Rule Formats for Nominal Process Calculi*

Luca Aceto[1], Ignacio Fábregas[2], Álvaro García-Pérez[3], Anna Ingólfsdóttir[4], and Yolanda Ortega-Mallén[5]

1 ICE-TCS, School of Computer Science, Reykjavik University, Iceland
   luca@ru.is
2 ICE-TCS, School of Computer Science, Reykjavik University, Iceland; and
   Departamento de Sistemas Informáticos y Computación, Universidad
   Complutense de Madrid, Spain; and
   IMDEA Software Institute, Madrid,Spain
   fabregas@ucm.es
3 ICE-TCS, School of Computer Science, Reykjavik University, Iceland; and
   IMDEA Software Institute, Madrid, Spain
   alvaro.garcia.perez@imdea.org
4 ICE-TCS, School of Computer Science, Reykjavik University, Iceland
   annai@ru.is
5 Departamento de Sistemas Informáticos y Computación, Universidad
   Complutense de Madrid, Spain
   yolanda@ucm.es

Abstract

The nominal transition systems (NTSs) of Parrow et al. describe the operational semantics of nominal process calculi. We study NTSs in terms of the nominal residual transition systems (NRTSs) that we introduce. We provide rule formats for the specifications of NRTSs that ensure that the associated NRTS is an NTS and apply them to the operational specification of the early pi-calculus. Our study stems from the recent Nominal SOS of Cimini et al. and from earlier works in nominal sets and nominal logic by Gabbay, Pitts and their collaborators.

1 Introduction

The goal of this paper is to develop the foundations of a framework for studying the meta-theory of structural operational semantics (SOS) [27] for process calculi with names and name-binding operations, such as the π-calculi [29]. To this end, we build on the large body

of work on rule formats for SOS, as surveyed in [2, 22], and on the nominal techniques of Gabbay, Pitts and their co-workers [8, 15, 25, 30].

Rule formats provide syntactic templates guaranteeing that the models of the calculi, whose semantics they specify, enjoy some desirable properties. A first design decision that has to be taken in developing a theory of rule formats for a class of languages is therefore the choice of the semantic objects specified by the rules. The target semantic model we adopt in our study is that of *nominal transition systems* (NTSs), which have been introduced by Parrow *et al.* in [23, 24] as a uniform model to describe the operational semantics of a variety of calculi with names and name-binding operations. Based on this choice, a basic sanity criterion for a collection of rules describing the operational semantics of a nominal calculus is that they specify an NTS, and we present a rule format guaranteeing this property (Thm. 28).

As a first stepping stone in our study, we introduce *nominal residual transition systems* (NRTSs), and study NTSs in terms of NRTSs (Section 2). More specifically, NRTSs enjoy one desirable property in the setting of nominal calculi, namely that their transition relation is equivariant (which means that it treats names uniformly). NTSs are NRTSs that, in addition to having an equivariant transition relation, satisfy a property Parrow *et al.* call *alpha-conversion of residuals* (see Def. 3 for the details). The latter property formalises a key aspect of calculi in which names can be scoped to represent local resources. To wit, one crucial feature of the π-calculus is scope opening [21]. Consider a transition $p \xrightarrow{\overline{a}(\nu b)} p'$ in which a process p exports a private/local channel name b along channel a. Since the name b is local, it 'can be subject to alpha-conversion' [23] and the transitions $p \xrightarrow{\overline{a}(\nu c)} p\{b/c\}$ should also be present for each 'fresh name' c.

In contrast to related work [7, 9], our approach uses nominal terms to connect the specification system with the semantic model. This has the advantage of capturing the requirement that transitions be 'up to alpha-equivalence' (typical in nominal calculi) without instrumenting alpha-conversion explicitly in the specification system.

We specify an NRTS by means of a nominal residual transition system specification (NRTSS), which describes the syntax of a nominal calculus in terms of a nominal signature (Section 3) and its semantics by means of a set of inference rules (Section 4). We develop the basic theory of the NRTS/NRTSS framework, building on the nominal algebraic datatypes of Pitts [25] and the nominal rewriting framework of Fernandez and Gabbay [9]. Based on this framework, we provide rule formats [2, 22] for NRTSSs (Section 5) that ensure that the induced transition relation is equivariant (Theorem 22) and enjoys alpha-conversion of residuals (Theorem 28), and is therefore an NTS. Section 6 presents an example of application of our rule formats to the setting of the π-calculus, and Section 7 discusses avenues for future work, as well as related work, and concludes.

2   Preliminaries

Nominal sets

We follow earlier foundational work by Gabbay and Pitts on nominal sets in [17, 25, 26]. We assume a countably infinite set $\mathbb{A}$ of *atoms* and consider Perm $\mathbb{A}$ as the group of *finite permutations of atoms* (hereafter *permutations*) ranged over by π, where we write ι for the *identity*, $\circ$ for *composition* and π^{-1} for the *inverse* of permutation π. We are particularly interested in *transpositions* of two atoms: $(a\,b)$ stands for the permutation that swaps a with b and leaves all other atoms fixed. Every permutation π is equal to the composition of a finite number of transpositions, i.e. $\pi = (a_1\,b_1) \circ \ldots \circ (a_n\,b_n)$ with $n \geq 0$.

An *action* of the group Perm $\mathbb{A}$ on a set S is a binary operation mapping each $\pi \in \text{Perm } \mathbb{A}$ and $s \in S$ to an element $\pi \cdot s \in S$, and satisfying the identity law $\iota \cdot s = s$ and the composition law $(\pi_1 \circ \pi_2) \cdot s = \pi_1 \cdot (\pi_2 \cdot s)$. A Perm $\mathbb{A}$-*set* is a set equipped with an action of Perm $\mathbb{A}$.

We say that a set of atoms A *supports* an object s iff $\pi \cdot s = s$ for every permutation π that leaves each element $a \in A$ invariant. In particular, we are interested in sets all of whose elements have finite support (Def. 2.2 of [25]).

▶ **Definition 1** (Nominal sets). A *nominal set* is a Perm $\mathbb{A}$-set all of whose elements are finitely supported.

For each element s of a nominal set, we write $\text{supp}(s)$ for the least set that supports s, called the *support* of s. (Intuitively, the action of permutations on a set S determines that a finitely supported $s \in S$ only depends on atoms in $\text{supp}(s)$, and no others.) The set $\mathbb{A}$ of atoms is a nominal set when $\pi \cdot a = \pi a$ since $\text{supp}(a) = \{a\}$ for each atom $a \in \mathbb{A}$. The set Perm $\mathbb{A}$ of finite permutations is also a nominal set where the permutation action on permutations is given by conjugation, i.e. $\pi \cdot \pi' = \pi \circ \pi' \circ \pi^{-1}$, and the support of a permutation π is $\text{supp}(\pi) = \{a \mid \pi a \neq a\}$.

Given two Perm $\mathbb{A}$-sets S and T and a function $f : S \to T$, the action of permutation π on function f is given by conjugation, i.e. $(\pi \cdot f)(s) = \pi \cdot f(\pi^{-1} \cdot s)$ for each $s \in S$. We say that a function $f : S \to T$ is *equivariant* iff $\pi \cdot f(s) = f(\pi \cdot s)$ for every $\pi \in \text{Perm } \mathbb{A}$ and every $s \in S$. The intuition is that an equivariant function f is atom-blind, in that f does not treat any atom preferentially. It turns out that a function f is equivariant iff $\text{supp}(f) = \emptyset$ (Rem. 2.13 of [25]). The function supp is equivariant (Prop. 2.11 of [25]).

Let S be a Perm $\mathbb{A}$-set, we write S_{fs} for the nominal set that contains the elements in S that are finitely supported. The *nominal function set* between nominal sets S and T is the nominal set $S \to_{\text{fs}} T$ of finitely supported functions from S to T—be they equivariant or not. Let S_1 and S_2 be nominal sets. The product $S_1 \times S_2$ is a nominal set (Prop. 2.14 of [25]). The permutation action for products is given componentwise (Eq (1.12) of [25]).

