Confluence of an Extension of Combinatory Logic by Boolean Constants[*]

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Abstract

We show confluence of a conditional term rewriting system CL-pc[1], which is an extension of Combinatory Logic by Boolean constants. This solves problem 15 from the RTA list of open problems. The proof has been fully formalized in the Coq proof assistant.

1 Introduction

Combinatory Logic is a term rewriting system defined by two rules:

$$\mathsf{K}xy \;\rightarrow\; x \qquad \mathsf{S}xyz \;\rightarrow\; xz(yz)$$

Using only S and K, it is possible to encode natural numbers via Church numerals. Any computable function may then be represented by a term in the system. However, a conditional C encoded in this way does not have a desirable property that $\mathsf{C}t_1t_2t_2 = t_2$ if t_1 encodes neither true nor false. It is therefore interesting to investigate extensions of Combinatory Logic incorporating a conditional directly. Perhaps the most natural such extension is CL-pc:

$$
\begin{array}{llll}
\mathsf{K}xy &\rightarrow\; x & \mathsf{CT}xy \;\rightarrow\; x & \mathsf{C}zxx \;\rightarrow\; x \\
\mathsf{S}xyz &\rightarrow\; xz(yz) & \mathsf{CF}xy \;\rightarrow\; y &
\end{array}
$$

The system CL-pc is known to be not confluent [7]. One may thus try other ways of adding a conditional and Boolean constants to Combinatory Logic.

We show confluence of a conditional term rewriting system CL-pc[1] defined by the rules:

$$
\begin{array}{llll}
\mathsf{K}xy &\rightarrow\; x & \mathsf{CT}xy \;\rightarrow\; x & \mathsf{C}zxy \;\rightarrow\; x \;\;\Leftarrow\;\; x = y \\
\mathsf{S}xyz &\rightarrow\; xz(yz) & \mathsf{CF}xy \;\rightarrow\; y &
\end{array}
$$

Confluence of this system[1] appears as problem 15 on the RTA list of open problems [5].

The equality in the side condition for the third rule for C in CL-pc[1] refers to equality in the system CL-pc[1] itself, thus the definition is circular. This circularity is an essential property of CL-pc[1] which distinguishes it from CL-pc.

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[1] Strictly speaking, in the literature the systems CL-pc, CL-pc[1] and CL-pc[L] also contain the rule $\mathsf{I}x \rightarrow x$. This rule could be added to our definitions without significantly changing the proofs. However, this would increase the number of cases to consider, making the proofs less readable. The formalization of our results uses the definitions from the literature.

A system related to CL-pc^1 is CL-pc^L, which consists of all rules of CL-pc^1 plus:

$$\mathsf{C}zxy \to y \quad \Leftarrow \quad x = y$$

It is known that CL-pc^L is confluent [4]. However, the confluence proof in [4] essentially depends on a "semantic" argument to first establish $\mathsf{T} \neq_{\text{CL-pc}^L} \mathsf{F}$. We provide a "syntactic" proof of confluence of both CL-pc^1 and CL-pc^L.

The systems CL-pc^1 and CL-pc^L are conditional linearizations of CL-pc. The notion of conditional linearization was introduced in the hope of providing a simpler proof of Chew's theorem [2, 9] which states that all compatible term rewriting systems have the unique normal form (UN) property. Compatibility imposes certain restrictions on the term rewriting system, but it does not require termination or left-linearity. In particular, Chew's theorem is applicable to many term rewriting systems which are not confluent. For instance, CL-pc satisfies the conditions of Chew's theorem, but it is not confluent. As shown in [4], to prove the unique normal form property of a term rewriting system, it suffices to prove confluence of one of its conditional linearizations. The proof of Chew's theorem in [9] is quite complicated and uses a related but different approach, relying on left-right separated conditional linearizations instead of the more straightforward ones from [4]. The original proof by Chew [2] uses yet another different but related method, but Chew's proof was later found to contain a gap.

In general, the methods of the present paper are broadly related to the problem of establishing the UN property for classes of term rewriting systems which include non-left-linear non-confluent systems. Aside of Chew's theorem, some other work in this direction has been carried out in e.g. [8, 12, 13, 10, 6].

In order to increase confidence in the correctness of the main result of this paper, we have formalized our proof of confluence of CL-pc^1 in the Coq proof assistant. The formalization is available online[2]. It follows closely the development presented here. We used the CoqHammer [3] tool and the automated reasoning tactics included with it.

2   Proof overview

In this section we present an informal overview of the proof, trying to convey the underlying intuitions. Section 3 presents formal definitions of the notions informally motivated here, and Section 4 provides details of the proof itself.

We assume familiarity with basic term-rewriting [1, 11]. By $\to^*$ we denote the transitive-reflexive closure of a relation $\to$, by $\to^{\equiv}$ its reflexive closure, by $\leftrightarrow$ the symmetric closure, and by $=$ the reflexive-transitive-symmetric closure. We use $\equiv$ to denote identity of terms. By $\to^!$ we denote reduction to normal form, i.e., $t \to^! s$ if $t \to^* s$ and s is in normal form. By $\cdot$ we denote composition of relations, e.g. $t \to \cdot \leftarrow t'$ holds iff there exists t_0 such that $t \to t_0$ and $t_0 \leftarrow t'$. We use the standard notions of subterms and subterm occurrences, which could be formally defined by introducing the notion of positions. If t is a redex, i.e. $t \equiv \sigma l$ for some term l and substitution σ, then a subterm s *occurs below a variable position* of the redex t if s occurs in a subterm of t occurring at the position of a variable in l. The contraction in $t_1 \to t_2$ *occurs at the root* if t_1 is the contracted redex.

Let u be a normal form w.r.t. a relation $\to$. The relation $\to$ (or the underlying rewrite system) is *u-normal* if for every t such that $t = u$ we have $t \to^* u$.

The most difficult part of our confluence proof is to show that CL-pc^1 is F-normal (Lemma 27). The confluence of CL-pc^1 (and CL-pc^L) is then obtained by a relatively simple argument similar to the one used in [4] to derive the confluence of CL-pc^L from $\mathsf{T} \neq_{\text{CL-pc}} \mathsf{F}$.

An important observation is that $q_1 =_{\text{CL-pc}^1} q_2$ and $q_1 =_{\text{CL-pc}} q_2$ are in fact equivalent (Lemma 2). Hence, we will use $=_{\text{CL-pc}^1}$ and $=_{\text{CL-pc}}$ interchangeably. In particular, we actually prove that for any term q, if $q =_{\text{CL-pc}} \mathsf{F}$ then $q \to^*_{\text{CL-pc}^1} \mathsf{F}$.

A naive approach to prove this could be to proceed by induction on the length of the conversion $q =_{\text{CL-pc}} \mathsf{F}$. In the inductive step we would need to prove:

1. if $q \to^*_{\text{CL-pc}^1} \mathsf{F}$ and $q \to_{\text{CL-pc}} q'$ then $q' \to^*_{\text{CL-pc}^1} \mathsf{F}$,
2. if $q \to^*_{\text{CL-pc}^1} \mathsf{F}$ and $q \,_{\text{CL-pc}}\!\!\leftarrow q'$ then $q' \to^*_{\text{CL-pc}^1} \mathsf{F}$.

The second part is obvious, but the first one is hard. The difficulty stems from the existence of a non-trivial overlap between the rules for C. If $t_1 =_{\text{CL-pc}^1} t_2$ then $\mathsf{CF}t_1t_2 \to_{\text{CL-pc}^1} t_1$ by the third rule of CL-pc^1 and $\mathsf{CF}t_1t_2 \to_{\text{CL-pc}} t_2$ by the second rule of CL-pc. We do not know enough about t_1 and t_2 to easily infer that they have a common reduct in CL-pc^1.

One may try to strengthen the inductive hypothesis in the hope of making the first part easier to prove. A naive attempt would be to claim that *all* reductions starting from q end in F, instead of claiming that *some* reduction ends in F. This would make the first part trivial, but the second one would not go through as this is false in general, e.g., consider $\mathsf{KF}\Omega$ where $\Omega \equiv (\mathsf{SII})(\mathsf{SII})$ and $\mathsf{I} \equiv \mathsf{SKK}$.

The idea is to consider, for a given conversion $q =_{\text{CL-pc}} \mathsf{F}$, a certain set $\mathcal{S}(q =_{\text{CL-pc}} \mathsf{F})$ of reductions, all starting from q. The set $\mathcal{S}(q =_{\text{CL-pc}} \mathsf{F})$ depends on the exact form of $q =_{\text{CL-pc}} \mathsf{F}$. Then our two parts of the proof for the inductive step become:

1. if $\mathcal{S}(q =_{\text{CL-pc}} \mathsf{F})$ is nonempty and all reductions in it end in F, and $q \to_{\text{CL-pc}^1} q'$, then $\mathcal{S}(q' \,_{\text{CL-pc}}\!\!\leftarrow q =_{\text{CL-pc}} \mathsf{F})$ is nonempty and all reductions in it end in F,
2. if $\mathcal{S}(q =_{\text{CL-pc}} \mathsf{F})$ is nonempty and all reductions in it end in F, and $q \,_{\text{CL-pc}}\!\!\leftarrow q'$, then $\mathcal{S}(q' \to_{\text{CL-pc}} q =_{\text{CL-pc}} \mathsf{F})$ is nonempty and all reductions in it end in F.

The hope is that if we define $\mathcal{S}(q =_{\text{CL-pc}} \mathsf{F})$ appropriately, then showing both parts will become feasible.

Essentially, the set $\mathcal{S}(q =_{\text{CL-pc}} \mathsf{F})$ will be encoded in the labeling of certain constants in q. The labels determine which contractions are permitted when a given constant appears as the leftmost constant in a redex[3]. At present the author does not know an "explicit" characterization of the set of reductions $\mathcal{S}(q =_{\text{CL-pc}} \mathsf{F})$ implicitly defined by the labelings described below.

Terms with the leftmost constant labeled will be called "significant", or s-terms, whereas others will not contain any labels and will be called "insignificant", or i-terms (cf. Definition 3). Reductions occurring in i-terms will be "insignificant", or i-reductions. A "significant" contraction, or s-contraction, will be a contraction of a term with the leftmost constant labeled, in a way permitted by the label of the leftmost constant. Contraction of a redex in which the leftmost constant is not labeled is not permitted in s-contractions. See Definition 4. The intuition is that we do not need to care about the expansions and contractions occurring in "insignificant" subterms of a given term, since they cannot influence the s-reductions starting from this term and ending in F.

