A New Perspective on the Mereotopology of RCC8

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Abstract

RCC8 is a set of eight jointly exhaustive and pairwise disjoint binary relations representing mereotopological relationships between ordered pairs of individuals. Although the RCC8 relations were originally presented as defined relations of Region Connection Calculus (RCC), virtually all implementations use the RCC8 Composition Table (CT) rather than the axioms of RCC. This raises the question of which mereotopology actually underlies the RCC8 composition table. In this paper, we characterize the algebraic and mereotopological properties of the RCC8 CT based on the metalogical relationship between the first-order theory that captures the RCC8 CT and Ground Mereotopology (MT) of Casati and Varzi. In particular, we show that the RCC8 theory and MT are relatively interpretable in each other. We further show that a nonconservative extension of the RCC8 theory that captures the intended interpretation of the RCC8 relations is logically synonymous with MT, and that a conservative extension of MT is logically synonymous with the RCC8 theory. We also present a characterization of models of MT up to isomorphism, and explain how such a characterization provides insights for understanding models of the RCC8 theory.

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1 Introduction

Representations of space, and their use in qualitative spatial reasoning, are widely recognized as key aspects in commonsense reasoning, with applications ranging from biology to geography. The predominant approach to spatial representation within the applied ontology community has used mereotopologies, which combine topological (expressing connectedness) with mereological (expressing parthood) relations. A variety of first-order mereotopological ontologies have been proposed, the most widespread being the Region Connection Calculus (RCC) [17], the ontology RT [1], and the ontologies introduced by Casati and Varzi [4]. Properties of RCC in particular have been studied extensively; [18, 5] present algebraic representations for the RCC theory, and [9] describes various mereotopological settings that satisfy axioms of RCC.

While theoretical work has focused on the first-order theories for mereotopologies, work within the qualitative spatial reasoning community has primarily used a formalism known as RCC8, which is a set of eight jointly exhaustive and pairwise disjoint binary relations representing mereotopological relationships between ordered pairs of individuals. Reasoning

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is supported through the use of a composition table, which specifies all possible mereotopological relationships between pairs of elements; deduction is implemented through constraint propagation algorithms.

Although the RCC8 relations were originally presented as defined relations within RCC, the theoretical analyses of RCC have not been helpful in understanding properties of formalisms that use the RCC8 relations. The reason is that virtually all implementations use the RCC8 composition table rather than the axioms of RCC, and the composition table has very different mereotopological properties than RCC. Of particular importance is the widespread use of RCC8 in efforts such as GeoSPARQL, which is an international standard for the representation of geospatial linked data developed by the Open Geospatial Consortium. A characterization of all solutions for a set of RCC8 constraints presumes an understanding of the possible models of some first-order logical theory.

In this paper, we investigate algebraic and mereotopological properties of the RCC8 composition table based on the metalogical relationship between the first-order theory that captures the RCC8 composition table and Ground Mereotopology (MT) of Casati and Varzi. After reviewing the basic axiomatizations of the mereotopological theories in Section 2, we discuss the relationship between the RCC8 theory and MT in Section 3. Our key result is that a nonconservative extension of the RCC8 theory, we called RCC8*, is logically synonymous with the MT theory, meaning MT and RCC8* axiomatize the same class of structures. In other words, MT and RCC8* are semantically equivalent, and only differ in signature (i.e., the non-logical symbols). Further, we present a conservative extension of MT which is logically synonymous with the RCC8 theory. We also show that the RCC8 theory and MT are relatively interpretable in each other. Finally, in Section 4, we present a characterization of models of MT up to isomorphism, and explain how such a characterization can be used in characterizing algebraic properties of models of the RCC8 theory.

2 Preliminaries: Mereotopological Theories

2.1 Region Connection Calculus

The Region Connection Calculus (RCC) is a first-order theory whose signature contains the single primitive binary relation $C(x, y)$ denoting "x is *connected* to y". Parthood is defined in terms of connection alone, being equivalent to the topological notion of enclosure. Representation theorems [18] have shown that the models of RCC are equivalent to mathematical structures known as Boolean contact algebras which consist of a standard Boolean algebra together with a binary relation C that is reflexive, anti-symmetric, and extensional.

2.2 RCC8

RCC8 is a set of eight binary relations representing mereotopological relationships between (ordered) pairs of individuals. These relations and their intended interpretations are illustrated in Figure 1. The RCC8 relations have been proven to be jointly exhaustive and pairwise disjoint (JEPD), that is, every ordered pair of individuals are related by exactly one RCC8 relation.

Originally, RCC8 relations were presented as defined relations of RCC (throughout the paper, free variables in a displayed formula are assumed to be universally quantified):

$$DC(x, y) \equiv \neg C(x, y). \tag{1}$$

$$EC(x, y) \equiv C(x, y) \wedge \neg O(x, y). \tag{2}$$

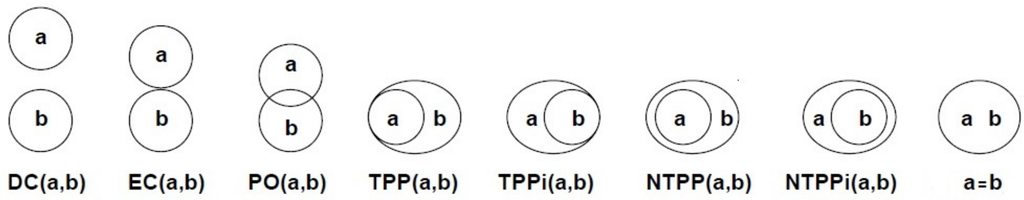

Figure 1 Illustration of RCC8 relations – $DC(a, b)$ (a is *disconnected* from b), $EC(a, b)$ (a is *externally connected* with b), $PO(a, b)$ (a *partially overlaps* b), $TPP(a, b)$ (a is a *tangential proper part* of b), $TPPi(a, b)$ (b is a *tangential proper part* of a), $NTPP(a, b)$ (a is a *nontangential proper part* of b), $NTPPi(a, b)$ (b is a *nontangential proper part* of a), $a = b$ (a is *identical* with b).

