Brief Announcement: Lower Bounds for Asymptotic Consensus in Dynamic Networks*†

Matthias Függer[1], Thomas Nowak[2], and Manfred Schwarz[3]

1 **CNRS, LSV, ENS Paris-Saclay, Inria, Paris, France**
`mfuegger@lsv.fr`
2 **Université Paris-Sud, Paris, France**
`thomas.nowak@lri.fr`
3 **ECS, TU Wien, Vienna, Austria**
`mschwarz@ecs.tuwien.ac.at`

Abstract

In this work we study the performance of asymptotic and approximate consensus algorithms in dynamic networks. The asymptotic consensus problem requires a set of agents to repeatedly set their outputs such that the outputs converge to a common value within the convex hull of initial values. This problem, and the related approximate consensus problem, are fundamental building blocks in distributed systems where exact consensus among agents is not required, e.g., man-made distributed control systems, and have applications in the analysis of natural distributed systems, such as flocking and opinion dynamics. We prove new nontrivial lower bounds on the contraction rates of asymptotic consensus algorithms, from which we deduce lower bounds on the time complexity of approximate consensus algorithms. In particular, the obtained bounds show optimality of asymptotic and approximate consensus algorithms presented in [Charron-Bost et al., ICALP'16] for certain classes of networks that include classical failure assumptions, and confine the search for optimal bounds in the general case.

Central to our lower bound proofs is an extended notion of valency, the set of reachable limits of an asymptotic consensus algorithm starting from a given configuration. We further relate topological properties of valencies to the solvability of exact consensus, shedding some light on the relation of these three fundamental problems in dynamic networks.

1998 ACM Subject Classification F.1.2 Modes of Computation

Keywords and phrases Asymptotic Consensus, Dynamic Networks, Contraction Rate, Time Commplexity, Lower Bounds

Digital Object Identifier 10.4230/LIPIcs.DISC.2017.51

1 Introduction

In the *asymptotic consensus* problem a set of agents, each starting from an initial value in $\mathbb{R}^d$, update their values such that all agents' values converge to a common value within the convex hull of initial values. The problem is closely related to the *approximate consensus* problem, in which agents have to irrevocably decide on values that lie within a predefined distance $\varepsilon > 0$ of each other. The latter is weaker than the *exact consensus* problem in which agents need to decide on the same value. Both the asymptotic and the approximate

* The research was partially funded by the Austrian Science Fund (FWF) project SIC (P26436) and ADynNet (P28182).
† The full version is available at `https://arxiv.org/abs/1705.02898`

consensus problems have not only a variety of applications in the design of man-made control systems like sensor fusion [1], clock synchronization [8], formation control [6], rendezvous in space [9], or load balancing [5], but also for analyzing natural systems like flocking [11], firefly synchronization [10], or opinion dynamics [7]. These problems often have to be solved under harsh-environmental restrictions: with limited computational power and local storage, under restricted communication abilities, and in presence of communication uncertainty.

In this work we study asymptotic consensus in round-based computational models with a dynamic communication topology whose directed communication graphs are chosen each round from a predefined set of communication graphs, the so-called *network model*. In previous work [2], Charron-Bost et al. showed that asymptotic consensus is solvable precisely within *rooted network models* in which all communication graphs contain rooted spanning trees. These rooted spanning trees need not have any edges in common and can change from one round to the next.

An interesting special case of rooted network models are network models whose graphs are *non-split*, that is, any two agents have a common incoming neighbor. Their prominent role is motivated by two properties: (i) They occur as communication graphs in benign classical distributed failure models. For example, in synchronous systems with crashes, in asynchronous systems with a minority of crashes, and synchronous systems with send omissions. (ii) In [2], Charron-Bost et al. showed that non-split graphs also play a central role in arbitrary rooted network models: they showed that any product of $n-1$ rooted graphs with n nodes is non-split, allowing to transform asymptotic consensus algorithms for non-split network models into their *amortized* variants for rooted models.

Contribution

In this work, we prove lower bounds on the contraction rate of any asymptotic consensus algorithm. All lower bounds hold regardless of the structure of the algorithm. In particular, algorithms can be full-information and agents can set their outputs outside the convex hull of received values. This, e.g., includes using higher-order filters in contrast to the 0-order filters of averaging algorithms.

The proof strategy is as follows: The central idea is the concept of the *valency of a configuration* of an asymptotic consensus algorithm, defined as the set of limits reachable from this configuration. By studying the changes in valency along executions, we infer bounds on the contraction rate. Notably, the lower bounds are valid for arbitrary dimensions.