The set $\mathcal{S}(q =_{\text{CL-pc}} \mathsf{F})$ will be encoded in a labeled variant[4] t of q, and it will consist of all s-reductions starting from t and ending in a normal form (w.r.t. s-contraction). Strictly

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[3] E.g. in the redex $\mathsf{CT}t_1t_2$ the constant C is the leftmost constant.
[4] By a "labeled variant" of a term q we mean a term with certain constants labeled which is identical with q when the labels are "erased".

speaking, we have just silently shifted from considering contractions in "plain" terms of the system CL-pc[1] to contractions in their labeled variants, in a different rewriting system which we have not yet defined. In particular, we will actually be interested in s-reductions ending in a labeled variant F_1 of F. However, it will be later shown that s-reductions defined on labeled terms may be "erased" to appropriate reductions in the system CL-pc[1]. In the next section we define the system CL-pc^s (Definition 4) over labeled terms (Definition 3) which will give precise rules of s-contraction. In this section we only give informal motivations.

The labels constrain the ways in which s-redexes may be contracted and encode permissible s-reductions to F_1. Each term decomposes into a "significant" prefix and an "insignificant" suffix (cf. 1 in Definition 8). The "significant" prefix contains all labeled constants and no unlabeled constants. The "insignificant" suffix consists of all "insignificant" subterms. All constants in the "insignificant" suffix are unlabeled. This is analogous to the existence of a needed prefix and a non-needed suffix in orthogonal TRSs [11, Section 9.2.2]. An "insignificant" subterm does not overlap with any needed redexes. In particular, it does not contain any needed redexes. No position inside an "insignificant" subterm (dynamically) traces to F_1 along any s-reduction to F_1 (cf. [11, Definition 8.6.7]). In contrast, each s-redex needs to be either s-contracted or erased by a rule for C_2 (see Definition 4) in any s-reduction to F_1. Each position of a labeled constant either traces to F_1 along a given s-reduction to F_1, or is erased in that s-reduction by a rule for C_2. An s-reduct of an s-term is always also an s-term (cf. 5 in Definition 8).

We write $t \to_s t'$ for one-step reduction in CL-pc^s. We use the abbreviation s-NF for CL-pc^s-normal form. We write $t \Downarrow_{\mathsf{F}_1}$ when, among other conditions to be defined later, t is complete, i.e. terminating and confluent, w.r.t. s-reductions with F_1 as the normal form (cf. Definition 8).

With the set $\mathcal{S}(q =_{\mathrm{CL\text{-}pc}} F)$ coded by labels, the two parts of the inductive step become:

1. if t is a labeled variant of q such that $t \Downarrow_{\mathsf{F}_1}$, and $q \to_{\mathrm{CL\text{-}pc}} q'$, then there exists a labeled variant t' of q' such that $t' \Downarrow_{\mathsf{F}_1}$ (cf. Corollary 15),

2. if t is a labeled variant of q such that $t \Downarrow_{\mathsf{F}_1}$, and $q \;_{\mathrm{CL\text{-}pc}}\!\leftarrow q'$, then there exists a labeled variant t' of q' such that $t' \Downarrow_{\mathsf{F}_1}$ (cf. Corollary 25).

Now we provide some explanations on how the terms will be labeled. For this purpose we analyze why the second part fails when we take $\mathcal{S}(q =_{\mathrm{CL\text{-}pc}} F)$ to be the set of all reductions starting from q. We indicate how to introduce the labeled variants so as to make the second part go through while still retaining the feasibility of showing the first part.

Suppose $q \;_{\mathrm{CL\text{-}pc}}\!\leftarrow q'$ at the root and we have already decided on the labeled variant t of q. We need to decide on a labeled variant t' of q', and assign appropriate meaning to the labels, in such a way that the second part goes through. In short, in t' we preserve the labelings of the subterms of q' which are copied to q in $q' \to_{\mathrm{CL\text{-}pc}} q$, we do not label the new subterms of q' which are erased in $q' \to_{\mathrm{CL\text{-}pc}} q$ (they become i-terms), and we ensure that i-terms and cannot influence *any* s-reduction from t' to F_1. First of all, if t is an i-term, i.e., $t \equiv q$, then we may take $t' \equiv q'$. So assume t is an s-term. Then there are the following possibilities.

- If $q' \equiv \mathsf{CT}qq_0 \to_{\mathrm{CL\text{-}pc}} q$ then q_0 is a new subterm. We take $t' \equiv \mathsf{C}_1\mathsf{T}_1tq_0$. The labeling C_1 of C will be interpreted as not permitting contraction by the third rule, i.e., in CL-pc^s we will only have the rules $\mathsf{C}_1\mathsf{T}_1xy \to x$ and $\mathsf{C}_1\mathsf{F}_1xy \to y$. This ensures that q_0 gets erased in every s-reduction of t' to F_1.

- The case when $q' \equiv \mathsf{CF}q_0q \to_{\mathrm{CL\text{-}pc}} q$ is analogous: we take $t' \equiv \mathsf{C}_1\mathsf{F}_1q_0t$.

- If $q' \equiv \mathsf{C}q_0qq \to_{\mathrm{CL\text{-}pc}} q$ by the third rule, then q_0 is a new term. We take $t' \equiv \mathsf{C}_2q_0tt$. In

the system CL-pc^s we have two rules for C_2

$$
\begin{aligned}
\mathsf{C}_2 zxy &\rightarrow x &\Leftarrow& \quad |x| =_{\text{CL-pc}^1} |y| \\
\mathsf{C}_2 zxy &\rightarrow y &\Leftarrow& \quad |x| =_{\text{CL-pc}^1} |y|
\end{aligned}
$$

where $|x| =_{\text{CL-pc}^1} |y|$ means that the "erasures" of the labeled terms substituted for x and y must be equal in CL-pc^1 for the rule to be applicable. These rules ensure that q_0 cannot influence any s-reduction of t' to F_1 – it gets erased in each.

The presence of the second rule for CL-pc^s is not a problem, because we will only consider terms terminating in CL-pc^s. Whenever the second rule is applicable, so is the first one, hence if all maximal s-reductions end in F_1, then there is an s-reduction ending in F_1 which does not use the second rule for C_2 (Lemma 26). It will be easy to "erase" an s-reduction not using the second rule for C_2 to obtain a reduction in CL-pc^1 (Lemma 7).

- If $q' \equiv \mathsf{K}qq_0 \rightarrow_{\text{CL-pc}^1} q$ then we take $t' \equiv \mathsf{K}_1 t q_0$. The rule for K_1 in CL-pc^s is $\mathsf{K}_1 xy \rightarrow x$.
- If $q' \equiv \mathsf{S}q_1 q_2 q_3 \rightarrow_{\text{CL-pc}^1} q_1 q_3(q_2 q_3) \equiv q$ then we run into a problem with our labeling approach, because the labeled variants of the distinct occurrences of q_3 may be distinct. Suppose t_1 is the labeled variant of q_1, the term t_2 of q_2, the term t_3 of the first q_3, and t'_3 of the second q_3. We cannot just arbitrarily choose e.g. t_3 and say that $\mathsf{S}_1 t_1 t_2 t_3$ is the labeled variant of q', because contracting $\mathsf{S}_1 t_1 t_2 t_3$ yields $t_1 t_3(t_2 t_3)$, not $t_1 t_3(t_2 t'_3)$, and now the second occurrence of q_3 has the wrong labeling.

  A solution is to remember both labeled variants of q_3. So the labeled variant of q' would be e.g. $\mathsf{S}_1 t_1 t_2 \langle t_3, t'_3 \rangle$. In CL-pc^s the rule for S_1 would be

  $$\mathsf{S}_1 x_1 x_2 \langle x_3, x'_3 \rangle \rightarrow x_1 x_3(x_2 x'_3).$$

  However, once we introduce such pairs, terms of the form $\mathsf{S}_1 t_1 t_2 \langle t_3, t'_3 \rangle$ may appear in the terms being expanded. This is not a problem for any of the rules of CL-pc^1 except the rule for S, because the right sides of all other rules are variables.

  Consider for instance $q \equiv q_0 q_3(\mathsf{S}q_1 q_2 q_3) \,_{\text{CL-pc}^1}\!\!\leftarrow \mathsf{S}q_0(\mathsf{S}q_1 q_2)q_3 \equiv q'$. Suppose $t \equiv t_0 t_3(\mathsf{S}_1 t_1 t_2 \langle t_3, t'_3 \rangle)$. Now the term q_3 has three possibly distinct labeled variants, and we need to remember all of them in a tuple. We will thus introduce a new labeling of S for every possible labeling of the right side $xz(yz)$ of the rule for S in the system CL-pc^1.

  By introducing the tuples in the labelings we in essence put constraints on the order in which s-redexes may be contracted (think of all reductions inside a tuple as "really" occuring after the surrounding S-redex is contracted). At present the author does not know a precise "explicit" characterization of these constraints.

Note that by labeling C differently in $\mathsf{CF}q_1 q_2$ and $\mathsf{C}q_0 qq$ we effectively eliminated in CL-pc^s the problematic non-trivial overlap occuring in CL-pc^1. Now a new "insignificant" term created in an expansion cannot later on appear in place of a "significant" term as a result of an "incompatible" contraction. A redex inside an "insignificant" subterm cannot suddenly become needed in an s-reduction – it is erased in any s-reduction to normal form.

We also need to ensure that we can handle the first part of the inductive step when $q \rightarrow_{\text{CL-pc}} q'$. Suppose t is the labeled variant of q. We need to find a labeled variant for q'. For simplicity assume that there is only one position in t which corresponds to the position of the contraction in q. If the contraction occurs inside an i-term in t, then it does not matter and we may label q' in the same way as q. If an s-term is contracted in a way permitted for significant contraction, then it is also obvious how to label q' – just take the labeled variant of q' to be the reduct of the labeled variant of q. But what if neither of the two holds?