	DC	**EC**	**PO**	**TPP**	**NTPP**	**TPPi**	**NTPPi**	=
DC	*	DC, EC, PO, TPP, NTPP	DC, EC, PO, TPP, NTPP	DC, EC, PO, TPP, NTPP	DC, EC, PO, TPP, NTPP	DC	DC	DC
EC	DC, EC, PO, TPPi, NTPPi	DC, EC, PO, TPP, TPPi, =	DC, EC, PO, TPP, NTPP	EC, PO, TPP, NTPP	PO, TPP, NTPP	DC, EC	DC	EC
PO	DC, EC, PO, TPPi, NTPPi	DC, EC, PO, TPPi, NTPPi	*	PO, TPP, NTPP	PO, TPP, NTPP	DC, EC, PO, TPPi, NTPPi	DC, EC, PO, TPPi, NTPPi	PO
TPP	DC	DC, EC	DC, EC, PO, TPP, NTPP	TPP, NTPP	NTPP	DC, EC, PO, TPP, TPPi, =	DC, EC, PO, TPPi, NTPPi	TPP
NTPP	DC	DC	DC, EC, PO, TPP, NTPP	NTPP	NTPP	DC, EC, PO, TPP, NTPP	*	NTPP
TPPi	DC, EC, PO, TPPi, NTPPi	EC, PO, TPPi, NTPPi	PO,TPPi,NTPPi	PO, TPP, TPPi, =	PO, TPP, NTPP	TPPi, NTPPi	NTPPi	TPPi
NTPPi	DC, EC, PO, TPPi, NTPPi	PO,TPPi,NTPPi	PO,TPPi,NTPPi	PO,TPPi,NTPPi	PO, TPP, NTPP, TPPi, NTPPi, =	NTPPi	NTPPi	NTPPi
=	DC	EC	PO	TPP	NTPP	TPPi	NTPPi	=

Figure 2 RCC8 Composition Table. “$*$” indicates that all RCC8 relations are possible.

$$PO(x, y) \equiv O(x, y) \land \neg P(x, y) \land \neg P(y, x). \tag{3}$$

$$(x = y) \equiv P(x, y) \land P(y, x). \tag{4}$$

$$TPPi(x, y) \equiv TPP(y, x). \tag{5}$$

$$NTPPi(x, y) \equiv NTPP(y, x). \tag{6}$$

$$TPP(x, y) \equiv PP(x, y) \land \neg NTPP(x, y). \tag{7}$$

$$NTPP(x, y) \equiv PP(x, y) \land \neg(\exists z)\, EC(z, x) \land EC(z, y). \tag{8}$$

In the axioms above, $C(x, y)$ denotes “x is *connected* to y,” $P(x, y)$ denotes “x is a *part* of y,” $O(x, y)$ denotes “x *overlaps* y,” $PP(x, y)$ denotes “x is a *proper part* of y”):

$$O(x, y) \equiv (\exists z)\, P(z, x) \land P(z, y). \tag{9}$$

$$PP(x, y) \equiv P(x, y) \land \neg P(y, x). \tag{10}$$

Given its origin within RCC, it is interesting to note that RCC8 is typically used independently of the RCC theory – the RCC axioms are not considered to be part of the RCC8 formalism, and in most reasoning tasks even the axiomatic descriptions of RCC8 relations are not explicitly used. Instead, the RCC8 Composition Table (CT) is used. The RCC8 CT (illustrated in Figure 2) is an 8×8 matrix such that for each ordered pair of RCC8 relations R_i, R_j, the cell $CT(R_i, R_j)$ indicates possible mereotopological relationships between two individuals a and c assuming that $R_i(a, b)$ and $R_j(b, c)$ holds. For example, $CT(EC, NTPP) = \{PO, TPP, NTPP\}$, meaning that if $EC(a, b)$ and $NTPP(b, c)$, then a is related to c by either PO or TPP or $NTPP$.

2.3 Combined Mereotopology

Even though the RCC8 CT is entailed by the RCC theory, they have very different mereotopological properties. In fact, the RCC8 CT seems to be closely related to Ground Mereotopology (also called MT), which is the weakest theory among the mereotopological theories proposed in [4]. The signature of the MT theory (which we will denote by T_{mt}) consist of two primitive binary relations, parthood (P) and connection (C). The axioms of the theory (Axioms 11 to 16) state that connection is a reflexive and symmetric relation, while parthood is a reflexive, transitive, and anti-symmetric relation.[1] In addition, if one individual is connected to another, then the first one is also connected to any individual which the second is part of.

$$C(x, x). \tag{11}$$

$$C(x, y) \supset C(y, x). \tag{12}$$

$$P(x, x). \tag{13}$$

$$P(x, y) \wedge P(y, x) \supset (x = y). \tag{14}$$

$$P(x, y) \wedge P(y, z) \supset P(x, z). \tag{15}$$

$$P(y, z) \wedge C(x, y) \supset C(x, z). \tag{16}$$

3 Relationship between MT and RCC8

Even though the RCC8 CT has been derived based on the RCC theory, they show very different mereotopological properties. For instance, while in models of RCC every individual is atomless (i.e., has a proper part) and externally connected to another individual, individuals that satisfy the RCC8 CT may have no proper part, or may not be connect to any other individual. These differences raise the question of which mereotopology actually underlies the RCC8 composition table.

We begin this section by describing the logical theory that captures RCC8 CT. We then show that the closest mereotopology to this theory is MT.

