})) : f\, x \sim g\, x$ in $A\, x$. When $j = \mathtt{0}$ this path is $\mathtt{happly}(\mathtt{funext}\, e)\, x$ (because $u_{\mathtt{0},x} = \mathtt{1}$ and $v_{\mathtt{0},x} = |\, e\, x\,|$); when $j = \mathtt{1}$ the path is $e\, x$ (because $u_{\mathtt{1},x} = |\, e\, x\,|$ and $v_{\mathtt{1},x} = \mathtt{1}$). Therefore we can define

$$\varepsilon\, e :\equiv \langle j \leqslant \mathtt{1}\rangle \lambda x \to \langle i \leqslant u_{j,x}\rangle (e\, x\, (i \cdot v_{j,x})) \tag{30}$$

to get the desired path $\mathtt{happly}(\mathtt{funext}\, e) \sim e$ in $\prod_{x:\Gamma}(f\, x \sim g\, x)$. Since for any $p : f \sim g$ it is the case that $p = \langle i \leqslant |\, p\,|\rangle (p\, i)$ and $\mathtt{funext}(\mathtt{happly}\, p) = \langle i \leqslant \mathtt{1}\rangle (p\, (i \cdot |\, p\,|))$, a similar argument to the one for ε shows that

$$\eta\, p :\equiv \langle j \leqslant \mathtt{1}\rangle \langle i \leqslant (\mathtt{1} - j) \cdot |\, p\,| + j\rangle (p\, (i \cdot (\mathtt{1} - j + j \cdot |\, p\,|))) \tag{31}$$

gives a path $p \sim \mathtt{funext}(\mathtt{happly}\, p)$ in $f \sim g$. ◀

5 Weak Fibrations

We leave to a sequel the study of universes and univalence [30, section 2.10] in this setting. However, initial investigation suggests that a weaker notion of fibration than in Definition 3.1 may be needed for this. To see why, we show that an essential aspect of univalence already holds for order-ringed toposes $(\mathcal{E}, \mathtt{R})$: *given a contractible object $C \in \mathcal{E}$, there is a family $\hat{C} \in \mathcal{E}(\mathtt{R}_+)$ that gives a Moore path of types of shape $\mathtt{1}$ with $\hat{C}\,\mathtt{0} \cong C$ and $\hat{C}\,\mathtt{1} \cong 1$.*

Recall that C is *contractible* [30, section 3.11] if there is some $c_0 : C$ (the centre of contraction) and a function $\kappa : \prod_{x:C}(x \sim c_0)$. Given such a C, one way to get $\hat{C} \in \mathcal{E}(\mathtt{R}_+)$ as above is to construct a cone over C and take the family of slices through the cone starting at the base and moving to its vertex. Specifically, for each $i : \mathtt{R}_+$ we can define

$$\hat{C}\, i \;:\equiv\; C/{\approx_i} \quad \text{where } \approx_i \text{ is the equivalence relation: } x \approx_i y \;:\equiv\; (x = y \;\vee\; \mathtt{1} \leqslant i)\,. \tag{32}$$

Write $[x]_i$ for the $\approx_i$-equivalence class of $x : C$. If $i \geqslant \mathtt{1}$, then $\approx_i$ is the indiscrete equivalence relation identifying all $x : C$ with c_0 and so $\hat{C}\, i = \{[c_0]_i\} \cong 1$. When $i = \mathtt{0}$, so long as $\mathtt{R}$ is *non-trivial*

$$\neg(\mathtt{0} = \mathtt{1}) \tag{33}$$

then $\approx_{\mathtt{0}}$ is the discrete equivalence relation ($x \approx_{\mathtt{0}} y \Leftrightarrow x = y$) and hence $\hat{C}\,\mathtt{0} = C/{\approx_{\mathtt{0}}} \cong C$.

If there were a universe for small fibrations (with respect to some given notion of smallness), then for the family $(\hat{C}\, i \mid i : \mathtt{R}_+)$ to determine a path in it, in addition to C being small, we would also need a tap structure for $\hat{C}$ (Definition 3.1). It is not clear why such a structure should exist. However, $\hat{C}$ is an instance of the following weaker notion of fibration where the equality (14) is replaced by a path:

▶ **Definition 5.1** (Weak fibration). Given an object $\Gamma \in \mathcal{E}$, a *weak fibration* over Γ is specified by a family $A \in \mathcal{E}(\Gamma)$ together with functions $\alpha : \prod_{x,y:\Gamma}(x \sim y) \to (A\,x \to A\,y)$ and $\alpha' : \prod_{x:\Gamma}\prod_{a:A\,x}(a \sim \alpha\,x\,x\,(\mathtt{idp}\,x)\,a)$. We write $\mathbf{wFib}(\Gamma)$ for the set of weak fibrations over Γ.