For instance, what if $t \equiv \mathsf{C}_1 t_0 t_1 t_2$ but $q \equiv \mathsf{C}q_0 q'q' \rightarrow_{\text{CL-pc}^1} q'$? This possibility is not problematic, provided that $t_0 \rightarrow^*_s \mathsf{T}_1$ or $t_0 \rightarrow^*_s \mathsf{F}_1$, which will be the case because t_0 was

"obtained" from T_1 or F_1 by a conversion with the intermediate terms labeled appropriately (cf. 2 in Definition 8 and 6 in Lemma 9). If e.g. $t_0 \to_s^* \mathsf{T}_1$ then we take t_1 to be the labeling of q'. We then have $\mathsf{C}_1 t_0 t_1 t_2 \to_s^* \mathsf{C}_1 \mathsf{T}_1 t_1 t_2$ and the contraction $\mathsf{C}_1 \mathsf{T}_1 t_1 t_2 \to_s t_1$ is permitted for "significant" contractions.

The last problematic case is when e.g. $t \equiv \mathsf{C}_2 \mathsf{F} t_1 t_2$ is the labeling of $q \equiv \mathsf{CF} q_1 q'$, and $q \to_{\text{CL-pc}} q'$ by the second rule. However, because $\mathsf{C}_2 \mathsf{F} t_1 t_2$ was "obtained" from $\mathsf{C}_2 q t' t'$ we will have $|t_1| =_{\text{CL-pc}^1} |t_2|$ (cf. 3 in Definition 8). Then the second rule for C_2 in CL-pc^s is applicable and we may take t_2 as the labeling of q'.

3   Definitions

This section is devoted to fixing notation and introducing definitions of various technical concepts. First, we clarify the formal definition of conditional term rewriting systems. For more background on conditional rewriting see e.g. [11].

▶ **Definition 1.** A *conditional rewrite rule* is a rule of the form $l \to r \Leftarrow P(x_1, \ldots, x_n)$, where l is not a variable, $\text{Var}(r) \subseteq \text{Var}(l)$, $x_1, \ldots, x_n \in \text{Var}(l)$, and $P(x_1, \ldots, x_n)$ is the *condition* of the rule, with P a fixed predicate on terms. The predicate P may refer to the conversion relation $=$ of the conditional term rewriting system being defined. A term t is a *redex* (*contractum*) by this rule if there is a substitution σ such that $t \equiv \sigma l$ ($t \equiv \sigma r$) and $P(\sigma(x_1), \ldots, \sigma(x_n))$ holds. A *conditional term rewriting system* R is a set of conditional rewrite rules. Because the conditions in the rules may refer to the conversion relation of R, the definition is circular. Formally, an R-contraction $q \to_R q'$ is defined in the following way. Define R_0 to be the system R but using the equality relation in place of $=$ in the conditions, and R_{n+1} to be the system R with the conversion relation $=_{R_n}$ of R_n used in place of $=$. We then define $q \to_R q'$ to hold if there is $n \in \mathbb{N}$ with $q \to_{R_n} q'$. The least such n is called the *level* of the contraction. If the conditions are continuous w.r.t. $=$ then the relation $\to_R$ is a fixpoint of the above construction, i.e., it is the contraction relation of the system R_∞ which uses $=_R$ in place of $=$. Let $\sim$ be a binary relation on terms. If for any substitution σ such that $P(\sigma(x_1), \ldots, \sigma(x_n))$ holds, and any σ' such that $\sigma(x) \sim \sigma'(x)$ for all variables x, also $P(\sigma'(x_1), \ldots, \sigma(x_1'))$ holds, then the condition $P(x_1, \ldots, x_n)$ is *stable under* $\sim$.

The following is a simple but crucial observation, which implies that it suffices to consider conversions in CL-pc. A generalization of this fact was already shown in [4, Lemma 3.7]. The proof is by induction on the maximum level of the contractions/expansions in $q =_{\text{CL-pc}^{\text{L}}} q'$.

▶ **Lemma 2.** *The following are equivalent:* $q =_{\text{CL-pc}} q'$, $q =_{\text{CL-pc}^1} q'$, *and* $q =_{\text{CL-pc}^{\text{L}}} q'$.

▶ **Definition 3.** We define *insignificant terms*, or *i-terms*, to be the terms of CL-pc^1, i.e., terms over the signature $\Sigma = \{@, \mathsf{C}, \mathsf{T}, \mathsf{F}, \mathsf{K}, \mathsf{S}\}$ where @ is a binary function symbol and the other symbols are constants. We write $t_1 t_2$ instead of $@(t_1, t_2)$. The set of *labeled terms*, or *l-terms*, is the set of terms over the signature consisting of the symbols of Σ, the *labeled constants* $\mathsf{C}_1, \mathsf{C}_2, \mathsf{T}_1, \mathsf{F}_1, \mathsf{K}_1$ and $\mathsf{S}^{n_0, \ldots, n_k}$ for each $k, n_1, \ldots, n_k \in \mathbb{N}_+$, and an n-ary function symbol P^n for each $n \in \mathbb{N}_+$. We write $\langle t_1, \ldots, t_n \rangle$ instead of $\mathsf{P}^n(t_1, \ldots, t_n)$. We adopt the convention $\langle t \rangle \equiv t$. If $t \equiv \langle t_1, \ldots, t_n \rangle$ with $n > 1$, then we say that t is a *tuple of length* n. Note that $\langle t \rangle \equiv t$ is just a notational convention. We say that $\langle t_1, \ldots, t_n \rangle$ is a tuple *only when* $n > 1$.

An *erasure* of an *l*-term is defined as follows:

- an *i*-term is an erasure of itself,
- C is an erasure of C_1 and C_2; T is an erasure of T_1; F is an erasure of F_1; K is an erasure of K_1; S is an erasure of $\mathsf{S}^{n_1, \ldots, n_k}$,

- if q_1, q_2 are erasures of t_1, t_2, respectively, then $q_1 q_2$ is an erasure of $t_1 t_2$,
- if q_i is an erasure of t_i, for some $1 \le i \le n$, then q_i is an erasure of $\langle t_1, \ldots, t_n \rangle$.

The *leftmost erasure* of t, denoted $|t|$, is the erasure in which we always choose $i = 1$ in the last point above. We write $t \succ q$ if *every* erasure of t is identical with q.

We define *significant terms*, or *s-terms*, inductively.

- Any labeled constant is an *s*-term.
- If t_1 is an *s*-term and t_2 is an *l*-term, then $t_1 t_2$ is an *s*-term.

In other words, an *s*-term is an *l*-term whose leftmost constant is labeled.

In what follows t, t_1, t_2, r, r_1, r_2, s, s_1, etc. stand for *l*-terms; and q, q_1, q_2, etc. stand for *i*-terms; unless otherwise qualified. Also, whenever we talk about terms without further qualification, we implicitly assume them to be *l*-terms.

▶ **Definition 4.** The system CL-pc^s is defined by the following *significant reduction rules*:

$$
\begin{array}{llll}
\mathsf{C}_1\mathsf{T}_1 xy & \to & x & \qquad \mathsf{C}_2 zxy \; \to \; x \; \Leftarrow \; |x| =_{\text{CL-pc}^1} |y| \\
\mathsf{C}_1\mathsf{F}_1 xy & \to & y & \qquad \mathsf{C}_2 zxy \; \to \; y \; \Leftarrow \; |x| =_{\text{CL-pc}^1} |y| \\
\mathsf{K}_1 xy & \to & x &
\end{array}
$$

$$
\mathsf{S}^{\vec{n}} x \langle y_1, \ldots, y_k \rangle \langle \vec{z}_0, \ldots, \vec{z}_k \rangle \quad \to \quad x\langle \vec{z}_0 \rangle \langle (y_1 \langle \vec{z}_1 \rangle), \ldots, (y_k \langle \vec{z}_k \rangle) \rangle \quad \Leftarrow \quad \varphi
$$

where

$$
\begin{aligned}
\varphi \quad \equiv \quad & |z_{i,j}| =_{\text{CL-pc}^1} |z_{i',j'}| \text{ for } i, i' = 0, \ldots, k, \ j = 1, \ldots, n_i, \ j' = 1, \ldots, n_{i'}, \text{ and} \\
& |y_i| =_{\text{CL-pc}^1} |y_j| \text{ for } i, j = 1, \ldots, k,
\end{aligned}
$$

and $\vec{n}$ stands for $n_0, \ldots, n_k$, and $\vec{z}_i$ stands for $z_{i,1}, \ldots, z_{i,n_i}$, for $i = 0, \ldots, k$. When dealing with terms whose leftmost constant is $\mathsf{S}^{n_0, \ldots, n_k}$, we will often use this kind of vector notation. Recall the convention $\langle t \rangle \equiv t$. Hence, if e.g. $n_0 = 1$, then $\langle \vec{z}_0 \rangle \equiv \langle z_{0,1} \rangle \equiv z_{0,1}$ in the above rule. The condition φ ensures that the leftmost erasures of all $z_{i,j}$ are convertible in CL-pc^1, and that the leftmost erasures of all y_i are convertible in CL-pc^1. Some examples of significant reduction rules for $\mathsf{S}^{\vec{n}}$ (omitting the conditions) are:

$$
\begin{aligned}
\mathsf{S}^{1,1} xy_1 \langle z_{0,1}, z_{1,1} \rangle & \to & xz_{0,1}(y_1 z_{1,1}) \\
\mathsf{S}^{1,2,1} x \langle y_1, y_2 \rangle \langle z_{0,1}, z_{1,1}, z_{1,2}, z_{2,1} \rangle & \to & xz_{0,1}\langle y_1 \langle z_{1,1}, z_{1,2} \rangle, y_2 z_{1,2} \rangle \\
\mathsf{S}^{2,2} xy_1 \langle z_{0,1}, z_{0,2}, z_{1,1}, z_{1,2} \rangle & \to & x\langle z_{0,1}, z_{0,2} \rangle (y_1 \langle z_{1,1}, z_{1,2} \rangle)
\end{aligned}
$$

For instance, the condition for the second of these rules states that $|y_1| =_{\text{CL-pc}^1} |y_2|$, $|z_{0,1}| =_{\text{CL-pc}^1} |z_{1,1}|$, $|z_{0,1}| =_{\text{CL-pc}^1} |z_{1,2}|$, $|z_{0,1}| =_{\text{CL-pc}^1} |z_{2,1}|$, $|z_{1,1}| =_{\text{CL-pc}^1} |z_{1,2}|$, etc.