3.1 The First-order Theory of RCC8

We denote the logical theory of RCC8 CT by T_{rcc8}. Following [2], we assume that for each cell in the RCC8 CT, T_{rcc8} contains an axiom of the following form

$$R_i(x, y) \wedge R_j(y, z) \supset T_1(x, z) \vee ... \vee T_n(x, z)$$

where $CT(R_i, R_j) = \{T_1, ..., T_n\}$. The following sentence, for example, is the axiom of T_{rcc8} which corresponds with $CT(TPP, EC)$:

$$TPP(x, y) \wedge EC(y, z) \supset DC(x, z) \vee EC(x, z).$$

Since RCC8 CT consists of 8×8 cells, T_{rcc8} must contain 64 axioms corresponding with the table. In addition to these axioms, we assume that T_{rcc8} contains an axiom that specifies RCC8 relations are jointly exhaustive:

$$DC(x, y) \vee EC(x, y) \vee PO(x, y) \vee NTPP(x, y) \vee TPP(x, y) \vee TPPi(x, y) \vee$$
$$NTPPi(x, y) \vee (x = y).$$

[1] In this paper, we consider a *theory* to be a set of first-order sentences closed under logical entailment. A collection of sentences of a theory which entail all other sentences in the theory are called *axioms* of the theory.

We also assume that for each RCC8 relation R_1, T_{rcc8} contains a sentence of the following form stating that RCC8 relations are pairwise disjoint (PD):

$$R_1(x,y) \supset \neg(R_2(x,y) \vee ... \vee R_7(x,y))$$

where $R_2, ..., R_7$ are RCC8 relations other than R_1. The following sentence, for example, is the PD axiom corresponding with DC:

$$DC(x,y) \supset \neg[EC(x,y) \vee PO(x,y) \vee TPP(x,y) \vee NTPP(x,y) \vee TPPi(x,y) \vee NTPPi(x,y) \vee (x = y)].$$

As there are 8 RCC8 relations, T_{rcc8} contains 8 PD axioms. All other sentences in T_{rcc8} are those which are entailed by the $64 + 1 + 8$ above-mentioned axioms.

3.2 MT Theory vs. RCC8 Theory

The RCC8 CT is commonly considered to be related to RCC because RCC8 relations were originally defined as part of the RCC theory, and the RCC8 CT was proved using the RCC theory. It turns out, however, that the RCC8 CT can also be deduced from a definitional extension of MT:

▶ **Definition 1** (adopted from [11]). Let T be a first-order theory and Π be a set containing sentences of the following form[2]

$$R(x_1, ..., x_n) \equiv \Phi(x_1, ..., x_n)$$

where R is a predicate which is not in $\Sigma(T)$ and Φ is a formula in $\mathcal{L}(T)$ in which at most variables $x_1, ..., x_n$ occur free. $T \cup \Pi$ is called a *definitional extension* of T.

Notice that Definitions 1 to 10 are defined in terms of C and P, which are primitives of T_{mt}, and so if we extend T_{mt} by Definitions 1 to 10, we get a definitional extension of T_{mt}. This extension entails T_{rcc8}.

▶ **Theorem 2.** *T_{rcc8} is entailed by a definitional extension of T_{mt}.*

Proof. Suppose Π denotes the set containing Definitions (1) to (10). Using an automated theorem prover, Prover9 [13], we showed that $T_{mt} \cup \Pi$ entails axioms of T_{rcc8}. Hence, $T_{mt} \cup \Pi$ entails T_{rcc8}. ◀

Recall that DC, EC, PO, TPP, $NTPP$, $TPPi$, $NTPPi$, and $=$ are primitives of T_{rcc8}. Using these primitives, one can extend T_{rcc8} with the following definitions for parthood and connection:

$$P(x,y) \equiv TPP(x,y) \vee NTPP(x,y) \vee (x = y). \tag{17}$$

$$C(x,y) \equiv \neg DC(x,y). \tag{18}$$

A more interesting result is that this definitional extension of T_{rcc8} actually entails MT:

▶ **Theorem 3.** *T_{mt} is entailed by a definitional extension of T_{rcc8}.*

[2] For a theory T, $\Sigma(T)$ denotes the *signature* of T, i.e., the set of non-logical symbols used in sentences of T; $\mathcal{L}(T)$ denotes the *language* of T, i.e., the set of all first-order formulae generated by symbols in $\Sigma(T)$; $Mod(T)$ denotes the class of all models of T.

Proof. Suppose Δ denotes the set containing Definitions (17) and (18). Using Prover9, we showed that $T_{rcc8} \cup \Delta$ entails Axioms (11) to (16). Since Axioms (11) to (16) axiomatize T_{mt}, we can conclude that $T_{rcc8} \cup \Delta$ entails T_{mt}. ◀

Using Theorems 2 and 3, it can be shown that T_{mt} and T_{rcc8} are *relatively interpretable* [7] in each other. Informally, a theory T_1 has a relative interpretation in another theory T_2 if every sentence in T_1 can be translated into a sentence in T_2. In other words, for all sentences $\Phi \in \mathcal{L}(T_1)$, if T_1 entails Φ, then T_2 entails a translation of Φ into the language of T_2. [10] show that if a definitional extension of T_2 entails T_1, translations for sentences of T_1 is obtained based on the formulas which define predicates of T_1 in the definitional extension. For instance, a translation of Axiom (16) of T_{mt} into the language of T_{rcc8} can be obtained by replacing C and P with the formulas defining them in Definitions (17) and (18). The result is the following sentence, which provably is a sentence in T_{rcc8}:

$$\neg DC(x,y) \wedge (TPP(y,z) \vee NTPP(y,z) \vee (y=z)) \supset \neg DC(x,z).$$

When T_1 is interpretable in T_2, every model of T_2 defines a model of T_1 using the translation definitions between T_1 and T_2 [7]. Consider a model $\mathcal{M}_1$ of T_{rcc8} with two elements $\mathbf{a}, \mathbf{b}$ that are externally connected:[3]

$\mathbf{EC}^{\mathcal{M}_1} = \{(\mathbf{a},\mathbf{b}),(\mathbf{b},\mathbf{a})\}$.