Using the contraction structure c_0, κ of C, we can get such a weak fibration over $\Gamma = \mathtt{R}_+$ with $A = \hat{C}$ and the functions α and α' well-defined by

$$\alpha\,i\,j\,p\,[x]_i :\equiv [\kappa\,x\,(i \cdot |\kappa\,x|)]_j \qquad (i, j : \mathtt{R}_+, p : i \sim j, x : C)$$
$$\alpha' i\,[x]_i :\equiv \langle j \leqslant i\rangle[\kappa\,x\,(j \cdot |\kappa\,x|)]_i \qquad (i : \mathtt{R}_+, x : C)$$

Restricting this to the end point $i = \mathtt{0}$ gives the identity transport function, which indeed corresponds under the isomorphism $\hat{C}\,\mathtt{0} \cong C$ to that for C regarded as a (weakly) fibrant object. More generally, given $\Gamma \in \mathcal{E}$, if we start with a family $C \in \mathbf{wFib}(\Gamma)$ all of whose fibres are contractible, the method outlined above constructs $\hat{C} \in \mathbf{wFib}(\Gamma \times \mathtt{R}_+)$ with $\hat{C}(\_, \mathtt{0})$ and C isomorphic weak fibrations over Γ and with $\hat{C}(\_, \mathtt{1})$ isomorphic to the terminal weak fibration over Γ.

Even though Definition 5.1 might seem to be a very weak notion of fibration, we have found that versions of Theorems 3.4 and 3.5 hold for it, albeit with more complicated proofs. The same is true for Theorem 3.9 except that one gets only *propositional* identity types, where there is a path between $(\mathtt{J}_A\,B\,\beta)\,\mathtt{Refl}_A$ and β, rather than an equality in $\mathcal{E}$ (cf. [6, section 9.1] and [31]).

6 A Gros Topos Model

In the previous sections we have seen how any order-ringed topos $(\mathcal{E}, \mathtt{R})$ (Definition 4.1) gives rise to a model of Martin-Löf type theory with intensional identity types given by Moore paths on $\mathtt{R}$. If $\mathtt{R}$ is trivial, that is, satisfies $\mathtt{0} = \mathtt{1}$, then existence of a Moore path just coincides with extensional equality in the topos $\mathcal{E}$. If the object $\mathtt{R}$ is non-trivial, but has decidable equality in $\mathcal{E}$ (for example, if it is the object of integers, or of rationals), then there is a path $\langle i \leqslant \mathtt{1}\rangle(\mathtt{if}\ i = \mathtt{0}\ \mathtt{then}\ \mathtt{true}\ \mathtt{else}\ \mathtt{false}) : \mathtt{true} \sim \mathtt{false}$ in the object of Booleans and so in this case the model of type theory we get is logically degenerate. Therefore, when searching for examples of order-ringed toposes, we should at least look for ones with $\mathtt{R}$ non-trivial and not decidable. In fact we will show that there are order-ringed toposes for which the associated model of type theory has identity types that are not necessarily truncated at any level. In other words, ones in which iterated identity types $\mathtt{Id}_A, \mathtt{Id}_{\mathtt{Id}_A}, \mathtt{Id}_{\mathtt{Id}_{\mathtt{Id}_A}}, \ldots$ can be homotopically non-trivial for any level of iteration. An interesting way of demonstrating that is by giving an $(\mathcal{E}, \mathtt{R})$ for which the homotopy types of a rich collection of topological spaces (including all the n-dimensional spheres, say) are faithfully represented by the internal, $\mathtt{R}$-based notion of homotopy on a corresponding collection of objects of the topos $\mathcal{E}$. We give one such order-ringed topos in this section.

The topos

We use a topos of sheaves $\mathbf{Sh}(\mathbf{T}, J)$ [19, III] for the following site $(\mathbf{T}, J)$. Let **Haus** denote the category of Hausdorff topological spaces and continuous functions. We take the small category $\mathbf{T}$ to be the least full subcategory of **Haus** containing the reals $\mathbb{R}$ and closed under the following operations:

(a) If $X, Y \in \mathbf{T}$, then $\mathbf{T}$ contains the product space $X \times Y$.
(b) If $X \in \mathbf{T}$ and $C \subseteq X$ is a closed subset, then C (with the subspace topology) is in $\mathbf{T}$.
(c) If $X, Y \in \mathbf{T}$ and X is locally compact, then $\mathbf{T}$ contains the exponential space Y^X (the set of continuous functions from X to Y endowed with the compact-open topology).