Note that the equality $=_{\text{CL-pc}^1}$ in the conditions refers to the system CL-pc^1, not CL-pc^s. Note also that all rules of CL-pc^s are linear, disregarding the side-conditions.

Reduction by a rule in CL-pc^s is called *significant reduction*, or *s-reduction*. One-step *s*-reduction is denoted by $\to_s$. Analogously, we use the terminology and notation of *s*-contraction, *s*-expansion, *s*-redex, *s*-normal form (*s*-NF), etc. Note that every *s*-redex is an *s*-term. We write $t \to_{s-} t'$ if $t \to_s t'$ and the *s*-contraction is not by the second rule for C_2 and it does not occur inside a tuple.

An *i-redex* is a CL-pc^1-redex which is also an *i*-term. An *l*-term t_1 is said to *i-reduce* to t_2, denoted $t_1 \to_i t_2$, if $t_1 \to_{\text{CL-pc}^1} t_2$ and the redex contracted in t_1 is an *i*-term. An *l*-term t_1 is said to *i-expand* to t_2 if $t_2 \to_i t_1$. We write $t_1 \to_{i,s} t_2$ if $t_1 \to_i t_2$ or $t_1 \to_s t_2$.

Actually, we will consider mostly *l*-terms whose all erasures are identical. For such a term an *s*-contraction by a rule for $\mathsf{S}^{\vec{n}}$ in CL-pc^s naturally corresponds to a CL-pc^1-contraction on its erasure. We could get rid of the side conditions in the rules for $\mathsf{S}^{\vec{n}}$ and

consider exclusively terms whose all erasures are identical. But then we would need to require i/s-contractions/expansions to always occur "in the same way" (modulo labeling) in all components of a tuple. This would complicate the inductive proofs concerning the relations $\to_s$, $\to_i$, etc. Hence, the role of the conditions in the rules for $\mathsf{S}^{\vec{n}}$ is purely technical.

▶ **Lemma 5.** *The system* $\mathrm{CL\text{-}pc}^s$ *is terminating.*

Proof. The number of labeled constants decreases with each s-contraction. ◄

▶ **Lemma 6.** *If* $t_1 \to_s t_2$ *then* $|t_1| =_{\mathrm{CL\text{-}pc}^1} |t_2|$.

The above simple lemma implies that the conditions in significant reduction rules are stable under s-reduction and s-expansion. It is obvious that they are also stable under i-reduction and i-expansion.

▶ **Lemma 7.** *If* $t \succ q$ *and* $t \to_{s-} t'$ *then there is* q' *with* $q \to_{\mathrm{CL\text{-}pc}^1} q'$ *and* $t' \succ q'$.

Proof. Because all erasures of t are identical and the second rule for C_2 is not used, the $s-$-reduction may be simulated by a $\mathrm{CL\text{-}pc}^1$-reduction in an obvious way. Because the $s-$-contraction does not occur inside a tuple, all erasures of t' are still identical. ◄

In the next definition we introduce the predicate $\Downarrow_{\mathsf{F}_1}$ and the notion of standard l-terms. Intuitively, an l-term t is standard if the labelings in t have the meaning we intend to assign them, i.e. if t is a term obtained by the process informally described in the previous section.

▶ **Definition 8.** An l-term t is *standard* if for every subterm t' of t the following hold:
1. t' is either an i-term, an s-term or a tuple,
2. if $t' \equiv \mathsf{C}_1 t_0 t_1 t_2$ and t_0 is in s-NF, then $t_0 \equiv \mathsf{T}_1$ or $t_0 \equiv \mathsf{F}_1$,
3. if $t' \equiv \mathsf{C}_2 t_0 t_1 t_2$ then $|t_1| =_{\mathrm{CL\text{-}pc}^1} |t_2|$,
4. if $t' \equiv \mathsf{S}^{n_0,\ldots,n_k} t_0 t_1 t_2$ then t_2 is a tuple of length $\sum_{i=0}^{k} n_k$ and if $k > 1$ then t_1 is a tuple of length k,
5. if t' is an s-term and $t' \to_s^* t''$, then t'' is also an s-term,
6. if $t' \equiv \langle t_1, \ldots, t_n \rangle$ with $n > 1$, then none of $t_1, \ldots, t_n$ is a tuple.

An l-term t is *strongly standard* if $t \to_s^* t'$ implies that t' is standard. We write $t \Downarrow_{\mathsf{F}_1}$ if t is strongly standard and has no s-NFs other than F_1, i.e. if $t \to_s^! t'$ then $t' \equiv \mathsf{F}_1$.

Point 1 in Definition 8 essentially ensures that a standard term may be decomposed into a "significant" prefix and an "insignificant" suffix. A labeled term which is not standard is e.g. CT_1, because it is neither an s-term, nor an i-term, nor a tuple. Other examples of non-standard terms are: $\mathsf{C}_1\mathsf{TFF}$, $\mathsf{C}_2\mathsf{CTF}$, $\mathsf{S}^{1,1,1}\mathsf{TTT}$, $\mathsf{K}_1\mathsf{FF}$, $\mathsf{C}_2\mathsf{CT}_1\mathsf{T}$, $\mathsf{S}^{1,1}\mathsf{CC}\langle \mathsf{T}_1, \mathsf{T}_1 \rangle$, $\mathsf{K}_1\langle \mathsf{T}_1, \mathsf{T}_1 \rangle \mathsf{T}_1$, $\langle \langle \mathsf{C}, \mathsf{C} \rangle, \mathsf{C} \rangle$. Examples of standard terms which are not strongly standard are: $\mathsf{S}^{1,1}\mathsf{C}_1\mathsf{C}\langle \mathsf{T}_1, \mathsf{T}_1 \rangle$, $\mathsf{S}^{1,1}\mathsf{C}_1\mathsf{C}\langle \mathsf{T}, \mathsf{T} \rangle \mathsf{T}$, $(\mathsf{K}_1\mathsf{C}_1\mathsf{T})\mathsf{TTT}$.

▶ **Lemma 9.**
1. *Any i-term is standard.*
2. *Any labeled constant is standard.*
3. *Every subterm of a standard term is also standard.*
4. *Every subterm of a term to which some strongly standard term s-reduces, is strongly standard.*
5. *If $t_1 t_2$ is standard then t_1 is not a tuple.*
6. *If $\mathsf{C}_1 t_0 t_1 t_2$ is a subterm of a strongly standard term, then $t_0 \to_s^* \mathsf{T}_1$ or $t_0 \to_s^* \mathsf{F}_1$.*

Proof. Follows from definitions. For the last point one also needs Lemma 5. ◄

4 Confluence proof

We now give technical details of our confluence proof. As outlined in Section 2, we show:

1. if $t \succ q$ and $t \Downarrow_{\mathsf{F}_1}$, and $q \to_{\mathrm{CL\text{-}pc}} q'$, then there is t' with $t' \succ q'$ and $t' \Downarrow_{\mathsf{F}_1}$ (Corollary 15),
2. if $t \succ q$ and $t \Downarrow_{\mathsf{F}_1}$, and $q \;_{\mathrm{CL\text{-}pc}}\!\!\leftarrow q'$, then there is t' with $t' \succ q'$ and $t' \Downarrow_{\mathsf{F}_1}$ (Corollary 25).

The first part is proven by showing that CL-pc-reductions in q may be simulated by i-reductions and s-reductions in t, and that i/s-reductions preserve $\Downarrow_{\mathsf{F}_1}$ (Lemma 13 and Lemma 14). For the second part, we show that CL-pc-expansions in q may be simulated by i-expansions and a-expansions (Definition 16) in t. The technical notion of a-expansion is needed to ensure that the new subterms of t' are labeled appropriately, in the way outlined in Section 2 (s-contraction by itself does not put any labeling restrictions on the terms erased in the contraction). Moreover, a-expansion is also needed to facilitate the proof that $t' \Downarrow_{\mathsf{F}_1}$ (see the discussion before Definition 16). Plain s-expansion does not necessarily preserve $\Downarrow_{\mathsf{F}_1}$, while a-expansion does (Lemma 24).

In other words, we show that CL-pc-reductions (expansions) in unlabeled terms may be simluated by i/s-reductions (i/a-expansions) in their labeled variants, and that i/s-reductions (i/a-expansions) preserve $\Downarrow_{\mathsf{F}_1}$. A conversion $q =_{\mathrm{CL\text{-}pc}} \mathsf{F}$ can then be translated into a conversion $t =_{i,s,a} \mathsf{F}_1$ with no s-expansions or a-reductions, and with $t \succ q$. For instance, a conversion in CL-pc

$$\mathsf{F} \leftarrow \mathsf{C}(\mathsf{KF}\Omega)\mathsf{FF} \leftarrow \mathsf{C}(\mathsf{KF}\Omega)\mathsf{F}(\mathsf{CTF}(\mathsf{KF}\Omega)) \to \mathsf{CFF}(\mathsf{CTF}(\mathsf{KF}\Omega)) \leftarrow$$
$$\mathsf{CFF}(\mathsf{C}(\mathsf{KTF})\mathsf{F}(\mathsf{KF}\Omega)) \to \mathsf{C}(\mathsf{KTF})\mathsf{F}(\mathsf{KF}\Omega) \to \mathsf{C}(\mathsf{KTF})\mathsf{FF} \to \mathsf{F} \leftarrow$$
$$\mathsf{KF}(\mathsf{CF}) \leftarrow \mathsf{SKCF} \leftarrow \mathsf{SKC}(\mathsf{KF}\Omega) \to \mathsf{SKCF}$$

will be translated to

$$\mathsf{F}_1 \;_a\!\!\leftarrow \mathsf{C}_2(\mathsf{KF}\Omega)\mathsf{F}_1\mathsf{F}_1 \;_a\!\!\leftarrow \mathsf{C}_2(\mathsf{KF}\Omega)\mathsf{F}_1(\mathsf{C}_1\mathsf{T}_1\mathsf{F}_1(\mathsf{KF}\Omega)) \to_i \mathsf{C}_2\mathsf{FF}_1(\mathsf{C}_1\mathsf{T}_1\mathsf{F}_1(\mathsf{KF}\Omega)) \;_a\!\!\leftarrow$$
$$\mathsf{C}_2\mathsf{FF}_1(\mathsf{C}_1(\mathsf{K}_1\mathsf{T}_1\mathsf{F})\mathsf{F}_1(\mathsf{KF}\Omega)) \to_s \mathsf{C}_1(\mathsf{K}_1\mathsf{T}_1\mathsf{F})\mathsf{F}_1(\mathsf{KF}\Omega) \to_i \mathsf{C}_1(\mathsf{K}_1\mathsf{T}_1\mathsf{F})\mathsf{F}_1\mathsf{F} \to_s^* \mathsf{F}_1 \;_a\!\!\leftarrow$$
$$\mathsf{K}_1\mathsf{F}_1(\mathsf{CF}) \;_a\!\!\leftarrow \mathsf{S}^{1,1}\mathsf{K}_1\mathsf{C}\langle\mathsf{F}_1,\mathsf{F}\rangle \;_{a,i}^*\!\!\leftarrow \mathsf{S}^{1,1}\mathsf{K}_1\mathsf{C}\langle\mathsf{K}_1\mathsf{F}_1\Omega,\mathsf{KF}\Omega\rangle \to_{s,i}^* \mathsf{S}^{1,1}\mathsf{K}_1\mathsf{C}\langle\mathsf{F}_1,\mathsf{F}\rangle$$