An element $s_1 \in S_1$ *is fresh in* $s_2 \in S_2$, written $s_1 \# s_2$, iff $\text{supp}(s_1) \cap \text{supp}(s_2) = \emptyset$. The freshness relation is equivariant (Eq. (3.2) of [25]).

Finally, we consider *atom abstractions* (Section 4 of [25]), which represent alpha-equivalence classes of elements.

▶ **Definition 2** (Atom abstraction). Given a nominal set S, the *atom abstraction* of atom a in element $s \in S$, written $\langle a \rangle s$, is the Perm $\mathbb{A}$-set $\langle a \rangle s = \{(b, (b\,a) \cdot s) \mid b = a \vee b \# s\}$, whose permutation action is $\pi \cdot \langle a \rangle s = \{(\pi \cdot b, \pi \cdot ((b\,a) \cdot s)) \mid \pi \cdot b = \pi \cdot a \vee \pi \cdot b \# \pi \cdot s\}$.

We write $[\mathbb{A}]S$ for the set of *atom abstractions* in elements of S, which is a nominal set (Def. 4.4 of [25]), since $\text{supp}(\langle a \rangle s) = \text{supp}(s) \setminus \{a\}$ for each atom a and element $s \in S$.

Nominal Transition Systems

Nominal transitions systems adopt the state/residual presentation for transitions of [4], where a residual is a pair consisting of an action and a state. In [23], Parrow *et al.* develop modal logics for process algebras à la Hennessy-Milner. Here we are mainly interested in the transition relation and we adapt Definition 1 in [23] by removing the predicates. We write $\mathcal{P}_\omega(\mathbb{A})$ for the *finite power set* of $\mathbb{A}$.

▶ **Definition 3** (Nominal transition system). A *nominal transition system* (NTS) is a quadruple $(S, Act, \text{bn}, \longrightarrow)$ where S and Act are nominal sets of *states* and *actions* respectively, $\text{bn} : Act \to \mathcal{P}_\omega(\mathbb{A})$ is an equivariant function that delivers the *binding names* in an action, and $\longrightarrow \subseteq S \times (Act \times S)$ is an equivariant binary transition relation from states to *residuals*

(we let $Act \times S$ be the set of residuals). The function bn is such that $\mathrm{bn}(\ell) \subseteq \mathrm{supp}(\ell)$ for each $\ell \in Act$. We often write $p \longrightarrow (\ell, p')$ in lieu of $(p, (\ell, p')) \in \longrightarrow$.

Finally, the transition relation $\longrightarrow$ must satisfy *alpha-conversion of residuals*, that is, if $a \in \mathrm{bn}(\ell)$, $b\#(\ell, p')$ and $p \longrightarrow (\ell, p')$ then also $p \longrightarrow ((a\,b) \cdot \ell, (a\,b) \cdot p')$, or equivalently $p \longrightarrow (a\,b) \cdot (\ell, p')$.

We will consider an NTS (without its associated binding-names function bn) as a particular case of a nominal residual transition system, which we introduce next.

▶ **Definition 4** (Nominal residual transition system)**.** A *nominal residual transition system* (NRTS) is a triple $(S, R, \longrightarrow)$ where S and R are nominal sets, and where $\longrightarrow \subseteq S \times R$ is an equivariant binary transition relation. We say S is the set of *states* and R is the set of *residuals*.

The connection between NTSs and NRTSs will be explained in more detail in Section 5.

3   Nominal terms

This section is devoted to the notion of nominal terms, which are syntactic objects that make use of the atom abstractions of Definition 2 and represent terms up to alpha-equivalence. As a first step, we introduce raw terms, devoid of any notion of alpha-equivalence. Our raw terms resemble those from the literature, mainly [8, 9, 25, 30], but with some important differences. In particular, our terms include both variables (i.e. unknowns) and moderated terms (i.e. explicit permutations over raw terms), and we consider atom and abstraction sorts. (The raw terms of [25] do not include moderated terms, and the ones in [9, 30] only consider moderated variables. In [8] the authors consider neither atom nor abstraction sorts.) We also adopt the classic presentation of free algebras and term algebras in [6, 18] in a different way from that in [8, 25]. The raw terms correspond to the standard notion of free algebra over a signature generated by a set of variables. We then adapt the Σ-structures of [8] to our sorting schema. Finally, the nominal terms are the interpretations of the ground terms in the initial Σ-structure; they coincide with the nominal algebraic terms of [25].

▶ **Definition 5** (Nominal signature and nominal sort)**.** A *nominal signature* (or simply a *signature*) Σ is a triple (Δ, A, F) where $\Delta = \{\delta_1, \ldots, \delta_n\}$ is a finite set of *base sorts*, A is a countable set of *atom sorts*, and F is a finite set of *function symbols*. The *nominal sorts* over Δ and A are given by the grammar $\sigma ::= \delta \mid \alpha \mid [\alpha]\sigma \mid \sigma_1 \times \ldots \times \sigma_k$, with $k \geq 0$, $\delta \in \Delta$ and $\alpha \in A$. The sort $[\alpha]\sigma$ is the *abstraction sort*. Symbol $\times$ denotes the *product sort*, which is associative; $\sigma_1 \times \ldots \times \sigma_k$ stands for the sort of the empty product when $k = 0$, which we may write as **1**. We write S for the set of nominal sorts. We arrange the function symbols in F based on the sort of the data that they produce. We write $f_{ij} \in F$ with $1 \leq i \leq n$ and $1 \leq j \leq m_i$ such that f_{ij} has arity $\sigma_{ij} \to \delta_i$, where δ_i is a base sort.

The theory of nominal sets extends to the case of (countably) many-sorted atoms (see Section 4.7 in [25]). We assume that $\mathbb{A}$ contains a countably infinite collection of atoms a_α, b_α, c_α, $\ldots$ for each atom sort α such that the sets of atoms $\mathbb{A}_\alpha$ of each sort are mutually disjoint. We write $\mathrm{Perm}_s \mathbb{A} = \{\pi \in \mathrm{Perm}\,\mathbb{A} \mid \forall \alpha \in A. \forall a \in \mathbb{A}_\alpha. \pi\,a \in \mathbb{A}_\alpha\}$ for the subgroup of finite permutations that respect the sorting. The sorted nominal sets are the $\mathrm{Perm}_s \mathbb{A}$-sets whose elements are finitely supported. In the sequel we may drop the s subscript in $\mathrm{Perm}_s \mathbb{A}$ and omit the 'sorted' epithet from 'sorted nominal sets'.

We let $\mathcal{V}$ be a set that contains a countably infinite collection of *variable names* (variables for short) x_σ, y_σ, z_σ, $\ldots$ for each sort σ, such that the sets of variables $\mathcal{V}_\sigma$ of each sort are mutually disjoint. We also assume that $\mathcal{V}$ is disjoint from $\mathbb{A}$.

▶ **Definition 6** (Raw terms). Let $\Sigma = (\Delta, A, F)$ be a signature. The set of *raw terms over signature Σ and set of variables $\mathcal{V}$* (*raw terms* for short) is given by the grammar

$$t_\sigma ::= \quad x_\sigma \mid a_\alpha \mid (\pi \bullet t_\sigma)_\sigma \mid ([a_\alpha]t_\sigma)_{[\alpha]\sigma} \mid (t_{\sigma_1}, \ldots, t_{\sigma_k})_{\sigma_1 \times \ldots \times \sigma_k} \mid (f_{ij}(t_{\sigma_{ij}}))_{\delta_i},$$

where term x_σ is a *variable* of sort σ, term a_α is an *atom* of sort α, term $(\pi \bullet t_\sigma)_\sigma$ is a *moderated term* (i.e. the explicit, or delayed, permutation π over term t_σ), term $([a_\alpha]t_\sigma)_{[\alpha]\sigma}$ is the *abstraction of atom a_α in term t_σ*, term $(t_{\sigma_1}, \ldots, t_{\sigma_k})_{\sigma_1 \times \ldots \times \sigma_k}$ is the *product of terms t_{σ_1}, ..., t_{σ_k}*, and term $(f_{ij}(t_{\sigma_{ij}}))_{\delta_i}$ is the *datum of base sort δ_i constructed from term $t_{\sigma_{ij}}$ and function symbol $f_{ij} : \sigma_{ij} \to \delta_i$*. When they are clear from the context or immaterial, we leave the arities and sorts implicit and write x, a, $\pi \bullet t$, $[a]t$, $(t_1, \ldots, t_k)$, $f(t)$, etc.