Now, consider a structure $\mathcal{N}_1$ with the same domain but in the signature of T_{mt} such that relations between elements are obtained based on $\mathcal{M}_1$ and Definitions (17) and (18). By Definitions (17) and (18), for any pair $\mathbf{x}, \mathbf{y}$:

$$\begin{aligned}
(\mathbf{x},\mathbf{y}) \in \mathbf{C}^{\mathcal{N}_1} \quad &\text{iff} \quad (\mathbf{x},\mathbf{y}) \notin \mathbf{DC}^{\mathcal{M}_1}, \\
(\mathbf{x},\mathbf{y}) \in \mathbf{P}^{\mathcal{N}_1} \quad &\text{iff} \quad \Big[(\mathbf{x},\mathbf{y}) \in \mathbf{TPP}^{\mathcal{M}_1} \text{ or} \\
&\qquad (\mathbf{x},\mathbf{y}) \in \mathbf{NTPP}^{\mathcal{M}_1} \text{ or } \mathbf{x}=\mathbf{y}\Big].
\end{aligned}$$

So, $\mathbf{C}^{\mathcal{N}_1} = \{(\mathbf{a},\mathbf{a}),(\mathbf{b},\mathbf{b}),(\mathbf{a},\mathbf{b}),(\mathbf{b},\mathbf{a})\}$ and $\mathbf{P}^{\mathcal{N}_1} = \{(\mathbf{a},\mathbf{a}),(\mathbf{b},\mathbf{b})\}$.

To study models of T_{rcc8} based on models of T_{mt}, we need a notion stronger than relative interpretation:

▶ **Definition 4** ([11]). Two theories T_1 and T_2 are logically synonymous iff they have a common definitional extension.

Considering Definition 4, it is easy to see that T_1 and T_2 are synonymous iff there exist two sets of translation definitions, Δ and Π, such that $T_1 \cup \Pi$ is a definitional extension of T_1, $T_2 \cup \Delta$ is a definitional extension of T_2, and $T_1 \cup \Pi$ and $T_2 \cup \Delta$ are logically equivalent.

T_{mt} and T_{rcc8} are not synonymous. In the following part of this section we will explain why, and present an extension of T_{rcc8} which is synonymous with T_{mt}.

3.3 MT and RCC8*

When two theories are synonymous, there is a one-to-one correspondence between their models such that the corresponding models can be defined based on each other [15]. Such a

[3] We denote *structures* by calligraphic uppercase letters, e.g. $\mathcal{M}, \mathcal{N}$; elements of a structure by **boldface** font, e.g., $\mathbf{a}, \mathbf{b}$; and the *extension of predicate* R in a structure $\mathcal{M}$ by $\mathbf{R}^{\mathcal{M}}$.

correspondence does not exists between models of T_{mt} and T_{rcc8}. Consider two models $\mathcal{M}_2$ and $\mathcal{M}_3$ of T_{rcc8}, both with two elements $\mathbf{a}, \mathbf{b}$ such that

$$\mathbf{TPP}^{\mathcal{M}_2} = \{(\mathbf{a}, \mathbf{b})\}, \qquad \mathbf{NTPP}^{\mathcal{M}_3} = \{(\mathbf{a}, \mathbf{b})\}.$$

Both $\mathcal{M}_2$ and $\mathcal{M}_3$ define the same model $\mathcal{N}_2$ of T_{mt}:

$$\mathbf{C}^{\mathcal{N}_2} = \{(\mathbf{a}, \mathbf{a}), (\mathbf{b}, \mathbf{b}), (\mathbf{a}, \mathbf{b}), (\mathbf{b}, \mathbf{a})\}, \quad \mathbf{P}^{\mathcal{N}_2} = \{(\mathbf{a}, \mathbf{a}), (\mathbf{b}, \mathbf{b}), (\mathbf{a}, \mathbf{b})\}.$$

$\mathcal{M}_2$ and $\mathcal{M}_3$ correspond with the same model of T_{mt} because the only way for MT to distinguish TPP from $NTPP$ is the existence of a third element that is externally connected to the inner element (i.e., $\mathbf{a}$). However, such an element does not exists in either of $\mathcal{M}_2$ and $\mathcal{M}_3$.

A similar issue arises when two individuals overlap, but they do not have a common part. Consider a model $\mathcal{M}_4$ of T_{rcc8} with two elements $\mathbf{a}, \mathbf{b}$ and $\mathbf{PO}^{\mathcal{M}_4} = \{(\mathbf{a}, \mathbf{b}), (\mathbf{b}, \mathbf{a})\}$.

$\mathcal{M}_4$ defines the following model of T_{mt}, which is isomorphic to $\mathcal{N}_1$ in the previous subsection:

$$\mathbf{C}^{\mathcal{N}_4} = \{(\mathbf{a}, \mathbf{a}), (\mathbf{b}, \mathbf{b}), (\mathbf{a}, \mathbf{b}), (\mathbf{b}, \mathbf{a})\}, \quad \mathbf{P}^{\mathcal{N}_4} = \{(\mathbf{a}, \mathbf{a}), (\mathbf{b}, \mathbf{b})\}.$$

Thus $\mathcal{M}_1$ and $\mathcal{M}_4$ correspond with the same model of T_{mt}. This is because within MT, 'overlap' is defined based on a third element that is a common part of the overlapping individuals. If such an element does not exists (as is the case with $\mathcal{M}_4$), MT cannot distinguish PO from EC.