Since $\mathsf{F}_1 \Downarrow_{\mathsf{F}_1}$ and we prove that i/s-reductions and i/a-expansions preserve $\Downarrow_{\mathsf{F}_1}$, we may conclude that $t \Downarrow_{\mathsf{F}_1}$. Then by the definition of $\Downarrow_{\mathsf{F}_1}$ we obtain a significant reduction $t \to_s^* \mathsf{F}_1$. In fact, the reduction may be assumed to be a $s-$-reduction (Lemma 26). By Lemma 7 this reduction $t \to_{s-}^* \mathsf{F}_1$ may be translated into a CL-pc[1]-reduction by erasing the labelings. Hence finally $q \to_{\mathrm{CL\text{-}pc}^1}^* \mathsf{F}$ (Lemma 27).

We first show that a CL-pc-contraction may be simulated by i-reductions and s-reductions.

▶ **Lemma 10.** *If t is strongly standard, $t \succ q$ and $q \to_{\mathrm{CL\text{-}pc}} q'$, then there exists a term t' such that $t \to_{i,s}^* t'$ and $t' \succ q'$.*

Proof. Induction on the size of t. First assume t is not a tuple and q is the CL-pc-redex contracted in $q \to_{\mathrm{CL\text{-}pc}} q'$. If $t \equiv q$ then $t \equiv q \to_i q'$ and we may take $t' \equiv q'$. If $t \not\equiv q$ then t is not an i-term because $t \succ q$. Hence by 1 in Definition 8 we conclude that t is an s-term. We have the following possibilities.

- If $q \equiv \mathsf{CT}q_1q_2 \to_{\mathrm{CL\text{-}pc}} q_1 \equiv q'$ then the leftmost constant in t is either C_1 or C_2.
  - If $t \equiv \mathsf{C}_1\mathsf{T}_1 t_1 t_2$ then $t \to_s t_1$ and $t_1 \succ q_1$, so we may take $t' \equiv t_1$.
  - The case $t \equiv \mathsf{C}_1\mathsf{T}t_1t_2$ is impossible by 2 in Definition 8.
  - If $t \equiv \mathsf{C}_2 t_0 t_1 t_2$ then $t_1 \succ q_1$ and $|t_1| =_{\mathrm{CL\text{-}pc}^1} |t_2|$ by 3 in Definition 8. Thus $t \to_s t_1$ and we may take $t' \equiv t_1$.
- If $q \equiv \mathsf{CF}q_1q_2 \to_{\mathrm{CL\text{-}pc}} q_2$ then the argument is analogous. Note that the presence of the second rule for C_2 is necessary here.

- If $q \equiv \mathsf{C}q_0q_1q_1 \to_{\text{CL-pc}} q_1$ then $t \equiv \mathsf{C}'t_0t_1t_2$ with $\mathsf{C}' \in \{\mathsf{C}_1, \mathsf{C}_2\}$, $t_0 \succ q_0$, $t_1 \succ q_1$ and $t_2 \succ q_1$.
  - If $\mathsf{C}' \equiv \mathsf{C}_1$ then $t_0 \to_s^* \mathsf{T}_1$ or $t_0 \to_s^* \mathsf{F}_1$ by Lemma 9. Hence $t \to_s^* t_1$ or $t \to_s^* t_2$. In the first case we may take $t' \equiv t_1$, and in the second we take $t' \equiv t_2$.
  - If $\mathsf{C}' \equiv \mathsf{C}_2$ then $t \to_s t_1$ because $|t_1| \equiv |t_2| \equiv q_1$. Thus we take $t' \equiv t_1$.
- If $q \equiv \mathsf{K}q_1q_2 \to_{\text{CL-pc}} q_1$ then $t \equiv \mathsf{K}_1t_1t_2 \to_s t_1$ with $t_1 \succ q_1$. We take $t' \equiv t_1$.
- If $q \equiv \mathsf{S}q_0q_1q_2 \to_{\text{CL-pc}} q_0q_2(q_1q_2)$ then $t \equiv \mathsf{S}^{\vec{n}}s\langle t_1, \ldots, t_k\rangle\langle \vec{r}_0, \ldots, \vec{r}_k\rangle$ where the conventions regarding the vector notation are as in Definition 4, and $s \succ q_0$, and $t_i \succ q_1$ for $i = 1, \ldots, k$, and $r_{i,j} \succ q_2$ for $i = 0, \ldots, k$, $j = 1, \ldots, i$. Thus

$$t \to_s s\langle \vec{r}_0\rangle\langle t_1\langle \vec{r}_1\rangle\rangle, \ldots, t_k\langle \vec{r}_k\rangle\rangle \succ q_0q_2(q_1q_2)$$

and we may take $t' \equiv s\langle \vec{r}_0\rangle\langle t_1\langle \vec{r}_1\rangle\rangle, \ldots, t_k\langle \vec{r}_k\rangle\rangle$.

If t is a tuple or q is not the contracted CL-pc-redex, then the claim follows from the inductive hypothesis. ◄

The following technical lemma shows that $\leftrightarrow_i$ may be postponed after $\to_s$.

▶ **Lemma 11.** *If* $t \leftrightarrow_i \cdot \to_s^* t'$ *then* $t \to_s^* \cdot \leftrightarrow_i^{\overline{\equiv}} t'$.

Proof. Suppose $t_1 \leftrightarrow_i t_2 \to_s t_3$. We proceed by induction on the definition of $t_2 \to_s t_3$.

If t_2 is the contracted s-redex then, because an i-redex (i-contractum) is an i-term, it is easy to see by inspecting Definition 4 that the i-redex (i-contractum) in t_2 must occur below a variable position of the s-redex. Since significant reduction rules are linear and their conditions are stable under i-reductions (i-expansions), the claim holds. Note that we need $\leftrightarrow_i^{\overline{\equiv}}$ instead of $\leftrightarrow_i$ in the conclusion, because the i-redex (i-contractum) may be erased by the s-contraction.

If t_2 is not the s-redex, then $t_2 \equiv s_1s_2$ or $t_2 \equiv \langle s_1, \ldots, s_n\rangle$ with $n > 1$, and the claim is easily established possibly appealing to the inductive hypothesis. ◄

The next lemmas show that i-reductions/expansions and s-reductions preserve $\Downarrow_{\mathsf{F}_1}$.

▶ **Lemma 12.** *If* t *is standard and* $t \leftrightarrow_i t'$ *then* t' *is standard.*

Proof. We check that the conditions in Definition 8 hold for every subterm s' of t'. Note that because i-redexes and i-contracta are i-terms, s' is an i-term or there is a subterm s of t such that $s \leftrightarrow_i^{\overline{\equiv}} s'$.

1. If s' is not an i-term, then there is a subterm s of t such that $s \leftrightarrow_i^{\overline{\equiv}} s'$. If s is an i-term or a tuple then so is s'. Otherwise, s is an s-term by 1 in Definition 8. Then s' is also an s-term.
2. Suppose $s' \equiv \mathsf{C}_1t_0't_1't_2'$ with t_0' in s-NF. Since s' is not an i-term, there is a subterm s of t such that $s \equiv \mathsf{C}_1t_0t_1t_2$ and $t_i \leftrightarrow_i^{\overline{\equiv}} t_i'$ for $i = 0, 1, 2$. Since t_0' is in s-NF and $t_0 \leftrightarrow_i^{\overline{\equiv}} t_0'$, the term t_0 is also in s-NF. Thus $t_0 \equiv \mathsf{T}_1$ or $t_0 \equiv \mathsf{F}_1$ by 2 in Definition 8. Hence $t_0' \equiv \mathsf{T}_1$ or $t_0' \equiv \mathsf{F}_1$.
3. Suppose $s' \equiv \mathsf{C}_2t_0't_1't_2'$. Since s' is not an i-term, there is a subterm s of t such that $s \equiv \mathsf{C}_1t_0t_1t_2$ and $t_i \leftrightarrow_i^{\overline{\equiv}} t_i'$ for $i = 0, 1, 2$. By 3 in Definition 8 we have $|t_1| =_{\text{CL-pc}^1} |t_2|$. Hence also $|t_1'| =_{\text{CL-pc}^1} |t_2'|$, because $t_i \leftrightarrow_i^{\overline{\equiv}} t_i'$ implies $|t_i| =_{\text{CL-pc}^1} |t_i'|$.
4. Suppose $s' \equiv \mathsf{S}^{n_0, \ldots, n_k}t_0't_1't_2'$. Since s' is not an i-term, there is a subterm s of t such that $s \equiv \mathsf{S}^{n_0, \ldots, n_k}t_0t_1t_2$ and $t_i \leftrightarrow_i^{\overline{\equiv}} t_i'$ for $i = 0, 1, 2$. By 4 in Definition 8 we conclude that t_2 is a tuple of length $n = \sum_{i=0}^k n_i$, and if $k > 1$ then t_1 is a tuple of length k. The same holds for t_2' and t_1', because a tuple cannot be an i-redex or an i-contractum.