Since the spaces in $\mathbf{T}$ are Hausdorff, equalizers of continuous functions give closed subspaces and hence by (a) and (b) we have that $\mathbf{T}$ is closed under taking finite limits in **Haus**. Hence $\mathbf{T}$ contains $\mathbb{R}_+$ (as well as $[0,1]$ and all the spheres S^n for any $n \in \mathbb{N}$). Then by (c) we have that $\mathbf{T}$ is closed under taking Moore path spaces (with their usual topology):

$$X \in \mathbf{T} \;\Rightarrow\; MX :\equiv \{(f,r) \in X^{\mathbb{R}_+} \times \mathbb{R}_+ \mid (\forall r' \geq r)\; f\,r' = f\,r\} \in \mathbf{T}\,. \tag{34}$$

We define a coverage [14, Definition A2.1.9] on $\mathbf{T}$ as follows. Following Dyer and Eilenberg [9] we say that a set S of closed subsets of a topological space X is a *local cover* if for each $x \in X$ there is a finite subset $\{C_1,\ldots,C_n\} \subseteq S$ with x in the interior of $C_1 \cup \cdots \cup C_n$. Of course every finite cover of X by closed subsets is trivially a local cover. Note that if $f : Y \to X$ in $\mathbf{T}$ and S is a local cover of X, then $f^{-1}S :\equiv \{f^{-1}C \mid C \in S\}$ is a local cover of Y. Hence we get a coverage on $\mathbf{T}$ with $J(X) :\equiv \{S \mid S \text{ is a local cover of } X\}$ for each $X \in \mathbf{T}$.

Given a functor $\Gamma : \mathbf{T}^{\mathrm{op}} \to \mathbf{Set}$, if $C \subseteq X$ is a closed subspace of a space in $\mathbf{T}$, then for each $x \in \Gamma\,X$ we will just write $x|_C$ for the element $\Gamma\,i\,x \in \Gamma\,C$, where the $\mathbf{T}$-morphism $i : C \to X$ is the inclusion function. Recall that Γ is a sheaf with respect to J iff for all $X \in \mathbf{T}$, $S \in J(X)$ and all S-indexed families $(x_C \in \Gamma\,C \mid C \in S)$ satisfying $(\forall C, D \in S)\; x_C|_{C\cap D} = x_D|_{C\cap D}$, there is a unique $x \in \Gamma\,X$ with $(\forall C \in S)\; x_C = x|_C$. The topos $\mathbf{Sh}(\mathbf{T}, J)$ is the full subcategory of the functor category $[\mathbf{T}^{\mathrm{op}}, \mathbf{Set}]$ whose objects are sheaves.

The ordered commutative ring object

Let $\mathrm{Y} : \mathbf{T} \to [\mathbf{T}^{\mathrm{op}}, \mathbf{Set}]$ denote the Yoneda embedding for the small category $\mathbf{T}$. Because elements of $J(X)$ are local covers of $X \in \mathbf{T}$, for each $S \in J(X)$ if we have a family of continuous functions $f_C : C \to X'$ for each $C \in S$ that agree where they overlap ($f_C|_{C\cap D} = f_D|_{C\cap D}$), then the unique function $f : X \to X'$ that agrees with each of them ($f_C = f|_C$) is necessarily continuous. Hence each representable presheaf $\mathrm{Y}\,X' = \mathbf{T}(\_, X')$ is a sheaf (in other words the coverage J is sub-canonical) and the Yoneda embedding gives a functor $\mathrm{Y} : \mathbf{T} \to \mathbf{Sh}(\mathbf{T}, J)$.