Note that if exact consensus is solvable in network model $\mathcal{N}$, an optimal contraction rate of 0 can be achieved. Otherwise, we show the following non trivial bounds:

- We show a tight lower bound of $1/3$ in non-split network models with $n = 2$ agents.
- We prove that the contraction rate is lower bounded by $1/2$ in a system with $n \geq 3$ agents and $\mathrm{deaf}(G) \subseteq \mathcal{N}$ where, for an arbitrary communication graph G, $\mathrm{deaf}(G) = \{F_1, \dots, F_n\}$ and F_i is derived from G by making agent i *deaf* in F_i, i.e., removing the incoming links of i in G. Additionally we show tightness for $d \in \{1, 2\}$ dimensional values.
- The study of the valencies' topological structure with respect to the network model where the asymptotic consensus algorithm is executed in, reveals that any asymptotic consensus algorithm must have a contraction rate of at least $1/(D+1)$, where D is the so-called α-diameter of $\mathcal{N}$. This generalizes the previous two lower bounds.

Tightness for $1/3$ and $1/2$ results from the combination with algorithms presented in [3] and [4]. Together with the algorithm for arbitrary dimensions d with contraction rate $\frac{d}{d+1}$ in non-split models [4] the bounds in Table 1 follow. Furthermore we extend our results on

Table 1 Summary of lower and upper bounds on contraction rates if consensus is not solvable. New lower bounds proved in this work are marked with a *. The three right columns distinguish between the case the network model is (i) non-split and contains $\mathrm{deaf}(G)$ for some communication graph G, (ii) is non-split, and (iii) is rooted.

		network model in which exact consensus is unsolvable		
agents	dimension	non-split with deaf graphs	$\subseteq$ non-split $\subseteq$	rooted
$n = 2$	$d \geq 1$	$\frac{1}{3}^*$	$\frac{1}{3}^*$	$\frac{1}{3}^*$
$n \geq 3$	$d \in \{1, 2\}$	$\frac{1}{2}^*$	$\left[\frac{1}{D+1}^*, \frac{1}{2}\right]$	$\left[\frac{1}{D+1}^*, \sqrt[n-1]{\frac{1}{2}}\right]$
	$d \geq 3$	$\left[\frac{1}{2}^*, \frac{d}{d+1}\right]$	$\left[\frac{1}{D+1}^*, \frac{d}{d+1}\right]$	$\left[\frac{1}{D+1}^*, \sqrt[n-1]{\frac{d}{d+1}}\right]$

contraction rates to derive new lower bounds on the decision time of approximate consensus algorithms.

Acknowledgments

We would like to thank Bernadette Charron-Bost for the many fruitful discussions and her valuable input which greatly helped improve the paper.

References

1 J. A. Benediktsson and P. H. Swain. Consensus theoretic classification methods. *IEEE Transactions on Systems, Man, and Cybernetics*, 22(4):688–704, 1992.

2 Bernadette Charron-Bost, Matthias Függer, and Thomas Nowak. Approximate consensus in highly dynamic networks: The role of averaging algorithms. In *Proceedings of ICALP*, pages 528–539, 2015.

3 Bernadette Charron-Bost, Matthias Függer, and Thomas Nowak. Fast, robust, quantizable approximate consensus. In *Proceedings of the 43rd International Colloquium on Automata, Languages, and Programming, ICALP16*, pages 137:1–137:14, 2016.

4 Bernadette Charron-Bost, Matthias Függer, and Thomas Nowak. Multidimensional asymptotic consensus in dynamic networks. *CoRR*, abs/1611.02496, 2016.

5 George Cybenko. Dynamic load balancing for distributed memory multiprocessors. *Journal of Parallel and Distributed Computing*, 7(2):279–301, 1989.

6 Magnus Egerstedt and Xiaoming Hu. Formation constrained multi-agent control. *IEEE Transactions on Robotics and Automation*, 17(6):947–951, 2001.

7 R. Hegselmann and U. Krause. Opinion dynamics and bounded confidence models, analysis, and simulation. *Journal of artificial societies and social simulation*, 5(3):1–33, 2002.

8 Qun Li and Daniela Rus. Global clock synchronization in sensor networks. *IEEE Transactions on Computers*, 55(2):214–226, 2006.

9 J. Lin, A.S. Morse, and B.D.O. Anderson. The multi-agent rendezvous problem. In Vijay Kumar, Naomi Leonard, and A. Stephen Morse, editors, *Cooperative Control: A Post-Workshop Volume 2003 Block Island Workshop on Cooperative Control*, pages 257–289. Springer, Heidelberg, 2005.

10 Renato E. Mirollo and Steven H. Strogatz. Synchronization of pulse-coupled biological oscillators. *SIAM Journal on Applied Mathematics*, 50(6):1645–1662, December 1990.

11 Tamás Vicsek, András Czirók, Eshel Ben-Jacob, Inon Cohen, and Ofer Shochet. Novel type of phase transition in a system of self-driven particles. *Physical Review Letters*, 75(6):1226–1229, 1995.