5. Suppose s' is an s-term. There is a subterm s of t such that $s \leftrightarrow_{\bar{i}}^{\equiv} s'$. Since s' is an s-term, so is s. Suppose $s' \rightarrow_s^* r'$. By Lemma 11 there is r such that $s \rightarrow_s^* r \leftrightarrow_{\bar{i}}^{\equiv} r'$. By 5 in Definition 8, the term r is an s-term. Hence, r' is also an s-term.

6. Suppose $s' \equiv \langle t_1', \ldots, t_n' \rangle$ with $n > 1$. Since s' is not an i-term, there is a subterm s of t such that $s \equiv \langle t_1, \ldots, t_n \rangle$ and $t_i \leftrightarrow_{\bar{i}}^{\equiv} t_i'$ for $i = 1, \ldots, n$. By 6 in Definition 8 none of $t_1, \ldots, t_n$ is a tuple. Hence, none of $t_1', \ldots, t_n'$ is a tuple either.   ◄

▶ **Lemma 13.** *If $t \Downarrow_{\mathsf{F}_1}$ and $t \leftrightarrow_i t'$ then $t' \Downarrow_{\mathsf{F}_1}$.*

Proof. Suppose $t' \rightarrow_s^* t_0'$. By Lemma 11 there is t_0 with $t \rightarrow_s^* t_0$ and $t_0 \leftrightarrow_{\bar{i}}^{\equiv} t_0'$. Because t is strongly standard, t_0 is standard. Hence t_0' is standard by Lemma 12. Therefore t' is strongly standard.

Suppose $t' \rightarrow_s^* t_0'$ with t_0' in s-NF. By Lemma 11 there is t_0 with $t \rightarrow_s^* t_0 \leftrightarrow_{\bar{i}}^{\equiv} t_0'$. Since t_0' is in s-NF, so is t_0, because an i-contraction or an i-expansion cannot create an s-redex. Since $t \Downarrow_{\mathsf{F}_1}$ we obtain $t_0 \equiv \mathsf{F}_1$. Thus $t_0' \equiv t_0 \equiv \mathsf{F}_1$.   ◄

▶ **Lemma 14.** *If $t \Downarrow_{\mathsf{F}_1}$ and $t \rightarrow_s t'$ then $t' \Downarrow_{\mathsf{F}_1}$.*

▶ **Corollary 15.** *If $t \Downarrow_{\mathsf{F}_1}$, $t \succ q$ and $q \rightarrow_{\mathrm{CL\text{-}pc}} q'$ then there is t' with $t' \succ q'$ and $t' \Downarrow_{\mathsf{F}_1}$.*

Proof. Follows from Lemma 10, Lemma 13 and Lemma 14.   ◄

With the above corollary we have finished the first half of the proof. Now we need to show an analogous corollary for CL-pc-expansions. First, we want to prove that CL-pc-expansions in unlabeled terms may be simulated by i-expansions and a-expansions in their strongly standard labeled variants. We have already shown in Lemma 13 that i-expansions preserve $\Downarrow_{\mathsf{F}_1}$. We need to show that a-expansions also preserve $\Downarrow_{\mathsf{F}_1}$.

One trivial reason why s-expansions do not necessarily preserve $\Downarrow_{\mathsf{F}_1}$ is that if $t \,_s\!\!\leftarrow t'$ then t' may be not standard even if t is, e.g., consider $\mathsf{F}_1 \,_s\!\!\leftarrow \mathsf{K}_1\mathsf{F}_1(\mathsf{CT}_1)$. A more profound reason is that with s-expansion we do not sufficiently "control" the expansion by a rule for C_2. E.g. $\mathsf{F}_1 \,_s\!\!\leftarrow \mathsf{C}_2\Omega\mathsf{F}_1(\mathsf{KF}\Omega)$. Then $\mathsf{C}_2\Omega\mathsf{F}_1(\mathsf{KF}\Omega) \rightarrow_s \mathsf{KF}\Omega$ but $\mathsf{KF}\Omega$ does not s-reduce to F_1.

Hence, we use a-expansions which put additional restrictions on the s-redexes, essentially implementing the labeling of expansions described in Section 2. They also allow to "delay" the reductions in a contractum of $\mathsf{C}_2 t_0 t_1 t_1$ to facilitate the proof of an analogon of Lemma 11.

Like in the proof of Lemma 13 we show that if $t' \rightarrow_a t$ then any reduction $t' \rightarrow_s^* s'$ may be simulated by a reduction $t \rightarrow_s^* s$ with $s' \rightarrow_a^{\equiv} s$. The most interesting case is when $t' \equiv E[\mathsf{C}_2 t_0 t_1 t_1] \rightarrow_a E[t_1] \equiv t$ (where E is a context), which is obtained from a CL-pc-expansion by the rule $\mathsf{C}xyy \rightarrow y$. We now informally describe the idea for the proof in this case. Thus suppose $t' \rightarrow_s^* s'$. If a contracted s-redex does not overlap with a descendant[5] of $\mathsf{C}_2 t_0 t_1 t_1$, then the s-reduction is simulated by the same s-reduction. If a descendant of $\mathsf{C}_2 t_0 t_1 t_1$ occurs inside a contracted s-redex, but it is different from this redex, then the descendant must occur below a variable position of the s-redex, because there are no non-root overlaps between the rules of significant reduction. Thus we may simulate this s-reduction by the same s-reduction. If a contracted s-redex occurs inside a descendant $\mathsf{C}_2 t_0' t_1' t_2'$ of $\mathsf{C}_2 t_0 t_1 t_1$, but it is different from this descendant, then it must occur in t_0', t_1' or t_2'. In this case we ignore the s-contraction while at all times maintaining the invariant: if $\mathsf{C}_2 t_0' t_1' t_2'$ is a descendant of $\mathsf{C}_2 t_0 t_1 t_1$ then $t_1 \rightarrow_s^* t_1'$ and $t_1 \rightarrow_s^* t_2'$, and the descendant of t_1 in the simulated reduction is always identical with t_1, i.e. t_1 (the a-contractum of $\mathsf{C}_2 t_0 t_1 t_1$) is not changed by

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[5] Note that because the rules of significant reduction are linear there may be at most one descendant.

the simulated s-reduction. Finally, if a descendant $C_2 t_0' t_1' t_2'$ of $C_2 t_0 t_1 t_1$ is s-contracted, then either $C_2 t_0' t_1' t_2' \to_s t_1'$ or $C_2 t_0' t_1' t_2' \to_s t_2'$. In any case we can s-reduce t_1 to t_1' or t_2'. In other words, we defer the choice of the simulated reduction path till the descendant of the a-redex is actually contracted.

▶ **Definition 16.** An l-term t' is an a-redex and t its a-contractum, if t is an s-term and one of the following holds:

- $t' \equiv C_1 T_1 tq$ and q is an i-term,
- $t' \equiv C_1 F_1 qt$ and q is an i-term,
- $t' \equiv C_2 q t_1 t_2$, $t \to_s^* t_1$, $t \to_s^* t_2$ and q is an i-term,
- $t' \equiv K_1 tq$ and q is an i-term,
- $t' \equiv S^{\vec{n}} t_0 \langle s_1, \ldots, s_k \rangle \langle \vec{r}_0, \ldots, \vec{r}_k \rangle$ where the conventions regarding vector notation are as in Definition 4, $|s_i| =_{\text{CL-pc}^1} |s_j|$ for $i, j = 1, \ldots, k$, $|r_{i,j}| =_{\text{CL-pc}^1} |r_{i',j'}|$ for $i, i' = 0, \ldots, k$, $j = 1, \ldots, n_i$, $j' = 1, \ldots, n_{i'}$, none of the s_i or $r_{i,j}$ is a tuple, and $t \equiv t_0 \langle \vec{r}_0 \rangle \langle s_1 \langle \vec{r}_1 \rangle, \ldots, s_k \langle \vec{r}_k \rangle \rangle$.

Because of the third point, an a-contractum of an a-redex is not unique. The notations $\to_a$, $\to_a^*$, $\to_{i,a}$, etc. are used accordingly. Note that any a-redex is an s-redex.

▶ **Lemma 17.** If $t' \to_a t$ then $t' \to_s \cdot {}_s^* \leftarrow t$, and hence $|t'| =_{\text{CL-pc}^1} |t|$.

The above simple lemma implies that the conditions in significant reduction rules are stable under a-reduction and a-expansion. Note that if $t' \to_a t$ then not necessarily $t' \to_s t$ because of the third point in Definition 16.

▶ **Lemma 18.** If t is standard, $t \succ q$ and $q _{\text{CL-pc}} \leftarrow q'$ then there is t' with $t' \to_{i,a}^* t$ and $t' \succ q'$.

Proof. Induction on the size of t. First assume t is not a tuple and q is the CL-pc-contractum expanded in $q _{\text{CL-pc}} \leftarrow q'$. If t is an i-term, then $t \equiv q _i \leftarrow q'$ and we may take $t' \equiv q'$. If t is not an i-term, then it is an s-term by 1 in Definition 8. We have the following possibilities, depending on the rule of CL-pc used in the expansion.