The raw terms are the inhabitants of the carrier of the free algebra over the set of variables $\mathcal{V}$ and over the S-sorted conventional signature that consists of the function symbols in F, together with a constant symbol for each atom a_α, a unary symbol that produces moderated terms for each permutation π and each sort σ, a unary symbol that produces abstractions for each atom a_α and sort σ, and a k-ary symbol that produces a product of sort $\sigma_1 \times \ldots \times \sigma_k$ for each sequence of sorts σ_1, ..., σ_k. (See [18] for a classic presentation of term algebras, initial algebra semantics and free algebras.)

We write $\mathbb{T}(\Sigma, \mathcal{V})_\sigma$ for the set of raw terms of sort σ. A raw term t is *ground* iff no variables occur in t. We write $\mathbb{T}(\Sigma)_\sigma$ for the set of ground terms of sort σ. The sets of raw terms (resp. ground terms) of each sort are mutually disjoint as terms carry sort information. Therefore we sometimes identify the family $(\mathbb{T}(\Sigma, \mathcal{V})_\sigma)_{\sigma \in \mathsf{S}}$ of S-indexed raw terms and the family $(\mathbb{T}(\Sigma)_\sigma)_{\sigma \in \mathsf{S}}$ of S-indexed ground terms with their respective ranges $\bigcup_{\sigma \in \mathsf{S}} \mathbb{T}(\Sigma, \mathcal{V})_\sigma$ and $\bigcup_{\sigma \in \mathsf{S}} \mathbb{T}(\Sigma)_\sigma$, which we abbreviate as $\mathbb{T}(\Sigma, \mathcal{V})$ and $\mathbb{T}(\Sigma)$ respectively.

The set $\mathbb{T}(\Sigma, \mathcal{V})$ of raw terms is a nominal set, with the Perm $\mathbb{A}$-action and the support of a raw term given by:

$$
\begin{aligned}
\pi \cdot x &= x & \mathrm{supp}(x) &= \emptyset \\
\pi \cdot a &= \pi\,a & \mathrm{supp}(a) &= \{a\} \\
\pi \cdot (\pi_1 \bullet t) &= (\pi \cdot \pi_1) \bullet (\pi \cdot t) & \mathrm{supp}(\pi \bullet t) &= \mathrm{supp}(\pi) \cup \mathrm{supp}(t) \\
\pi \cdot [a]t &= [\pi\,a](\pi \cdot t) & \mathrm{supp}([a](t)) &= \{a\} \cup \mathrm{supp}(t) \\
\pi \cdot (t_1, \ldots, t_k) &= (\pi \cdot t_1, \ldots, \pi \cdot t_k) & \mathrm{supp}((t_1, \ldots, t_k)) &= \mathrm{supp}(t_1) \cup \ldots \cup \mathrm{supp}(t_k) \\
\pi \cdot (f(t)) &= f(\pi \cdot t), & \mathrm{supp}(f(t)) &= \mathrm{supp}(t).
\end{aligned}
$$

It is straightforward to check that the permutation action for raw terms is sort-preserving (remember that permutations are also sort-preserving). The set $\mathbb{T}(\Sigma)$ of ground terms is also a nominal set since it is closed with respect to the Perm $\mathbb{A}$-action given above.

▶ **Example 7** (π-calculus). Consider a signature Σ for the π-calculus [7, 29] given by a single atom sort ch of channel names, and base sorts pr and ac for processes and actions respectively. The function symbols (adapted from [29]) are the following:

$$
\begin{aligned}
F = \{ \ & null : \mathbf{1} \to \mathsf{pr}, & par : (\mathsf{pr} \times \mathsf{pr}) \to \mathsf{pr}, & \quad tauA : \mathbf{1} \to \mathsf{ac}, \\
& tau : \mathsf{pr} \to \mathsf{pr}, & sum : (\mathsf{pr} \times \mathsf{pr}) \to \mathsf{pr}, & \quad inA : (\mathsf{ch} \times \mathsf{ch}) \to \mathsf{ac}, \\
& in : (\mathsf{ch} \times [\mathsf{ch}]\mathsf{pr}) \to \mathsf{pr}, & rep : \mathsf{pr} \to \mathsf{pr}, & \quad outA : (\mathsf{ch} \times \mathsf{ch}) \to \mathsf{ac}, \\
& out : (\mathsf{ch} \times \mathsf{ch} \times \mathsf{pr}) \to \mathsf{pr}, & new : [\mathsf{ch}]\mathsf{pr} \to \mathsf{pr}, & \quad boutA : (\mathsf{ch} \times \mathsf{ch}) \to \mathsf{ac} \ \}.
\end{aligned}
$$

Recalling terminology from [7, 29], *null* stands for inaction, *tau(p)* for the internal action after which process p follows, *in(a, [b]p)* for the input at channel a where the input name is bound to b in the process p that follows, *out(a, b, p)* for the output of name b through

channel a after which process p follows, $par(p, q)$ for parallel composition, $sum(p, q)$ for nondeterministic choice, $rep(p)$ for parallel replication, and $new([a]p)$ for the restriction of channel a in process p (a is private in p). Actions and processes belong to different sorts. We use $tauA$, $outA(a, b)$, $inA(a, b)$ and $boutA(a, b)$ respectively for the internal action, the output action, the input action and the bound output action.

The set of terms of the π-calculus corresponds to the subset of ground terms over Σ of sort pr and ac in which no moderated (sub-)terms occur. For instance, the process $(\nu b)(\overline{a}b.0)$ corresponds to the ground term $new([b](out(a, b, null)))$, whose support is $\{a, b\}$. Both free and bound channel names (such as the a and b respectively in the example process) are represented by atoms. The set of ground terms also contains generalised processes and actions with moderated (sub-)terms $\pi \bullet p$, which stand for a delayed permutation π that ought to be applied to a term p, e.g. $new(\pi \bullet ([b](out(a, b, null))))$.   ◀

Raw terms allow variables to occur in the place of any ground subterm. The variables represent *unknowns*, and should be mistaken neither with free nor bound channel names. For instance, the raw term $new([b](out(a, b, x)))$ represents a process $(\nu b)(\overline{a}b.P)$ where the x is akin to the meta-variable P, which stands for some unknown process. The process $(\nu b)(\overline{a}b.P)$ unifies with $(\nu b)(\overline{a}b.0)$ by replacing P with 0. In the nominal setting, the raw term $new([b](out(a, b, x)))$ unifies with ground term $new([b](out(a, b, null)))$, by means of a *substitution* φ such that $\varphi(x) = null$. Formally, substitutions are defined below.

▶ **Definition 8** (Substitution). A *substitution* $\varphi : \mathcal{V} \rightarrow_{\text{fs}} \mathbb{T}(\Sigma, \mathcal{V})$ is a sort-preserving, finitely supported function from variables to raw terms. The *domain* $\text{dom}(\varphi)$ *of a substitution* φ is the set $\{x \mid \varphi(x) \neq x\}$. A substitution φ is *ground* iff $\varphi(x) \in \mathbb{T}(\Sigma)$ for every variable $x \in \text{dom}(\varphi)$.