It is interesting to observe that although $\mathcal{M}_2$ and $\mathcal{M}_4$ are models of T_{rcc8}, they do not satisfy the original (axiomatic) definitions of PO, TPP or O (i.e., Definitions (3), (7), and (9)); that is definitions which are part of the RCC theory, and the RCC8 CT is derived based on them. Notice also that no model of T_{mt} defines $\mathcal{M}_2$ and $\mathcal{M}_4$ because without the existence of a third element TPP and O are not definable in MT.

Since a one-to-one correspondence between models of RCC8 and MT does not exists, they are not synonymous. To get synonymy, we need to extend T_{rcc8} by axioms that eliminate those models of T_{rcc8} which are not definable by any model of T_{mt}. Based on the examples we just discussed, undefinable models are those that do not satisfy the axiomatic definitions of TPP or O: That is, models (like $\mathcal{M}_2$) in which an element is related to another element by TPP, but there is no other element that externally connects with the inner element; or models (like $\mathcal{M}_4$) in which two elements are related by O, but they do not have a common part. To eliminate such models, we extend T_{rcc8} by the following axioms:

$$TPP(x, y) \supset (\exists z) EC(z, x) \wedge EC(z, y). \tag{19}$$

$$O(x, y) \supset (\exists z)\, P(z, x) \wedge P(z, y). \tag{20}$$

We call the resulting theory RCC8* and denote it by T_{rcc8^*}.

▶ **Theorem 5.** *T_{mt} is logically synonymous with T_{rcc8^*}.*

Proof. Suppose Π contains Definitions (1) to (10); and Δ contains Definitions (17), (18), (21), (22).

$$O(x, y) \equiv \neg DC(x, y) \wedge \neg EC(x, y). \tag{21}$$

$$PP(x, y) \equiv TPP(x, y) \vee NTPP(x, y). \tag{22}$$

Using Prover9, we showed

$$T_{mt} \cup \Pi \models T_{rcc8^*} \cup \Delta \qquad \text{and} \qquad T_{rcc8^*} \cup \Delta \models T_{mt} \cup \Pi\,.$$

Hence, $T_{mt} \cup \Pi$ and $T_{rcc8^*} \cup \Delta$ are logically equivalent. So, by definition, T_{mt} and T_{rcc8^*} are logically synonymous. ◀

According to [15], synonymous theories axiomatize the same class of structures. Thus, T_{mt} and T_{rcc8^*} are semantically equivalent and only differ in signature.

All relations in RCC8 CT can be deduced from T_{rcc8^*} as it is an extension of T_{rcc8}. In addition, for every entry $CT(R_i, R_j)$ of the RCC8 CT and every RCC8 relation $S \notin CT(R_i, R_j)$ we proved a sentence of the following form (proofs are done by Prover9):

$$R_i(x, y) \wedge R_j(y, z) \supset \neg S(x, z)\,.$$

Thus, the additional axioms of RCC8* does not change RCC8 CT, but only eliminate those models of T_{rcc8} that do not satisfy the axiomatic definitions of RCC8 relations.

3.4 Extending MT

As we explained in Section 3.3, logical synonymy between MT and RCC8 does not achieved because of the way $NTPP$, TPP, PO and EC are defined within the MT theory: The difference between $NTPP$ and TPP is defined with respect to a third element. Hence, only models of MT with more than two elements can distinguish between $NTPP$ and TPP. However, within the RCC8 theory $NTPP$ and TPP are distinguishable even by models of size two. A similar arguments applies to PO and EC. Thus, a one-to-one correspondence between models of MT and RCC8 does not exist.

In Section 3.3 we demonstrate how extending T_{rcc8} to T_{rcc8*} gives a one-to-one correspondence between models of T_{mt} and T_{rcc8*}, meaning that T_{mt} and T_{rcc8*} are logically synonymous. Another way of getting logical synonymy is to extend MT with Axioms (23) to (30), which specify properties of $NTPP$ and O (Axioms (23) to (26) are borrowed from [8]). We call the resulting theory MTNO and denote it by T_{mtno}.

$$NTPP(x, y) \wedge P(y, z) \supset NTPP(x, z). \tag{23}$$
$$P(x, y) \wedge NTPP(y, z) \supset NTPP(x, z). \tag{24}$$
$$NTPP(x, y) \supset PP(x, y). \tag{25}$$
$$C(x, y) \wedge NTPP(y, z) \supset O(x, z). \tag{26}$$
$$O(x, x). \tag{27}$$
$$O(x, y) \supset O(y, x). \tag{28}$$
$$O(x, y) \supset C(x, y). \tag{29}$$
$$O(x, y) \wedge P(y, z) \supset O(x, z). \tag{30}$$

Since O and $NTPP$ are primitive relations in T_{mtno}, PO and TPP can be defined based on them, without introducing a third element:

$$PO(x, y) \equiv O(x, y) \wedge \neg P(x, y) \wedge \neg P(y, x). \tag{31}$$
$$TPP(x, y) \equiv P(x, y) \wedge \neg P(y, x) \wedge \neg NTPP(x, y). \tag{32}$$

Therefore, a one-to-one correspondence between models of T_{mtno} and T_{rcc8} should exist, and it should be possible to show that the two theories are logically synonymous.

▶ **Theorem 6.** *T_{mtno} is logically synonymous with T_{rcc8}.*

Proof. To show that T_{mtno} and T_{rcc8} are synonymous we need to show that there exist conservative definitions Π and Δ for T_{mtno} and T_{rcc8} such that $T_{mtno} \cup \Pi$ and $T_{rcc8} \cup \Delta$ are logically equivalent.