- If $q' \equiv CT q_1 q_2 \to_{\text{CL-pc}} q_1 \equiv q$ then we take $t' \equiv C_1 T_1 t q_2$ and we have $t' \to_a t$ and $t' \succ q'$.
- If $q' \equiv CF q_1 q_2 \to_{\text{CL-pc}} q_2 \equiv q$ then we may take $t' \equiv C_1 F_1 q_1 t$.
- If $q' \equiv C q_0 q_1 q_1 \to_{\text{CL-pc}} q_1$ then we may take $t' \equiv C_2 q_0 tt$.
- If $q' \equiv K q_0 q_1 \to_{\text{CL-pc}} q_0$ then we may take $t' \equiv K_1 t q_1$.
- If $q' \equiv S q_0 q_1 q_2 \to_{\text{CL-pc}} q_0 q_2 (q_1 q_2)$ then $t \succ q_0 q_2 (q_1 q_2)$ and t is an s-term. Hence $t \equiv t_a t_b t_c$ with $t_a \succ q_0$, $t_b \succ q_2$ and $t_c \succ q_1 q_2$. Recalling the convention $\langle s \rangle \equiv s$ for any term s, we may assume

$$t_b \equiv \langle s_1, \ldots, s_m \rangle, t_c \equiv \langle t_1, \ldots, t_k \rangle, \text{ for } k, m \in \mathbb{N}_+,$$

$$\text{if } k = 1 \text{ then } t_1 \text{ is not a tuple, and if } m = 1 \text{ then } s_1 \text{ is not a tuple.} \qquad (\star)$$

In other words, if e.g. t_b is a tuple, then $t_b \equiv \langle s_1, \ldots, s_m \rangle$ for some $s_1, \ldots, s_m$. If t_b is not a tuple then we take $s_1 \equiv t_b$ and consider $t_b \equiv \langle t_b \rangle \equiv \langle s_1 \rangle$. This is chiefly to reduce the number of cases to consider. Let $1 \le i \le k$. Because $t_b \succ q_2$, we have $s_i \succ q_2$ for $i = 1, \ldots, m$. Also none of $s_1, \ldots, s_m$ is a tuple, by condition 6 in Definition 8, or by $(\star)$ if $m = 1$. Since $t_c \succ q_1 q_2$, we have $t_i \succ q_1 q_2$. Also t_i cannot be a tuple, by condition 6 in Definition 8, or by $(\star)$ if $k = 1$. Thus $t_i \equiv u_i \langle \vec{r}_i \rangle$ where $\vec{r}_i$ stands for $r_{i,1}, \ldots, r_{i,n_i}$, and $u_i \succ q_1$ and $r_{i,j} \succ q_2$ for $j = 1, \ldots, n_i$, where none of the $r_{i,j}$ is a tuple, by definition (if $n_i = 1$) or by condition 6 in Definition 8. By Lemma 9 also none of $u_1, \ldots, u_k$ is a tuple. We may thus take $t' \equiv S^{m,n_1,\ldots,n_k} t_a \langle u_1, \ldots, u_k \rangle \langle \vec{r}_0, \vec{r}_1, \ldots, \vec{r}_k \rangle$ where $\vec{r}_0$ stands for $s_1, \ldots, s_m$. We have $t' \to_a t$ and $t' \succ q'$.

If t is a tuple or q is not the CL-pc-contractum, then the claim follows from the inductive hypothesis. ◀

▶ **Lemma 19.** *If $t \, {}_a\!\leftarrow \cdot \rightarrow_s t'$ and t is standard then $t \rightarrow_s^* \cdot \, {}_a^{\equiv}\!\leftarrow t'$.*

Proof. Suppose $t' \rightarrow_s t'_1$, $t' \rightarrow_a t$ and t is standard. By induction on the definition of $t' \rightarrow_s t'_1$ we show that there is t_1 with $t \rightarrow_s^* t_1$ and $t'_1 \rightarrow_a^{\equiv} t_1$. The base case is when the s-contraction in $t' \rightarrow_s t'_1$ occurs at the root.

If the s-contraction occurs at the root, but the a-contraction in $t' \rightarrow_a t$ does not occur at the root, then it is easy to see by inspecting the definitions that the a-redex in t'_1 must occur below a variable position of the s-redex. Since significant reduction rules are linear and their conditions are stable under a-reduction, the claim holds in this case.

Assume that both the s-contraction and the a-contraction occur at the root. If $t' \equiv \mathsf{C}_2 q s_1 s_2 \rightarrow_a t$ then $t \rightarrow_s^* s_1$, $t \rightarrow_s^* s_2$ and the s-contraction of t' yields either s_1 or s_2. We may thus take either $t_1 \equiv s_1$ or $t_1 \equiv s_2$, and we have $t \rightarrow_s^* t_1 \equiv t'_1$. If $t' \equiv \mathsf{C}_1 \mathsf{T}_1 tq \rightarrow_a t$ then the s-contraction must be by the first rule of CL-pc^s, so $t'_1 \equiv t$ and we may take $t_1 \equiv t'_1 \equiv t$. All other cases are analogous.

If neither the s-contraction nor the a-contraction occurs at the root, then the claim is easily established, possibly appealing to the inductive hypothesis.

Finally, assume that the a-contraction occurs at the root, but the s-contraction does not occur at the root. We have the following possibilities.

- If $t' \equiv \mathsf{C}_1 \mathsf{T}_1 tq \rightarrow_a t$ then the s-contraction must occur inside t. So $t \rightarrow_s t_1$ for some term t_1. Note that t is an s-term by definition of a-contraction. Therefore t_1 is also an s-term, by 5 in Definition 8. Thus t_1 satisfies the required conditions.
- If $t' \equiv \mathsf{C}_2 q s_1 s_2 \rightarrow_a t$ then $t \rightarrow_s^* s_1$, $t \rightarrow_s^* s_2$ and the s-contraction must occur inside s_1 or s_2. We may take $t_1 \equiv t$ and we still have $t'_1 \rightarrow_a t_1$.
- The cases $t' \equiv \mathsf{C}_1 \mathsf{F}_1 qt \rightarrow_a t$ and $t' \equiv \mathsf{K}_1 tq \rightarrow_a t$ are analogous to the first case.
- If $t' \equiv \mathsf{S}^{\vec{n}} t_0 \langle s_1, \ldots, s_k \rangle \langle \vec{r}_0, \ldots, \vec{r}_k \rangle$ then $|s_i| =_{\text{CL-pc}^1} |s_j|$, $|r_{i,j}| =_{\text{CL-pc}^1} |r_{i',j'}|$ for i, j, i', j' as in Definition 16, none of the s_i or $r_{i,j}$ is a tuple, and $t \equiv t_0 \langle \vec{r}_0 \rangle \langle s_1 \langle \vec{r}_1 \rangle, \ldots, s_k \langle \vec{r}_k \rangle \rangle$. The s-contraction must occur inside one of the s_i or the $r_{i,j}$, or in t_0. For instance, assume $s_1 \rightarrow_s s'_1$. Since s_1 is a subterm of t and it is not a tuple, it cannot s-reduce to a tuple by Definition 8. Hence s'_1 is not a tuple. Take $t_1 \equiv t_0 \langle \vec{r}_0 \rangle \langle s'_1 \langle \vec{r}_1 \rangle, s_2 \langle \vec{r}_2 \rangle, \ldots, s_k \langle \vec{r}_k \rangle \rangle$. Note that $t \rightarrow_s t_1$. Thus t_1 is an s-term, because t is an s-term and it s-reduces only to s-terms, by 5 in Definition 8. ◀

▶ **Corollary 20.** *If $t \, {}_a\!\leftarrow \cdot \rightarrow_s^* t'$ and t is strongly standard then $t \rightarrow_s^* \cdot \, {}_a^{\equiv}\!\leftarrow t'$.*

▶ **Lemma 21.** *If r is a strongly standard a-contractum of an a-redex r', and s' is a proper subterm of r', then s' is standard.*

Proof. It suffices to show that s' is a subterm of some standard term.

- Suppose $r' \equiv \mathsf{C}_1 \mathsf{T}_1 rq \rightarrow_a r$ with q an i-term. Both $\mathsf{C}_1 \mathsf{T}_1 r$ and q are standard and s' is a subterm of one of them. The cases $r' \equiv \mathsf{C}_1 \mathsf{F}_1 qr$ and $r' \equiv \mathsf{K}_1 rq$ are analogous.
- Suppose $r' \equiv \mathsf{C}_2 q r_1 r_2 \rightarrow_a r$ with q an i-term. Because $r \rightarrow_s^* r_i$ and r is strongly standard, r_1, r_2 are standard. Also q is an i-term. This implies that $\mathsf{C}_2 q r_1$ is also standard. Since s' occurs in $\mathsf{C}_2 q r_1$ or r_2, it is standard.
- Suppose $r' \equiv \mathsf{S}^{\vec{n}} t_0 \langle s_1, \ldots, s_k \rangle \langle \vec{r}_0, \ldots, \vec{r}_k \rangle \rightarrow_a t_0 \langle \vec{r}_0 \rangle \langle s_1 \langle \vec{r}_1 \rangle, \ldots, s_k \langle \vec{r}_k \rangle \rangle$. The term t_0 and each of s_i and $r_{i,j}$ (with i, j as in Definition 16) is standard. Note that none of s_i or $r_{i,j}$ is a tuple by Definition 16. Since each s_i is also standard, by inspecting Definition 8 we may conclude that $\langle s_1, \ldots, s_k \rangle$ is standard. Similarly $\langle \vec{r}_0, \ldots, \vec{r}_k \rangle$ is standard. Also

$\mathsf{S}^{\vec{n}} t_0 \langle s_1, \ldots, s_k \rangle$ is standard. This implies that s' is standard, because it occurs in $\mathsf{S}^{\vec{n}} t_0 \langle s_1, \ldots, s_k \rangle$ or $\langle \vec{r}_0, \ldots, \vec{r}_k \rangle$. ◀

▶ **Lemma 22.** *If s is an s-term and $s' \to_a s$ then s' is also an s-term.*

Proof. Induction on the structure of s. ◀

▶ **Lemma 23.** *If t is strongly standard and $t' \to_a t$ then t' is standard.*

Proof. We check that the conditions in Definition 8 hold for every subterm s' of t'. We may assume that s' does not occur in t, as otherwise the claim follows from the fact that t is standard. Therefore, s' occurs in the a-redex contracted in $t' \to_a t$, or the a-redex occurs inside s'. If s' is a proper subterm of the a-redex, then our claim holds by Lemma 21. Hence, we may assume that the a-redex r' is a subterm of s'. Then $s' \to_a s$ with s a subterm of t (so s is strongly standard).