The set of substitutions is a nominal set. The *extension to raw terms* $\overline{\varphi}$ *of substitution* φ is the unique homomorphism induced by φ from the free algebra $\mathbb{T}(\Sigma, \mathcal{V})$ to itself, which coincides with the function given by:

$$
\begin{array}{rclcrcl}
\overline{\varphi}(x) & = & \varphi(x) & & \overline{\varphi}([a]t) & = & [a](\overline{\varphi}(t)) \\
\overline{\varphi}(a) & = & a & & \overline{\varphi}(t_1, \ldots, t_k) & = & (\overline{\varphi}(t_1), \ldots, \overline{\varphi}(t_k)) \\
\overline{\varphi}(\pi \bullet t) & = & \pi \bullet \overline{\varphi}(t) & & \overline{\varphi}(f(t)) & = & f(\overline{\varphi}(t)).
\end{array}
$$

Given substitutions φ and γ we write $\varphi \circ \gamma$ for their composition, which is defined as follows: For every variable x, $(\varphi \circ \gamma)(x) = \overline{\varphi}(t)$ where $\gamma(x) = t$. It is straightforward to check that $(\overline{\varphi \circ \gamma})(t) = \overline{\varphi}(\overline{\gamma}(t))$. We note that our definition of substitution is different form those in both [8, 30], where the authors consider a function that performs the delayed permutations of the moderated terms *on-the-fly*.

▶ **Lemma 9** (Extension to raw terms is equivariant). *Let φ be a substitution and π a permutation. Then, $\pi \cdot \overline{\varphi} = \overline{\pi \cdot \varphi}$.*

It is straightforward to check that the support of $\overline{\varphi}$ coincides with the support of φ. By the above lemma, the set of extended substitutions is also a nominal set, since it is closed with respect to the Perm $\mathbb{A}$-action. Hereafter we sometimes write $\varphi(t)$, where t is a raw term, instead of $\overline{\varphi}(t)$. We may also write φ^π instead of $\pi \cdot \overline{\varphi}$ or $\overline{\pi \cdot \varphi}$ for short.

The following result highlights the relation between substitution and the permutation action.

▶ **Lemma 10** (Substitution and permutation action). *Let φ be a substitution, π a permutation and t a raw term. Then, $\pi \cdot \varphi(t) = \varphi^\pi(\pi \cdot t)$.*

Our goal is to give meaning to ground terms in nominal sets. To this end, we need a suitable class of algebraic structures that can be used to give an *interpretation* of those ground terms.

▶ **Definition 11** (Σ-structure). Let $\Sigma = (\Delta, A, F)$ be a signature. A Σ-*structure M* consists of a nominal set $M[\![\sigma]\!]$ for each sort σ defined as follows

$$
\begin{aligned}
M[\![\alpha]\!] &= \mathbb{A}_\alpha \\
M[\![[\alpha]\sigma]\!] &= [\mathbb{A}_\alpha](M[\![\sigma]\!]) \\
M[\![\sigma_1 \times \ldots \times \sigma_k]\!] &= M[\![\sigma_1]\!] \times \ldots \times M[\![\sigma_k]\!],
\end{aligned}
$$

where the $M[\![\delta_i]\!]$ with $\delta_i \in \Delta$ are given, as well as an equivariant function $M[\![f_{ij}]\!] : M[\![\sigma_{ij}]\!] \to M[\![\delta_i]\!]$ for each symbol $(f_{ij})_{\sigma_{ij} \to \delta_i} \in F$.

The notion of Σ-structure adapts that of Σ-structure in [8] to our sorting convention with atom and abstraction sorts. The Σ-structures characterise a range of interpretations of ground terms into elements of nominal sets, such that any sort σ gives rise to the expected nominal set, i.e. atom sorts give rise to sets of atoms, abstraction sorts give rise to sets of atom abstractions, and product sorts give rise to finite products of nominal sets.

Next we define the *interpretation of a ground term in a Σ-structure*, which resembles the *value of a term* in [8].

▶ **Definition 12** (Interpretation of ground terms in a Σ-structure). Let Σ be a signature and M be a Σ-structure. The *interpretation $M[\![p]\!]$ of a ground term p in M* is given by:

$$
\begin{aligned}
M[\![a]\!] &= a & M[\![(p_1, \ldots, p_k)]\!] &= (M[\![p_1]\!], \ldots, M[\![p_k]\!]) \\
M[\![\pi \bullet p]\!] &= \pi \cdot M[\![p]\!] & M[\![f(p)]\!] &= M[\![f]\!](M[\![p]\!]). \\
M[\![[a]p]\!] &= \langle a \rangle (M[\![p]\!])
\end{aligned}
$$

The next lemma states that interpretation in a Σ-structure is equivariant and highlights the relation between interpretation and moderated terms.

▶ **Lemma 13** (Interpretation and moderated terms). *Let M be a Σ-structure. Interpretation in M is equivariant, that is, $\pi \cdot M[\![p]\!] = M[\![\pi \cdot p]\!]$ for every ground term p and permutation π. Moreover, $M[\![\pi \bullet p]\!] = M[\![\pi \cdot p]\!]$.*

Finally, we introduce the Σ-structure NT, which formalises the set of *nominal terms*.

▶ **Definition 14** (Σ-structure for nominal terms). Let Σ be a signature. The Σ-*structure NT for nominal terms* is given by the least tuple $(NT[\![\delta_1]\!], \ldots, NT[\![\delta_n]\!])$ satisfying

$$NT[\![\delta_i]\!] = NT[\![\sigma_{i1}]\!] + \ldots + NT[\![\sigma_{im_i}]\!] \quad \text{for each base sort } \delta_i \in \Delta, \text{ and}$$

$NT[\![f_{ij}]\!] = \mathrm{inj}_j : NT[\![\sigma_{ij}]\!] \to NT[\![\delta_i]\!]$, for each function symbol $f_{ij} \in F$.

In the conditions above, the 'less than or equal to' relation for tuples is pointwise set inclusion. The $NT[\![f_{ij}]\!]$ is the jth injection of the ith component in $(NT[\![\delta_1]\!], \ldots, NT[\![\delta_n]\!])$.

Nominal terms represent alpha-equivalence classes of raw terms by using the atom abstractions of Definition 2.

▶ **Definition 15** (Nominal terms). Let Σ be a signature. The set $\mathbb{N}(\Sigma)_\sigma$ of *nominal terms over Σ of sort σ* is the domain of interpretation of the ground terms of sort σ in the Σ-structure NT, that is, $\mathbb{N}(\Sigma)_\sigma = NT[\![\sigma]\!]$.

We sometimes write p, ℓ instead of $NT[\![p]\!]$, $NT[\![\ell]\!]$ when it is clear from the context that we are referring to the interpretation into nominal terms of ground terms p and ℓ.

It can be checked that the nominal sets $\mathbb{N}(\Sigma)_\sigma$ coincide (up to isomorphism) with the nominal algebraic datatypes of Definition 8.9 in [25], and therefore by Theorem 8.15 in [25] the nominal terms represent alpha-equivalence classes of raw terms.

4   Specifications of NRTSs

The NRTSs of Definition 4 are meant to be a model of computation for calculi with name-binding operators and state/residual presentation. In this section we present syntactic specifications for NRTSs. We start by defining nominal residual signatures.

▶ **Definition 16** (Nominal residual signature). A *nominal residual signature* (a residual signature for short) is a quintuple $\Sigma = (\Delta, A, \sigma, \rho, F)$ such that (Δ, A, F) is a nominal signature and σ and ρ are distinguished nominal sorts over Δ and A, which we call *state sort* and *residual sort* respectively. We say that $\mathbb{N}(\Sigma)_\sigma$ is the set of *states* and $\mathbb{N}(\Sigma)_\rho$ is the set of *residuals*.

Let $\mathcal{T} = (S, R, \longrightarrow)$ be an NRTS and $\Sigma = (\Delta, A, \sigma, \rho, F)$ be a residual signature. We say that $\mathcal{T}$ is an NRTS *over signature* Σ iff the sets of states S and residuals R coincide with the sets of nominal terms of state sort $\mathbb{N}(\Sigma)_\sigma$ and residual sort $\mathbb{N}(\Sigma)_\rho$ respectively.