Suppose Π contains Definitions (1) to (6) and (31) to (32); and Δ contains Definitions (17), (18), (21), (22). Using Prover9, we showed that

$$T_{mtno} \cup \Pi \models T_{rcc8} \cup \Delta \qquad \text{and} \qquad T_{rcc8} \cup \Delta \models T_{mtno} \cup \Pi\,.$$

Hence, $T_{mtno} \cup \Pi$ and $T_{rcc8} \cup \Delta$ are logically equivalent. ◀

4 Model-Theoretic Characterization of MT

Is the equivalence between RCC8* and MT simply an intellectual curiosity, or does it give us new insights into RCC8? If we consider that RCC8 is primarily used in constraint satisfaction problems, in which one constructs a satisfying interpretation of a set of expressions in the signature of RCC8, then the set of all possible solutions of RCC8 problems, excluding those eliminated by RCC8*, is equivalent to the set of all possible models of T_{mt}. In this section, we provide a characterization of the models of T_{mt} up to isomorphism, by first specifying a class of mathematical structures, and then showing that T_{mt} axiomatizes this class of structures.

4.1 Representation Theorem for Models of T_{mt}

We begin by introducing the two classes of mathematical structures that capture the intended interpretations of the the connection and parthood relations in MT. The connection relation in T_{mt} corresponds to a class of graphs:

▶ **Definition 7.** A graph with loops is a pair $\mathcal{G} = \langle V, E\rangle$ of sets such that:

1. $E \subseteq V \times V$.
2. For each $\mathbf{v} \in V$, $\mathbf{v} \in N(\mathbf{v})$, where $N(\mathbf{x})$, $\mathbf{x} \in V$, denotes the set of neighbors of $\mathbf{x}$ and is defined as

$$N(\mathbf{x}) = \{\mathbf{y} \,:\, (\mathbf{x},\mathbf{y}) \in E\}\,.$$

$\mathfrak{M}^{graph\_loops}$ is the class of structures which are graphs with loops.

It is well-known that Ground Mereology, the subtheory of T_{mt} which describes the parthood relation, is synonymous with the theory of partial orderings [4]. That is, the parthood relation in models of T_{mt} forms a partial ordering:

▶ **Definition 8.** A partial ordering is a pair $\mathcal{Q} = \langle V, \preceq\rangle$ s.t. $\preceq$ is a reflexive, antisymmetric, and transitive binary relation. For each $\mathbf{x} \in V$ and each set $X \subseteq V$ the upper set, denoted by $U(\mathbf{x})$ and $U(X)$ respectively, is defined as

$$U(\mathbf{x}) = \{\mathbf{y} \,:\, \mathbf{x} \preceq \mathbf{y}\} \qquad U(X) = \bigcup_{\mathbf{x}\in X} U(\mathbf{x})\,.$$

$\mathfrak{M}^{par\_orders}$ denotes the class of partial orderings.

We pull all of these ideas together to define the class of mathematical structures which we will eventually show are equivalent to the models of T_{mt}:

▶ **Definition 9.** $\mathfrak{M}^{mt}$ is the following class of structures. $\mathcal{M} \in \mathfrak{M}^{mt}$ iff $\mathcal{M} = \langle V, E, \preceq \rangle$ such that

1. $\mathcal{G} = \langle V, E \rangle$ and $\mathcal{G} \in \mathfrak{M}^{graph\_loops}$;
2. $\mathcal{Q} = \langle V, \preceq \rangle$ and $\mathcal{Q} \in \mathfrak{M}^{par\_orders}$;
3. $U(N(\mathbf{x})) \subset N(\mathbf{x})$, for each $\mathbf{x} \in V$.

Condition (3) constrains how the two graph and partial ordering substructures are related to each other – the neighborhood of a vertex in the graph is closed under upper sets in the partial ordering. An example of a structure in $\mathfrak{M}^{mt}$ can be seen in the graph of Figure 3(i) and the corresponding partial ordering in Figure 3(ii); note that the vertices in the graph are the elements of the partial ordering.

The following theorem shows that there is a one-to-one correspondence between the models of T_{mt} and class of structures $\mathfrak{M}^{mt}$ that capture the intended semantics of the mereotopology of MT.

▶ **Theorem 10.** *There exists a bijection*

$$\varphi : Mod(T_{mt}) \rightarrow \mathfrak{M}^{mt}$$

such that

1. *the domain of $\mathcal{M}$ and $\varphi(\mathcal{M})$ are the same;*
2. $(\mathbf{x}, \mathbf{y}) \in \mathbf{C}^{\mathcal{M}}$ *iff* $(\mathbf{x}, \mathbf{y}) \in E^{\varphi(\mathcal{M})}$*;*
3. $(\mathbf{x}, \mathbf{y}) \in \mathbf{P}^{\mathcal{M}}$ *iff* $\mathbf{x} \preceq^{\varphi(\mathcal{M})} \mathbf{y}$.

Suppose $\mathcal{M} \in Mod(T_{mt})$ and $\mathcal{N} = \varphi(\mathcal{M})$. Then $\mathcal{N} \in \mathfrak{M}^{mt}$, and the domain of $\mathcal{M}$ and $\mathcal{N}$ are the same. For each element $\mathbf{x}$ in $\mathcal{M}$ and $\mathcal{N}$, the neighbors of $\mathbf{x}$ in the graph (i.e., $N(\mathbf{x})$) in $\mathcal{N}$ are those which are connected to $\mathbf{x}$ in $\mathcal{M}$. Also, $U(\mathbf{x})$ contains those elements which $\mathbf{x}$ is part of them in $\mathcal{M}$. Thus, Condition (3) in Definition 9 basically captures the monotonicity axiom in T_{mt} (Axiom 16) which says that every element that has a part which is connected to $\mathbf{x}$ is also connected to $\mathbf{x}$.