The axioms for *partially* ordered commutative rings make sense in any category with finite limits once we have an interpretation of the binary operation $\leqslant$, the constants $\mathtt{0}, \mathtt{1}$ and the operations $+, \mathtt{-}, \cdot$; and satisfaction of those axioms is preserved by functors that preserve finite limits. Since $\{(x,y) \in \mathbb{R} \times \mathbb{R} \mid x \leq y\}$ is a closed subset of $\mathbb{R} \times \mathbb{R}$ and the usual ring operations on $\mathbb{R}$ are continuous, it follows that $\mathbb{R}$ is a partially ordered commutative ring object in the finitely complete category $\mathbf{T}$. Then since Y preserves finite limits, the representable sheaf $\mathtt{R} :\equiv \mathrm{Y}\mathbb{R}$ is a partially ordered commutative ring object in the topos $\mathbf{Sh}(\mathbf{T}, J)$. In fact it is totally ordered, that is, satisfies (2). This is simply because we have a local cover $\{\{(x,y) \mid x \leq y\}, \{(x,y) \mid y \leq x\}\} \in J(\mathbb{R} \times \mathbb{R})$.[2]

Homotopy types in $\mathbf{Sh}(\mathbf{T}, J)$

As in section 3 we get a CwF **Fib** based on $\mathbf{Sh}(\mathbf{T}, J)$ with intensional identity types given by Moore paths from $\mathtt{R} = \mathrm{Y}\mathbb{R}$. Consider the congruence on the category $\mathbf{Sh}(\mathbf{T}, J)$ that identifies morphisms $\gamma, \delta : \Delta \to \Gamma$ when there is a global section of $\mathtt{Id}_{\Delta \to \Gamma}\,\gamma\,\delta$. Let $\mathbf{Ho}(\mathbf{Sh}(\mathbf{T}, J))$ denote the associated quotient category.

[2] This fact is the motivation for considering a site based on covers by closed subsets rather than the more familiar case of open covers used for Giraud's gros topos [10, IV, 2.5]. However, as Spitters has pointed out [private communication], in view of [13, Theorem 8.1] we could have used the reals in Johnstone's *topological topos* instead of $\mathbf{Sh}(\mathbf{T}, J)$ in this section.

By Theorem 4.2, two morphisms $\gamma, \delta : \Delta \to \Gamma$ are identified in $\mathbf{Ho}(\mathbf{Sh}(\mathbf{T}, J))$ iff $\prod_{x:\Delta}(\gamma\, x \sim \delta\, x)$ has a global section. This is equivalent to requiring the existence of a morphism $H : \Delta \to \wp\Gamma$ in $\mathbf{Sh}(\mathbf{T}, J)$ satisfying $\partial^- \circ H = \gamma$ and $\partial^+ \circ H = \delta$, where

$$\begin{aligned} &\wp\Gamma :\equiv \{(f, i) : (\mathtt{R}_+ \to \Gamma) \times \mathtt{R}_+ \mid (\forall j \geqslant i)\ f\, j = f\, i\} \\ &\partial^-, \partial^+ : \wp\Gamma \to \Gamma \qquad \partial^-(f, i) :\equiv f\, \mathtt{0} \qquad \partial^+(f, i) :\equiv f\, i \end{aligned} \tag{35}$$

is the total object of the family of Moore path objects (6) with associated source and target morphisms. Note that since $\mathtt{R} = \mathrm{Y}\mathbb{R}$, it follows that we also have $\mathtt{R}_+ \cong \mathrm{Y}(\mathbb{R}_+)$. Furthermore, as well as preserving finite limits the Yoneda embedding preserves any exponentials that happen to exist. It follows that for each representable sheaf $\mathrm{Y}X$ its total path object is representable by the Moore path space of X (34): $\wp(\mathrm{Y}X) \cong \mathrm{Y}(MX)$; and under this isomorphism the source and target morphisms ∂^-, ∂^+ correspond to the representable morphisms induced by the usual source and target functions for MX. Since Y is full and faithful, it follows that for two continuous functions $f, g : X \to X'$ in $\mathbf{T}$, the morphisms Yf and $\mathrm{Y}g$ get identified in $\mathbf{Ho}(\mathbf{Sh}(\mathbf{T}, J))$ iff f and g are homotopic in the classical sense. Thus Y induces a full and faithful embedding of the category $\mathbf{Ho}(\mathbf{T})$ of homotopy types of spaces in $\mathbf{T}$ into $\mathbf{Ho}(\mathbf{Sh}(\mathbf{T}, J))$. Since $\mathbf{T}$ contains all the spheres S^n, we deduce that identity types in this particular $\mathbf{Fib}$ are not necessarily truncated at any level of iteration.

7 Related Work

The classical topological notion of Moore path is a standard, if somewhat niche topic within homotopy theory. The Schedule Theorem of Dyer and Eilenberg [9] for globalising Hurewicz fibrations is a nice example of their usefulness. They have been used in connection with higher-dimensional category theory by Kapranov and Voevodsky [16] and by Brown [5].