Our next goal is to introduce syntactic specifications of NRTSs, which we call nominal residual transition system specifications. To this end, we will make use of residual formulas and freshness assertions over raw terms, which are defined below.

▶ **Definition 17** (Residual formula and freshness assertion). A *residual formula* (a *formula* for short) over a residual signature Σ is a pair (s, r), where $s \in \mathbb{T}(\Sigma, \mathcal{V})_\sigma$ and $r \in \mathbb{T}(\Sigma, \mathcal{V})_\rho$. We use the more suggestive $s \longrightarrow r$ in lieu of (s, r). A formula $s \longrightarrow r$ is *ground* iff s and r are ground terms.

A *freshness assertion* (an *assertion* for short) over a signature Σ is a pair (a, t) where $a \in \mathbb{A}$ and $t \in \mathbb{T}(\Sigma, \mathcal{V})$. We will write $a \mathbin{\not\#} t$ in lieu of (a, t). An assertion is *ground* iff t is a ground term.

▶ Remark. Formulas and assertions are raw syntactic objects, similar to raw terms, which will occur in the rules of the nominal residual transition system specifications to be defined, and whose purpose is to represent respectively transitions and freshness relations involving nominal terms. A formula $s \longrightarrow r$ (resp. an assertion $a \mathbin{\not\#} t$) unifies with a ground formula $\varphi(s) \longrightarrow \varphi(r)$ (resp. a ground assertion $a \mathbin{\not\#} \varphi(t)$), which in turn represents a transition $NT[\![\varphi(s)]\!] \longrightarrow NT[\![\varphi(r)]\!]$ (resp. a freshness relation $a\#NT[\![\varphi(t)]\!]$). For the assertions, notice how the symbols $\mathbin{\not\#}$, $\#$ and $NT[\![\ ]\!]$ interact. The ground assertion $a \mathbin{\not\#} [a]a$ represents the freshness relation $a\#NT[\![[a]a]\!]$, which is true. On the other hand, the freshness relation $a\#[a]a$ is false because $a \in \mathrm{supp}([a]a)$.

Permutation action and substitution extend to formulas and assertions in the expected way. Formulas and assertions are elements of nominal sets. Their support is the union of the supports of the raw terms in them, hence we write $\mathrm{supp}(t \longrightarrow t')$ and $\mathrm{supp}(a \mathbin{\not\#} t)$. We will also write $b\#(t \longrightarrow t')$ and $b\#(a \mathbin{\not\#} t)$ for freshness relations involving formulas and assertions respectively.

▶ **Definition 18** (Nominal residual transition system specification)**.** Let Σ be a residual signature $(\Delta, A, \sigma, \rho, F)$. A *transition rule over* Σ (a *rule*, for short) is of the form

$$\frac{\{u_i \longrightarrow u_i' \mid i \in I\} \quad \{a_j \not\approx v_j \mid j \in J\}}{t \longrightarrow t'} \; ,$$

abbreviated as $H, \nabla/t \longrightarrow t'$, where $H = \{u_i \longrightarrow u_i' \mid i \in I\}$ is a finitely supported set of formulas over Σ (we call H the set of *premisses*) and where $\nabla = \{a_j \not\approx v_j \mid j \in J\}$ is a finite set of assertions over Σ (we call ∇ the *freshness environment*). We say formula $t \longrightarrow t'$ over Σ is the *conclusion*, where t is the *source* and t' is the *target*. A rule is an *axiom* iff it has an empty set of premisses. Note that axioms might have a non-empty freshness environment.

A *nominal residual transition system specification over* Σ (abbreviated to NRTSS) is a set of transition rules over Σ.

Permutation action and substitution extend to rules in the expected way; they are applied to each of the formulas and freshness assertions in the rule.

Notice that the rules of an NRTSS are elements of a nominal set. The support of a rule $H, \nabla/t \longrightarrow t'$ is the union of the support of H, the support of ∇ and the support of $t \longrightarrow t'$. In the sequel we write supp(Ru) for the support of rule Ru, and $a\#$Ru for a freshness relation involving atom a and rule Ru. Observe that the set H of premisses of a rule may be infinite, but its support must be finite. However, the freshness environment ∇ must be finite in order to make the simplification rules of Definition 23 to follow terminating. These simplification rules will be used in Section 5 to define the rule format in Definition 27.

Let $\mathcal{R}$ be an NRTSS. We say that the formula $s \longrightarrow r$ *unifies* with rule Ru in $\mathcal{R}$ iff Ru has conclusion $t \longrightarrow t'$ and $s \longrightarrow r$ is a substitution instance of $t \longrightarrow t'$. If s and r are ground terms, we also say that transition $NT[\![s]\!] \longrightarrow NT[\![r]\!]$ unifies with Ru.

▶ **Definition 19** (Proof tree)**.** Let Σ be a residual signature and $\mathcal{R}$ be an NRTSS over Σ. A proof tree in $\mathcal{R}$ of a transition $NT[\![s]\!] \longrightarrow NT[\![r]\!]$ is an upwardly branching rooted tree without paths of infinite length whose nodes are labelled by transitions such that
 (i) the root is labelled by $NT[\![s]\!] \longrightarrow NT[\![r]\!]$, and
 (ii) if $K = \{NT[\![q_i]\!] \longrightarrow NT[\![q_i']\!] \mid i \in I\}$ is the set of labels of the nodes directly above a node with label $NT[\![p]\!] \longrightarrow NT[\![p']\!]$, then there exist a rule

$$\frac{\{u_i \longrightarrow u_i' \mid i \in I\} \qquad \{a_j \not\approx v_j \mid j \in J\}}{t \longrightarrow t'}$$

in $\mathcal{R}$ and a ground substitution φ such that $\varphi(t \longrightarrow t') = p \longrightarrow p'$ and, for each $i \in I$ and for each $j \in J$, $\varphi(u_i \longrightarrow u_i') = q_i \longrightarrow q_i'$ and $a_j\#NT[\![\varphi(v_j)]\!]$ hold.
We say that $NT[\![s]\!] \longrightarrow NT[\![r]\!]$ is *provable* in $\mathcal{R}$ iff it has a proof tree in $\mathcal{R}$. The transition relation specified by $\mathcal{R}$ consists of all the transitions that are provable in $\mathcal{R}$.

The nodes of a proof tree are labelled by transitions, which contain nominal terms (i.e. syntactic objects that use the atom abstractions of Definition 2). The use of nominal terms captures the convention in typical nominal calculi of considering terms 'up to alpha-equivalence'.

The fact that the nodes of a proof tree are labelled by nominal terms is the main difference between our approach and previous work in nominal structural operational semantics [1], nominal rewriting [9, 30] and nominal algebra [15]. In all these works, the 'up-to-alpha-equivalence' transitions are explicitly instrumented within the model of computation by adding to the specification system inference rules that perform alpha-conversion of raw terms.

5   Rule formats for NRTSSs

This section defines two rule formats for NRTSSs that ensure that:

(i) an NRTSS induces an equivariant transition relation, and thus an NRTS of Definition 4;

(ii) an NRTSS induces a transition relation which, together with an equivariant function bn, corresponds to an NTS of Definition 3 [23]. For the latter, we need to ensure that the induced transition relation is equivariant and satisfies *alpha-conversion of residuals* (recall, if $p \longrightarrow (\ell, p')$ is provable in $\mathcal{R}$ and a is in the set of binding names of ℓ, then for every atom b that is fresh in (ℓ, p') the transition $p \longrightarrow (a\,b) \cdot (\ell, p')$ is also provable).

As a first step, we introduce a rule format ensuring equivariance of the induced transition relation.