Theorem 10 gives a characterization of the models of T_{mt} up to isomorphism. Furthermore, since T_{mt} and T_{rcc8^*} are synonymous, this provides a characterization of the models of T_{rcc8^*}.

4.2 Characterization of $\mathfrak{M}^{mt}$

Although Definition 9 gives us a precise specification of the models of T_{mt}, it only provides an implicit characterization; we now outline an explicit characterization that gives us a complete understanding of the possible structures in $\mathfrak{M}^{mt}$, and so models of T_{mt}. The key to the characterization of $\mathfrak{M}^{mt}$ lies in understanding the graphs. In particular, we identify three distinct subgraphs in any structure in $\mathfrak{M}^{mt}$. The first graph is an instance of the following class:

▶ **Definition 11.** Suppose $\mathcal{Q} \in \mathfrak{M}^{par\_orders}$ and $\mathcal{Q} = \langle V, \preceq \rangle$. $\mathcal{U}_{\mathcal{Q}} = \langle V, E \rangle$ is the lower bound graph for $\mathcal{Q}$ iff : $(\mathbf{x}, \mathbf{y}) \in E$ iff exists $\mathbf{z} \in V$ s.t. $\mathbf{z} \preceq \mathbf{x}$, $\mathbf{z} \preceq \mathbf{y}$.

The lower bound graph for the partial ordering in Figure 3(ii) can be seen in Figure 3(iii). Note that the upper sets of elements form cliques in the graph. Lower bound graphs are well-understood within graph theory [14, 12, 3], with two different characterizations. The first is to consider them strictly from a graph-theoretic perspective: $\mathcal{G} = \langle V, E \rangle$ is a lower bound graph iff its vertex clique cover number is equal to its edge clique cover number, where the vertex clique cover number of $\mathcal{G}$ is the minimum number of cliques needed to cover V and the edge clique cover number of $\mathcal{G}$ is the minimum number of cliques needed to cover E.

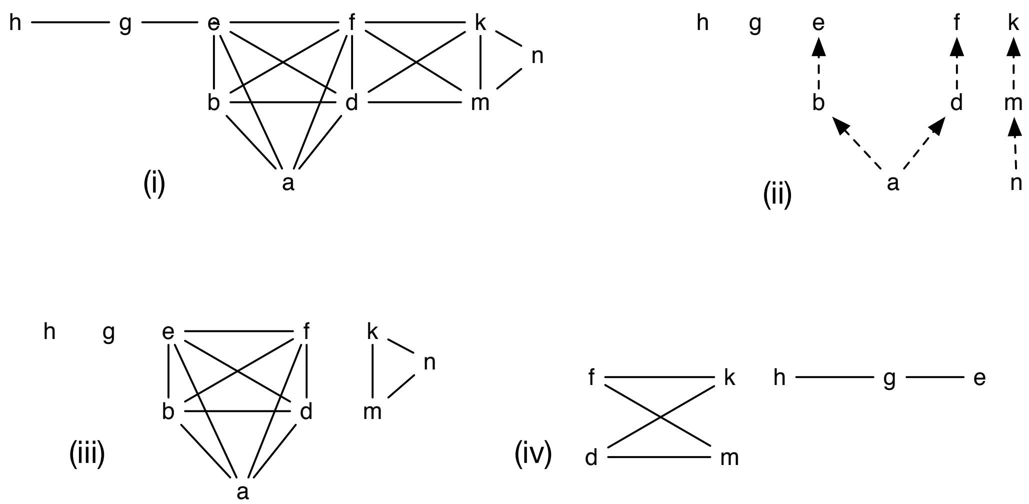

Figure 3 Examples structures and substructures of $\mathfrak{M}^{mt}$. The loops for each vertex in a graph are suppressed to enhance readability.

The second way to look at lower bound graphs is that they are isomorphic to the extension of the "overlaps" relation O. The other two subgraphs within a structure in $\mathfrak{M}^{mt}$ will be used to characterize the relationship between nonoverlapping (externally connected) elements.

We first need to define a few other classes of graphs before we get to the characterization theorem.

▶ **Definition 12.** Let $P = \langle V, \leq \rangle$ be a poset. The graph $G_{\mathbb{P}} = (V, E_P)$ is the comparability graph for P iff $(x, y) \in E_P$ whenever $x < y$ or $y < x$. $G = (V, E)$ is a comparability graph iff there is a poset P such that $G \cong G_P$.

▶ **Definition 13.** A graph G is a permutation graph iff G and $\overline{G}$ are comparability graphs.

This definition is actually the statement of a characterization theorem from [6]; the original definition of permutation graphs with respect to the representation of the elements of a permutation can be found in [16].

▶ **Definition 14.** Suppose $\mathcal{Q} \in \mathfrak{M}^{par\_orders}$. A graph $\mathcal{H}$ is an upper bipartite permutation graph for $\mathcal{Q}$ iff $\mathcal{H} = (V_1 \cup V_2, E)$ is a bipartite permutation graph such that V_1, V_2 are upper sets in $\mathcal{Q}$.

The first subgraph in Figure 3(iv) is an upper bipartite permutation graph for the partial ordering in Figure 3(ii), in which the upper sets are $V_1 = \{\mathbf{d}, \mathbf{f}\}$ and $V_2 = \{\mathbf{m}, \mathbf{k}\}$.

The third subgraph is not an instance of any special class of graphs, but rather can be an arbitrary graph, the only condition being that the vertices are all maximal elements in $\mathcal{Q}$.

▶ **Definition 15.** Suppose $\mathcal{Q} \in \mathfrak{M}^{par\_orders}$. A crown for $\mathcal{Q}$ is a graph $G = (V, E)$ such that all vertices in V are maximal elements of $\mathcal{Q}$ and which are not externally connected to proper parts of any other element.