Although our use of constructive algebra within toposes to make models of intensional type theory appears to be new, we are not the only ones to consider using some form of path with strictly unitary and associative composition to model identity types with a judgemental computation rule. Van den Berg and Garner [32] use topological Moore paths and a simplicial version of them to get instances of their notion of *path object category* for modelling identity types. The results of section 2 show that any ordered abelian group in a topos induces a path object category structure on that topos; and since the notion of fibration we use in section 3 in fact coincides with the one used in [32] (see Proposition 6.1.5 of that paper), one can get alternative, more abstract proofs of Theorems 3.4 and 3.9 from the results of Van den Berg and Garner. However, the concrete calculations in the internal language that we give are quite simple by comparison; and this approach proves its worth in section 4, whose results are new as far as we know.

The forthcoming PhD thesis of North [25] uses a category-theoretic abstraction of the notion of Moore paths, called *Moore relation systems*, as part of a complete analysis of when a weak factorization system gives a model (in terms of display map categories, rather than CwFs) of identity-, Σ- and Π-types. A Moore relation system is a piece of category theory comparable to our use of ordered abelian groups in categorical logic in section 2; it would be interesting to see if it can be extended in the way we extended from groups to rings in section 4 in order to validate function extensionality.

Spitters [27, section 3] attempts to use a somewhat different formulation of Moore path in the cubical topos [6]. His notion is the reflexive-transitive closure of the usual path types given by the bounded interval. For better properties and to get a closer relationship with our version, one would like to quotient these cubical "Spitters-Moore" path objects up to degenerate paths; but the undecidability of degeneracy seems to stop one being able to do that while retaining the (uniform) Kan-fibrancy of such path objects. Here we can side-step such issues, since notably our models manage to avoid using a notion of Kan fibrancy at all.

8 Future Work

There are two directions in which we would like to develop this work. First, the calculations about type structure in this paper using constructive algebraic Moore paths, particularly those using the bounded abstraction notation (Definition 2.4), are strikingly simple compared with ones based upon Kan filling [17, 3, 6]. It would be interesting and possibly useful to embody them in a constructive type theory with Moore path types and in which any type families describable in the theory are automatically fibrant. The inspiration for this is the Cubical Type Theory of Cohen, Coquand, Huber and Mörtberg [6].

Secondly, are there order-ringed toposes for which the associated CwF **Fib** constructed as in section 3 contains universes that satisfy Voevodsky's *univalence axiom* [30, section 2.10]? Our initial analysis of what is needed to satisfy univalence (see section 5) suggests that one just needs an order-ringed topos containing a fibration that is weakly generic for (weak) fibrations whose fibres are small with respect to some given external Grothendieck universe. Such universes in the classical simplicial sets [18] and constructive cubical sets [6] models of type theory make use of a modified form of the Hofmann-Streicher [12] construction in *presheaf* categories. To be sure, there are order-ringed toposes $(\mathcal{E}, \mathtt{R})$ for which $\mathcal{E}$ is a presheaf topos (with $\mathtt{R}$ non-trivial and not decidable). For example one can take $\mathcal{E} = [\mathbf{C}, \mathbf{Set}]$ where $\mathbf{C}$ is a suitable small full subcategory of the category of ordered commutative rings in **Set** and where $\mathtt{R}$ is the forgetful functor $\mathbf{C} \to \mathbf{Set}$. However, this $\mathtt{R}$ is not representable and $\mathbf{C}^{\mathrm{op}}$ is not cartesian; and that seems to block the construction of a (weak) tap structure for the generic family over the Hofmann-Streicher universe of small fibrations on representables. Given the crucial, disjunctive axiom (2) upon which everything in this paper depends, one is led to consider sheaf toposes instead of presheaf toposes, as we did in section 6. Streicher [29] points out that the basic Hofmann-Streicher universe construction works for sheaf toposes through a suitable use of sheafification. However, as yet we have not been able to extend that to work for the sheafification of the presheaf universe of small families *with tap structure* over representables, for the site in section 6. Although that presheaf universe is not a sheaf, it is probably a *stack of groupoids* [28, Tag 02ZH]. (To see this one would have to prove a version of globalisation [9] for fibrations in $\mathbf{Sh}(\mathbf{T}, J)$.) In which case it might be fruitful to consider stack-based models of univalent type theory of the kind considered by Coquand, Mannaa and Ruch [7], or higher-dimensional versions thereof.

Acknowledgements. We are very grateful to Benno van den Berg and Bas Spitters for discussions about the material in this paper.

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