▶ **Definition 20** (Equivariant format). Let $\mathcal{R}$ be an NRTSS. $\mathcal{R}$ is in *equivariant format* iff the rule $(a\,b) \cdot \text{Ru}$ is in $\mathcal{R}$, for every rule Ru in $\mathcal{R}$ and for each $a, b \in \mathbb{A}$.

▶ **Lemma 21.** *Let $\mathcal{R}$ be an NRTSS in equivariant format. For every rule* Ru *in $\mathcal{R}$ and for every permutation π, the rule $\pi \cdot \text{Ru}$ is in $\mathcal{R}$.*

▶ **Theorem 22** (Rule format for NRTSs). *Let $\mathcal{R}$ be an NRTSS. If $\mathcal{R}$ is in equivariant format then $\mathcal{R}$ induces an NRTS.*

Before introducing a rule format ensuring alpha-conversion of residuals, we adapt to our freshness environments the simplification rules and the entailment relation of Definition 10 and Lemma 15 in [9], which we will use in the definition of the rule format.

▶ **Definition 23** (Simplification of freshness environments). Consider a signature Σ. The following rules, where a, b are assumed to be distinct atoms and ∇ is a freshness environment over Σ, define *simplification of freshness environments*:

$$\{a \not\mathrel{\#} b\} \cup \nabla \implies \nabla \qquad\qquad \{a \not\mathrel{\#} (p_1, \ldots, p_k)\} \cup \nabla \implies \{a \not\mathrel{\#} p_i, \ldots, a \not\mathrel{\#} p_k\} \cup \nabla$$
$$\{a \not\mathrel{\#} \pi \bullet t\} \cup \nabla \implies \{\pi^{-1} \cdot a \not\mathrel{\#} t\} \cup \nabla \qquad \{a \not\mathrel{\#} [a]p\} \cup \nabla \implies \nabla$$
$$\{a \not\mathrel{\#} [b]p\} \cup \nabla \implies \{a \not\mathrel{\#} p\} \cup \nabla \qquad \{a \not\mathrel{\#} f(p)\} \cup \nabla \implies \{a \not\mathrel{\#} p\} \cup \nabla.$$

The rules define a reduction relation on freshness environments. We write $\nabla \implies \nabla'$ when ∇' is obtained from ∇ by applying one simplification rule, and $\implies^*$ for the reflexive and transitive closure of $\implies$.

▶ **Lemma 24.** *The relation $\implies$ is confluent and terminating.*

A freshness assertion is *reduced* iff it is of the form $a \not\mathrel{\#} a$ or $a \not\mathrel{\#} x$. We say that $a \not\mathrel{\#} a$ is *inconsistent* and $a \not\mathrel{\#} x$ is *consistent*. An environment ∇ is *reduced* iff it consists only of reduced assertions. An environment containing a freshness assertion that is not reduced can always be simplified using one of the rules in Definition 23. Therefore, by Lemma 24, an environment ∇ reduces by $\implies^*$ to a unique reduced environment, which we call the *normal form* of ∇, written $\langle\nabla\rangle_{nf}$. An environment ∇ is *inconsistent* iff $\langle\nabla\rangle_{nf}$ contains some inconsistent assertion. We say ∇ *entails* ∇' (written $\nabla \vdash \nabla'$) iff either ∇ is an inconsistent environment, or $\langle\nabla'\rangle_{nf} \subseteq \langle\nabla\rangle_{nf}$. We write $\vdash \nabla$ iff $\emptyset \vdash \nabla$.

▶ **Lemma 25.** *Let ∇ be an environment over Σ. Then, for every ground substitution φ, the conjunction of the freshness relations represented by $\varphi(\langle\nabla\rangle_{nf})$ holds iff the conjunction of the freshness relations represented by $\varphi(\nabla)$ hold.*

In particular, if $\vdash \nabla$ then for every ground substitution φ the freshness relations represented by $\varphi(\nabla)$ hold.

We are interested in NTS [23], which consider signatures with base sorts ac and pr, with a single atom sort ch and with source and residual sorts pr and ac $\times$ pr respectively. We let Σ_{NTS} be any such signature parametric on a set F of function symbols that we keep implicit. We let bn $: \mathbb{N}(\Sigma)_{\mathsf{ac}} \to \mathcal{P}_\omega(\mathbb{A}_{\mathsf{ch}})$ be the binding-names function of a given NTS. From now on we restrict the attention to the NTS of [23], and the definitions and results to come apply to NRTS/NRTSS over a signature Σ_{NTS}. We require that the rules of an NRTSS only contain ground actions ℓ and therefore function bn is always defined over $NT[\![\ell]\!]$. (Recall that we write bn(ℓ) instead of bn$(NT[\![\ell]\!])$ since it is clear in this context that the ℓ stands for a nominal term.) The rule format that we introduce in Definition 27 relies on identifying the rules that give rise to transitions with actions ℓ such that bn(ℓ) is non-empty. To this end, we adapt the notion of strict stratification from [3, 14].

▶ **Definition 26** (Partial strict stratification). Let $\mathcal{R}$ be an NRTSS over a signature Σ_{NTS} and bn be a binding-names function. Let S be a partial map from pairs of ground processes and actions to ordinal numbers. S is a *partial strict stratification of $\mathcal{R}$ with respect to* bn iff
(i) $S(\varphi(t), \ell) \neq \perp$, for every rule in $\mathcal{R}$ with conclusion $t \longrightarrow (\ell, t')$ such that bn(ℓ) is non-empty and for every ground substitution φ, and
(ii) $S(\varphi(u_i), \ell_i) < S(\varphi(t), \ell)$ and $S(\varphi(u_i), \ell_i) \neq \perp$, for every rule in $\mathcal{R}$ with conclusion $t \longrightarrow (\ell, t')$ such that $S(\varphi(t), \ell) \neq \perp$, for every premiss $u_i \longrightarrow (\ell_i, u_i')$ of $\mathcal{R}$ and for every ground substitution φ.
We say a pair (p, ℓ) of ground process and action *has order $S(p, \ell)$*.

The choice of S determines which rules will be considered by the rule format for NRTSSs of Definition 27 below, which guarantees that the induced transition relation satisfies alpha-conversion of residuals and, therefore, the associated transition relation together with function bn are indeed an NTS. We will intend the map S to be such that the only rules whose source and label of the conclusion have defined order are those that may take part in proof trees of transitions with some binding atom in the action.

▶ **Definition 27** (Alpha-conversion-of-residuals format). Let $\mathcal{R}$ be an NRTSS over a signature Σ_{NTS}, bn be a binding-names function and S be a partial strict stratification of $\mathcal{R}$ with respect to bn. Assume that all the actions occurring in the rules of $\mathcal{R}$ are ground. Let

$$\frac{\{u_i \longrightarrow (\ell_i, u_i') \mid i \in I\} \qquad \nabla}{t \longrightarrow (\ell, t')} \ \mathrm{Ru}$$

be a rule in $\mathcal{R}$. Let D be the set of variables that occur in the source t of Ru but do not occur in the premisses $u_i \longrightarrow (\ell_i, u_i')$ with $i \in I$, the environment ∇ or the target t' of the rule. The rule Ru is in *alpha-conversion-of-residuals format with respect to S (ACR format with respect to S* for short) iff for each ground substitution φ such that $S(\varphi(t), \ell) \neq \perp$, there exists a ground substitution γ such that dom$(\gamma) \subseteq D$, and for every atom a in the set $\mathbb{A} \setminus \{c \in \mathrm{supp}(t) \mid \langle \{c \not\mathrel{\#} t\} \rangle_{nf} = \emptyset\}$ and for every atom $b \in$ bn(ℓ), the following hold:
(i) $\{a \not\mathrel{\#} t'\} \cup \nabla \vdash \{a \not\mathrel{\#} u_i' \mid i \in I\}$,
(ii) $\{a \not\mathrel{\#} t'\} \cup \nabla \cup \{a \not\mathrel{\#} u_i \mid i \in I\} \vdash \{a \not\mathrel{\#} \gamma(t)\}$, and
(iii) $\nabla \cup \{b \not\mathrel{\#} u_i \mid i \in I \wedge b \in$ bn$(\ell_i)\} \vdash \{b \not\mathrel{\#} \gamma(t)\}$.
An NRTSS $\mathcal{R}$, together with a binding-names function bn is in *ACR format* iff $\mathcal{R}$ is in equivariant format and there exists a partial strict stratification S such that all the rules in $\mathcal{R}$ are in ACR format with respect to S.