1. Suppose r is the a-contractum of r' and $s' \to_a s$. By Definition 16, the term r is an s-term. Thus s cannot be an i-term. If s is a tuple, then so is s'. Otherwise, s is an s-term, by 1 in Definition 8. Hence s' is also an s-term by Lemma 22.
2. Suppose $s' \equiv \mathsf{C}_1 t_0' t_1' t_2'$ and t_0' is in s-NF. If $s' \equiv r'$ then $s' \equiv \mathsf{C}_1 \mathsf{T}_1 t_1' t_2'$ or $s' \equiv \mathsf{C}_1 \mathsf{F}_1 t_1' t_2'$, hence $t_0' \equiv \mathsf{T}_1$ or $t_0' \equiv \mathsf{F}_1$. If r' is a proper subterm of s', then r' must be a subterm of t_1' or t_2', because a-redexes are not in s-NF. Thus, $s' \to_a s \equiv \mathsf{C}_1 t_0' t_1 t_2$ for some terms t_1, t_2, where s is a subterm of t. Hence, $t_0' \equiv \mathsf{T}_1$ or $t_0' \equiv \mathsf{F}_1$ by 2 in Definition 8.
3. Suppose $s' \equiv \mathsf{C}_2 t_0' t_1' t_2'$. If $s' \equiv r'$ then $s \to_s^* t_1'$ and $s \to_s^* t_2'$. Hence $|t_1'| =_{\text{CL-pc}^1} |s| =_{\text{CL-pc}^1} |t_2'|$. If $s' \not\equiv r'$ then $s \equiv \mathsf{C}_2 t_0 t_1 t_2$ with $t_i' \to_a^{\equiv} t_i$. Because s is standard, $|t_1| =_{\text{CL-pc}^1} |t_2|$ by 3 in Definition 8. Thus also $|t_1'| =_{\text{CL-pc}^1} |t_2'|$ by Lemma 17.
4. Suppose $s' \equiv \mathsf{S}^{n_0,\ldots,n_k} t_0' t_1' t_2'$. If $s' \equiv r'$, then $s' \equiv \mathsf{S}^{n_0,\ldots,n_k} t_0' \langle s_1, \ldots, s_k \rangle \langle \vec{r}_0, \ldots, \vec{r}_k \rangle$, as in Definition 16, so the claim holds. If r' is a proper subterm of s', then $s' \equiv \mathsf{S}^{n_0,\ldots,n_k} t_0' t_1' t_2' \to_a s \equiv \mathsf{S}^{n_0,\ldots,n_k} t_0 t_1 t_2$ where $t_i' \to_a^{\equiv} t_i$ for $i = 0, 1, 2$, and s is a subterm of t. By 4 in Definition 8, the term t_2 is a tuple of length $\sum_{i=0}^{k} n_i$, and if $k > 1$ then t_1 is a tuple of length k. Since an a-contractum is an s-term, and hence not a tuple, t_2 is not an a-contractum, and if $k > 1$ then t_1 is not an a-contractum. Thus we may conclude that t_2' is a tuple of length $\sum_{i=0}^{k} n_i$, and if $k > 1$ then t_1' is a tuple of length k.
5. Suppose s' is an s-term and $s' \to_s^* s_1'$. Because also $s' \to_a s$ and s is strongly standard, by Corollary 20 there is s_1 with $s_1' \to_a^{\equiv} s_1$ and $s \to_s^* s_1$. By Definition 16, the term s is an s-term, so s_1 is also an s-term by 5 in Definition 8. So s_1' is an s-term by Lemma 22.
6. Suppose $s' \equiv \langle t_1', \ldots, t_n' \rangle$ with $n > 1$. We have $s' \to_a s \equiv \langle t_1, \ldots, t_n \rangle$ where $t_i' \to_a^{\equiv} t_i$ for $i = 1, \ldots, n$. By 6 in Definition 8, none of $t_1, \ldots, t_n$ is a tuple. Thus it is easy to see by inspecting Definition 16 that none of $t_1', \ldots, t_n'$ can be a tuple. ◀

▶ **Lemma 24.** *If $t \Downarrow_{\mathsf{F}_1}$ and $t' \to_a t$ then $t' \Downarrow_{\mathsf{F}_1}$.*

Proof. Suppose $t' \to_s^* t_0'$. By Corollary 20 there is t_0 with $t \to_s^* t_0$ and $t_0' \to_a^{\equiv} t_0$. Since t is strongly standard, so is t_0. Therefore, t_0' is standard by Lemma 23.

Suppose $t' \to_s^* t_0'$ with t_0' in s-NF. By Corollary 20 there is t_0 with $t \to_s^* t_0$ and $t_0' \to_a^{\equiv} t_0$. Since an a-redex is an s-redex and t_0' is in s-NF, we conclude that $t_0' \equiv t_0$. But then $t_0' \equiv t_0 \equiv \mathsf{F}_1$, because $t \Downarrow_{\mathsf{F}_1}$. ◀

▶ **Corollary 25.** *If $t \Downarrow_{\mathsf{F}_1}$, $t \succ q$ and q $_{\text{CL-pc}}\!\leftarrow q'$ then there is t' with $t' \Downarrow_{\mathsf{F}_1}$ and $t' \succ q'$.*

Proof. Follows from Lemma 18, Lemma 13 and Lemma 24. ◀

▶ **Lemma 26.** *If t has no s-NFs other than F_1 then $t \to_{s-}^* \mathsf{F}_1$.*

Proof. Since s-reduction is terminating, by reducing s-redexes outside any tuples and not using the second rule for C_2 we will ultimately obtain a term t' with all s-redexes inside tuples, and such that $t \to_{s-}^* t'$. Note that an s-redex in t' may only occur inside a tuple, because any s-redex by the second rule for C_2 is also an $s-$-redex by the first rule for C_2. If t' is in s-NF then $t' \equiv \mathsf{F}_1$. Otherwise, any s-NF of t' must contain a tuple, because s-reduction inside a tuple cannot erase this tuple or create an s-redex outside of it. But since any s-NF of t' is an s-NF of t, this contradicts the fact that t has no s-NFs other than F_1.   ◄

We now have everything we need to show the central lemma of the confluence proof.

▶ **Lemma 27.** *The system* $\mathrm{CL\text{-}pc}^1$ *is* F*-normal, i.e., if* $q =_{\mathrm{CL\text{-}pc}^1} \mathsf{F}$ *then* $q \to_{\mathrm{CL\text{-}pc}^1}^* \mathsf{F}$.

Proof. If $q =_{\mathrm{CL\text{-}pc}^1} \mathsf{F}$ then by Lemma 2 we have $q =_{\mathrm{CL\text{-}pc}} \mathsf{F}$. Note that $\mathsf{F}_1 \Downarrow_{\mathsf{F}_1}$ and $|\mathsf{F}_1| \equiv \mathsf{F}$. Thus, using Corollary 15 and Corollary 25 it is easy to show by induction on the length of $q =_{\mathrm{CL\text{-}pc}} \mathsf{F}$ that there is t with $t \succ q$ and $t \Downarrow_{\mathsf{F}_1}$. By Lemma 26 we have $t \to_{s-}^* \mathsf{F}_1$. But then, because $t \succ q$, using Lemma 7 it is easy to show by induction on the length of $t \to_{s-}^* \mathsf{F}_1$ that $q \equiv |t| \to_{\mathrm{CL\text{-}pc}^1}^* |\mathsf{F}_1| \equiv \mathsf{F}$.   ◄

It remains to derive the confluence of $\mathrm{CL\text{-}pc}^1$ and $\mathrm{CL\text{-}pc}^{\mathrm{L}}$ from Lemma 27. We use a trick with an auxiliary term rewriting system R, in a way similar to how the confluence of $\mathrm{CL\text{-}pc}^{\mathrm{L}}$ is derived from the condition $\mathsf{T} \neq_{\mathrm{CL\text{-}pc}^{\mathrm{L}}} \mathsf{F}$ in [4]. The idea is to eliminate the non-trivial overlap between the rules of $\mathrm{CL\text{-}pc}^1$ by imposing additional side conditions.

▶ **Definition 28.** The term rewriting system R is defined by the following rules:

$$
\begin{array}{llll}
\mathsf{K}xy & \to & x & \qquad\qquad \mathsf{CT}xy \;\to\; x \\
\mathsf{S}xyz & \to & xz(yz) & \qquad\qquad \mathsf{C}zxy \;\to\; y \quad \Leftarrow \quad z =_{\mathrm{CL\text{-}pc}^1} \mathsf{F} \\
& & & \qquad\qquad \mathsf{C}zxy \;\to\; x \quad \Leftarrow \quad z \neq_{\mathrm{CL\text{-}pc}^1} \mathsf{F} \wedge x =_{\mathrm{CL\text{-}pc}^1} y
\end{array}
$$

▶ **Lemma 29.** *If* $q \to_{\mathrm{CL\text{-}pc}} q'$ *then* $q \to_R q'$.

▶ **Lemma 30.** *If* $q \to_R q'$ *then* $q \to_{\mathrm{CL\text{-}pc}^1}^* q'$.

Proof. Follows from definitions and Lemma 27.   ◄

▶ **Lemma 31.** *The system* R *is confluent.*

Proof. Because $\mathsf{T} \neq_{\mathrm{CL\text{-}pc}^1} \mathsf{F}$ by Lemma 27, the system R is weakly orthogonal (i.e. it is left-linear and all its critical pairs are trivial). By Lemma 30 the conditions are stable under reduction. Weakly orthogonal conditional term rewriting systems whose conditions are stable under reduction are confluent [11, Chapter 4].   ◄

▶ **Theorem 32.** *The systems* $\mathrm{CL\text{-}pc}^1$ *and* $\mathrm{CL\text{-}pc}^{\mathrm{L}}$ *are confluent.*

Proof. Since $q_1 \to_{\mathrm{CL\text{-}pc}^1} q_2$ implies $q_1 \to_{\mathrm{CL\text{-}pc}^{\mathrm{L}}} q_2$, it suffices to show that if $q_1 =_{\mathrm{CL\text{-}pc}^{\mathrm{L}}} q_2$ then there is q with $q_1 \to_{\mathrm{CL\text{-}pc}^1}^* q$ and $q_2 \to_{\mathrm{CL\text{-}pc}^1}^* q$. So suppose $q_1 =_{\mathrm{CL\text{-}pc}^{\mathrm{L}}} q_2$. Then by Lemma 2 we have $q_1 =_{\mathrm{CL\text{-}pc}} q_2$. By Lemma 29 we obtain $q_1 =_R q_2$. By Lemma 31 there is q with $q_1 \to_R^* q$ and $q_2 \to_R^* q$. By Lemma 30 we have $q_1 \to_{\mathrm{CL\text{-}pc}^1}^* q$ and $q_2 \to_{\mathrm{CL\text{-}pc}^1}^* q$.   ◄

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