The second subgraph in Figure 3(iv) is a crown, since its vertices consist entirely of elements $\{\mathbf{e}, \mathbf{g}, \mathbf{h}\}$ that are maximal in $\mathbb{Q}$.

The lower bound graph, the upper bipartite permutation graphs, and the crowns must be combined to form a graph that satisfies the conditions in Definition 9.

▶ **Definition 16.** A graph $\mathcal{G} = \langle V, E \rangle$ is edge-decomposable into a set of graphs H iff
1. $\mathcal{H}_i \subset \mathcal{G}$, for each $\mathcal{H}_i \in H$;
2. $E_i \cap E_j = \emptyset$, for each $\mathcal{H}_i = \langle V_i, E_i \rangle$ and $\mathcal{H}_j = \langle V_j, E_j \rangle$;
3. $E = \bigcup_i E_i$.

Thus, a graph $\mathcal{G}$ is edge-decomposable into a set of subgraphs iff the set of edges in $\mathcal{G}$ can be partitioned. We will use the notation $\mathcal{G} = \mathcal{H}_1 \uplus ... \uplus \mathcal{H}_n$ to indicate that $\mathcal{G}$ is edge-decomposable into $\mathcal{H}_1, ..., \mathcal{H}_n$.

▶ **Theorem 17.** *$\mathcal{M} \in \mathfrak{M}^{mt}$ iff $\mathcal{M} = \langle V, E, \preceq \rangle$ such that*
1. *$\mathcal{Q} = \langle V, \preceq \rangle$ and $\mathcal{Q} \in \mathfrak{M}^{par\_orders}$;*
2. *$\mathcal{G} = \langle V, E \rangle$ and $\mathcal{G} \in \mathfrak{M}^{graph\_loops}$;*
3. *$\mathcal{G} = \mathcal{U}_{\mathcal{Q}} \uplus \mathcal{G}_u \uplus \mathcal{G}_m$ such that*
   (a) *$\mathcal{U}_{\mathcal{Q}}$ is the lower bound graph for $\mathcal{Q}$;*
   (b) *$\mathcal{G}_u$ is decomposable into a set of upper bipartite permutation graphs for $\mathcal{Q}$;*
   (c) *$\mathcal{G}_m$ is a crown for $\mathcal{Q}$.*

Suppose a structure $\mathcal{M} \in \mathfrak{M}^{mt}$ is composed of the graph $\mathcal{G}$ depicted in Figure 3(i) and the corresponding partial ordering $\mathcal{Q}$ depicted in Figure 3(ii). The graph $\mathcal{G}$ is edge-decomposable into $\mathcal{U}_{\mathcal{Q}}$, $\mathcal{G}_u$, and $\mathcal{G}_m$, where $\mathcal{U}_{\mathcal{Q}}$ is the lower bound graph depicted in Figure 3(iii), while $\mathcal{G}_u$ and $\mathcal{G}_m$ are depicted in Figure 3(iv). Suppose $\mathcal{N} \in Mod(T_{mt})$ is the model corresponding with $\mathcal{M}$. Intuitively speaking, $\mathcal{U}_{\mathcal{Q}}$ is the subgraph of $\mathcal{G}$ in which two vertices $\mathbf{x}, \mathbf{y}$ are neighbors whenever $\mathbf{x}$ and $\mathbf{y}$ overlap in $\mathcal{N}$. That is, $\mathcal{U}_{\mathcal{Q}}$ captures the connection relation between overlapping elements of $\mathcal{N}$. $\mathcal{G}_u \cup \mathcal{G}_m$ represents (externally) connected non-overlapping elements of $\mathcal{N}$; that is, $\mathbf{x}$ and $\mathbf{y}$ are neighbours in $\mathcal{G}_u$ whenever in $\mathcal{N}$ they are connected but do not overlap.

Theorem 17 is a characterization theorem for $Mod(T_{mt})$ because it tells us how to construct all possible models of T_{mt} up to isomorphism. We can take an arbitrary lower bound graph, together with a set of upper bipartite permutation graphs, and an arbitrary graph, and combine these graphs together to yield a model of T_{mt}. Given the synonymy of T_{mt} and T_{rcc8*}, this Theorem also characterizes all possible solutions of a set of RCC8 constraints; by synonymy, any solution is isomorphic to a mereology together with a graph that is decomposable into the three subgraphs specified in Theorem 17.

5 Summary

Constraint satisfaction with spatial calculi such as RCC8 has been the predominant application of mereotopology within commonsense reasoning. Yet in some way, this has diminished the role played by the different mereotopology ontologies that were the original sources. It has long been known that the first-order theory of RCC8 is interpretable by the mereotopology ontologies, not only RCC, but also including the rather weak ontology T_{mt}. This perspective has been considered sufficient for showing that RCC8 was in some sense sound with respect to its mereotopological foundations. On the other hand, it has been thought that the first-order theory of RCC8 was too weak to be considered to be a mereotopological ontology in its own right. In this paper, we have shown that indeed the RCC8 theory is mutually interpretable with T_{mt}. Furthermore, by extending the RCC8 theory with sentences that fully capture the intended interpretations of the RCC8 relations, we obtain a theory that is logically synonymous with T_{mt}. Finally, we have provided a characterization of the models of T_{mt} up to isomorphism, by first specifying a class of mathematical structures, and then showing that T_{mt} axiomatizes this class of structures. This characterization gives us insights into the set of all possible solutions for a set of RCC8 constraints. The characterization also lays the groundwork for a new approach to location ontologies, in which we embed the models of a mereotopology of physical objects in a mereotopology of abstract spatial regions.

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