Given a transition $p \longrightarrow (\ell, q)$ that unifies with the conclusion of Ru, the rule format ensures that any atom a fresh in (ℓ, q) is also fresh in p, and also that the binding atom b is fresh in p. We have obtained the constraints of the rule format by considering the variable flow in each node of a proof tree and the freshness relations that we want to ensure. Constraints (i) and (ii) cover the case for the freshness relation $a\#p$ and Constraint (iii) covers the case for the freshness relation $b\#p$. The purpose of substitution γ is to ignore the variables that occur in the source of a rule but are dropped everywhere else in the rule. Constraints (i) and (ii) are not required for atoms a that for sure are fresh in p, and this explains why the a in the rule format ranges over $\mathbb{A} \setminus \{c \in \text{supp}(t) \mid \langle\{c \not\mathrel{\#} t\}\rangle_{nf} = \emptyset\}$. For instance, take rule ResB from Section 6. Condition $\{c \not\mathrel{\#} (boutA(a,b), new([c]y)), c \not\mathrel{\#} boutA(a,b)\} \vdash \{c \not\mathrel{\#} (boutA(a,b), y)\}$ does not hold because $c \not\mathrel{\#} [c]y$ does not entail that $c \not\mathrel{\#} y$. However, for a transition $NT[\![new([c]p)]\!] \longrightarrow NT[\![(\ell, new([c]p'))]\!]$, c is fresh in $NT[\![new([c]p)]\!]$ even if c is not fresh in $NT[\![p]\!]$.

▶ **Theorem 28** (Rule format for NTSs). *Let $\mathcal{R}$ be an NRTSS. If $\mathcal{R}$, together with the binding-names function* bn*, is in ACR format then the NRTS induced by $\mathcal{R}$ and* bn *constitute an NTS—that is, the transition relation induced by $\mathcal{R}$ is equivariant and satisfies alpha-conversion of residuals.*

Sketch of the proof. Given a transition $NT[\![\varphi(t)]\!] \longrightarrow NT[\![\varphi((\ell, t'))]\!]$, we first prove the freshness relations $a\#NT[\![\varphi(\gamma(t))]\!]$ and $b\#NT[\![\varphi(\gamma(t))]\!]$. Both relations are proven by induction on $S(\varphi(\gamma(t)), \ell)$, and by analysing the variable flow in the rule unifying with $\varphi(t) \longrightarrow \varphi(\ell, t')$. For the first relation, we assume $a\#NT[\![\varphi(t')]\!]$, use Constraint (i) to prove that $a\#NT[\![\varphi(u'_i)]\!]$ for each target u'_i of a premiss, apply the induction hypothesis to obtain $a\#NT[\![\varphi(\gamma(u_i)]\!]$ for each source of a premiss u_i, and use Constraint (ii) to conclude that $a\#NT[\![\varphi(\gamma(t))]\!]$. For the second relation, the induction hypothesis ensures that $b\#NT[\![\varphi(\gamma(u_i))]\!]$ for each source u_i of a premiss having b as a binding name, and we use Constraint (iii) to conclude that $b\#NT[\![\varphi(\gamma(t))]\!]$. From these two freshness relations it is straightforward to prove that $NT[\![\varphi(t)]\!] \longrightarrow (a\,b) \cdot NT[\![\varphi((\ell, t'))]\!]$ and we are done. ◀

6   Example of application to the early π-calculus

Consider the NRTSS $\mathcal{R}$ for the early π-calculus [21] over a signature Σ_{NTS} where F is the set of function symbols from Example 7. Below we collect an excerpt of the rules, where $a, b, c \in \mathbb{A}_{\mathsf{ch}}$ and ℓ is a ground action:

$$\frac{b \not\mathrel{\#} [c]x}{in(a, [c]x) \longrightarrow (inA(a,b), (c\,b) \bullet x)} \text{ In} \qquad \frac{x \longrightarrow (outA(a,b), y) \qquad b \not\mathrel{\#} a}{new([b]x) \longrightarrow (boutA(a,b), y)} \text{ Open}$$

$$\frac{x_1 \longrightarrow (\ell, y_1)}{sum(x_1, x_2) \longrightarrow (\ell, y_1)} \text{ SumL} \qquad \ell \notin \{boutA(a,b)\} \frac{x_1 \longrightarrow (\ell, y_1)}{par(x_1, x_2) \longrightarrow (\ell, (par(y_1, x_2)))} \text{ ParL}$$

$$\frac{x_1 \longrightarrow (boutA(a,b), y_1) \qquad b \not\mathrel{\#} x_2}{par(x_1, x_2) \longrightarrow (boutA(a,b), (par(y_1, x_2)))} \text{ ParResL}$$

$$\frac{}{out(a,b,x) \longrightarrow (outA(a,b), x)} \text{ Out} \qquad \frac{x \longrightarrow (\ell, y)}{rep(x) \longrightarrow (\ell, (par(y, rep(x))))} \text{ Rep}$$

$$\frac{x_1 \longrightarrow (boutA(a,b), y_1) \qquad x_2 \longrightarrow (inA(a,b), y_2) \qquad b \not\mathrel{\#} x_2}{par(x_1, x_2) \longrightarrow (tauA, new([b](par(y_1, y_2))))} \text{ CloseL}$$

An input process $NT[\![in(a, [c]p)]\!]$ can perform a transition to a process $NT[\![(c\,b) \cdot p]\!]$ that is obtained by substituting a channel name b received through channel a for channel name c in p. In the rule IN, the moderated term $(c\,b) \bullet x$ needs to be used in order to indicate that permutation $(c\,b)$ will be performed over the term substituted for variable x.

The rule CLOSEL specifies the interaction of a process such as $NT[\![new([b](out(a, b, p)))]\!]$, which exports a private channel name b through channel a, composed in parallel with an input process such as $NT[\![in(a, [c]q)]\!]$ that reads through channel a. The private name b is exported and the resulting process $NT[\![new([b](par(p, (c\,b) \cdot q)))]\!]$ is the parallel composition of processes p and q where atom b is restricted. For illustration, consider the raw terms $t \equiv new([b](out(a, b, p)))$ and $t' \equiv (boutA(a, b), p)$. The transition $NT[\![t]\!] \longrightarrow NT[\![t']\!]$ is provable in $\mathcal{R}$ by the following proof tree:

$$\dfrac{\dfrac{}{NT[\![out(a, b, p)]\!] \longrightarrow NT[\![(outA(a, b), p)]\!]}\ \text{OUT}}{NT[\![new([b](out(a, b, p)))]\!] \longrightarrow NT[\![(boutA(a, b), p)]\!]}\ \text{OPEN, as } b\#a.$$

Notice that the nodes of the proof tree above are labelled by transitions involving nominal terms. Therefore, if we were to start with the raw term $q \equiv new([c](out(a, c, p)))$—which is alpha-equivalent to t—then the transition $NT[\![q]\!] \longrightarrow NT[\![t']\!]$ would have the same proof tree as above, since $NT[\![t]\!]$ and $NT[\![q]\!]$ are the same nominal term. This contrasts with the related work [7, 9], which considers raw terms in the model of computation and instruments alpha-conversion explicitly in the specification system.

We use the rule format of Definition 27 to show that $\mathcal{R}$, together with equivariant function $\text{bn}(\ell) = \{b \mid \ell = boutA(a, b)\}$ specifies an NTS. We consider the following partial strict stratification:

$$\begin{aligned}
S(out(a, b, p), outA(a, b)) &= 0 \\
S(par(p, q), \ell) &= 1 + \max\{S(p, \ell), S(q, \ell)\} \quad \ell \in \{boutA(a, b), outA(a, b)\} \\
S(sum(p, q), \ell) &= 1 + \max\{S(p, \ell), S(q, \ell)\} \quad \ell \in \{boutA(a, b), outA(a, b)\} \\
S(rep(p), \ell) &= 1 + S(p, \ell) \quad \ell \in \{boutA(a, b), outA(a, b)\} \\
S(new([c]p), \ell) &= 1 + S(p, \ell) \quad \ell \in \{boutA(a, b), outA(a, b)\} \text{ and } c \notin \{a, b\} \\
S(new([b]p), boutA(a, b)) &= 1 + S(p, outA(a, b)) \\
S(p, \ell) &= \bot \quad \text{o.w.}
\end{aligned}$$

We check that $\mathcal{R}$ is in ACR format as follows. The only rules in $\mathcal{R}$ whose sources and actions unify with pairs of processes and actions that have defined order are OUT, OPEN and PARRESL, and the instance of rule PARL where $\ell = outA(a, b)$, and the instances of rules SUML, REP and RES where $\ell \in \{boutA(a, b), outA(a, b)\}$ (and the corresponding instances of the symmetric versions PARRESR, PARR and SUMR, which are omitted in the excerpt). We will only check the ACR-format for rules OUT, SUML and OPEN.

For rule OUT, we have an empty set of premises and the set D of atoms that are in $\text{supp}(out(a, b, x))$ but are not in $\text{supp}(outA(a, b), x)$ is empty. Therefore we can do away with substitution γ. There is no atom a such that $\langle\{a \not\approx out(a, b, x)\} = \emptyset\rangle_{nf}$ and the set $\text{bn}(outA(a, b))$ is empty. We only need to check that for every atom c, $\{c \not\approx (outA(a, b), x)\} \vdash \{c \not\approx out(a, b, x)\}$. For atoms $c \in \text{supp}(outA(a, b), x)$ the obligation of the rule format vacuously holds, and therefore it is enough to pick an atom c fresh in the rule and check that $\{c \not\approx (outA(a, b), x)\} \vdash \{c \not\approx out(a, b, x)\}$, which is straightforward.

For rule SUML, we first check the instance where $\ell = boutA(a, b)$. We have premiss $x_1 \longrightarrow (boutA(a, b), y_1)$ and the set D contains x_2. We pick γ such that $\gamma(x_2) = null$. There is no atom a such that $\langle\{a \not\approx sum(x_1, x_2)\}\rangle_{nf} = \emptyset$ and the set $\text{bn}(boutA(a, b))$ contains atom

b. Again, it is enough to pick atom c fresh in the rule and check that

$$\{c \not\!\!\# (boutA(a,b), y_1)\} \vdash \{c \not\!\!\# (boutA(a,b), y_1)\} \quad \text{and}$$
$$\{c \not\!\!\# (boutA(a,b), y_1), c \not\!\!\# x_1\} \vdash \{c \not\!\!\# \gamma(sum(x_1, x_2))\} \quad \text{and}$$
$$\{b \not\!\!\# x_1\} \vdash \{b \not\!\!\# \gamma(sum(x_1, x_2))\},$$

which holds since $\gamma(sum(x_1, x_2)) = sum(x_1, null)$ and $b \not\!\!\# null$ reduces to the empty set.

Now we check the instance where $\ell = outA(a, b)$. We have premiss $x_1 \longrightarrow (outA(a, b), y_1)$ and the set D and the substitution γ are the same as before. There is no atom a such that $\langle \{a \not\!\!\# sum(x_1, x_2)\} \rangle_{nf} = \emptyset$ and the set $bn(outA(a, b))$ is empty. Again, it is enough to pick atom c fresh in the rule and check that $\{c \not\!\!\# (outA(a, b), y_1)\} \vdash \{c \not\!\!\# (outA(a, b), y_1)\}$ and $\{c \not\!\!\# (outA(a, b), y_1), c \not\!\!\# x_1\} \vdash \{c \not\!\!\# \gamma(sum(x_1, x_2))\}$, which holds as before.

For rule OPEN the set D is empty and $\langle \{b \not\!\!\# new([b]x)\} \rangle_{nf} = \emptyset$. It is enough to pick atom c fresh in the rule (and therefore different from b) and check that

$$\{c \not\!\!\# (boutA(a,b), y), b \not\!\!\# a\} \vdash \{c \not\!\!\# (boutA(a,b), y)\} \quad \text{and}$$
$$\{c \not\!\!\# (boutA(a,b), y), b \not\!\!\# a, c \not\!\!\# x\} \vdash \{c \not\!\!\# new([b]x)\} \quad \text{and}$$
$$\{b \not\!\!\# x, b \not\!\!\# a\} \vdash \{b \not\!\!\# new([b]x)\},$$

which holds because $b \not\!\!\# new([b]x)$ reduces to the empty set.

Atoms a, b and c in the specification of $\mathcal{R}$ range over $\mathbb{A}_{\mathsf{ch}}$, and thus $\mathcal{R}$ is in equivariant format. Therefore $\mathcal{R}$ is in ARC format. By Theorem 28 the NRTS induced by $\mathcal{R}$, together with function bn, constitute an NTS of Definition 3.

7 Conclusions and future work

The work we have presented in this paper stems from the recently proposed Nominal SOS (NoSOS) framework [7] and from earlier proposals for nominal logic in [8, 15, 30]. It is by no means the only approach studied so far in the literature that aims at a uniform treatment of binders and names in programming and specification languages. Other existing approaches that accommodate variables and binders within the SOS framework are those proposed by Fokkink and Verhoef in [13], by Middelburg in [19, 20], by Bernstein in [5], by Ziegler, Miller and Palamidessi in [31] and by Fiore and Staton in [10] (originally, by Fiore and Turi in [11]). The aim of all of the above-mentioned frameworks is to establish sufficient syntactic conditions guaranteeing the validity of a semantic result (congruence in the case of [5, 10, 19, 31] and conservativity in the case of [13, 20]). In addition, Gabbay and Mathijssen present a nominal axiomatisation of the λ-calculus in [16]. None of these approaches addresses equivariance nor the property of alpha-conversion of residuals in [23].

Our current proposal aims at following closely the spirit of the seminal work on nominal techniques by Gabbay, Pitts and their co-workers, and paves the way for the development of results on rule formats akin to those presented in the aforementioned references. Amongst those, we consider the development of a congruence format for the notion of bisimilarity presented in [23, Def. 2] to be of particular interest. The logical characterisation of bisimilarity given in [23] opens the intriguing possibility of employing the divide-and-congruence approach from [12] to obtain an elegant congruence format and a compositional proof system for the logic.

In the NTSs of Parrow *et al.* [23], scope opening is modelled by the property of alpha-conversion of residuals. We are currently exploring an alternative in which scope opening is encoded by a *residual abstraction* of sort $[\mathsf{ch}](\mathsf{ac} \times \mathsf{pr})$. We have developed mutual, one-to-one translations between the NTSs and the NRTSs with residual abstractions. The generality of

our NRTSs also allows for neat specifications of variants of the π-calculus such as Sangiorgi's internal π-calculus [28].

Developing rule formats for SOS is always the result of a trade-off between ease of application and generality. Our rule format for alpha-conversion of residuals in Definition 27 is no exception and might be generalised in various ways. For instance, the quantification on atom a in conditions (i) and (ii), and the use of substitution γ might be made more general by a finer analysis of the variable flow in a rule. Another generalisation of the rule format would consider possibly open raw actions.

Finally, we are developing rule formats for properties other than alpha-conversion of residuals. One such rule format ensures a *non-dropping property* for NRTSs to the effect that, in each transition, the support of a state is a subset of the support of its derivative.

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