A 2-Categorical Approach to Composing Quantum Structures*

David Reutter[1] and Jamie Vicary[2]

1 Department of Computer Science, University of Oxford, UK
david.reutter@cs.ox.ac.uk

2 Department of Computer Science, University of Oxford, UK
jamie.vicary@cs.ox.ac.uk

Abstract

We present an infinite number of construction schemes for quantum structures, including unitary error bases, Hadamard matrices, quantum Latin squares and controlled families, many of which have not previously been described. Our results rely on the type structure of biunitary connections, 2-categorical structures which play a central role in the theory of planar algebras. They have an attractive graphical calculus which allows simple correctness proofs for the constructions we present. We apply these techniques to construct a unitary error basis that cannot be built using any previously known method.

1998 ACM Subject Classification G.2.1 Combinatorics, F.1.1 Models of Computation

Keywords and phrases quantum constructions, 2-category, graphical calculus, planar algebra

Digital Object Identifier 10.4230/LIPIcs.CALCO.2017.20

1 Introduction

Biunitary connections (or simply *biunitaries*) were introduced by Ocneanu [39] in 1989, and have since been developed by Jones, Morrison and others [23, 24, 34] as a central tool in the classification of subfactors. They belong to the theory of *planar algebras*, an area of mathematics related to 2-category theory which studies the linear representation theory of algebraic structures in the plane. We can describe a biunitary informally as a planar algebra element U with two inputs and two outputs, drawn below and above the vertex respectively, which is *vertically unitary* (Figure 1(a)), and which is *horizontally unitary* up to a scalar factor λ (Figure 1(b)).

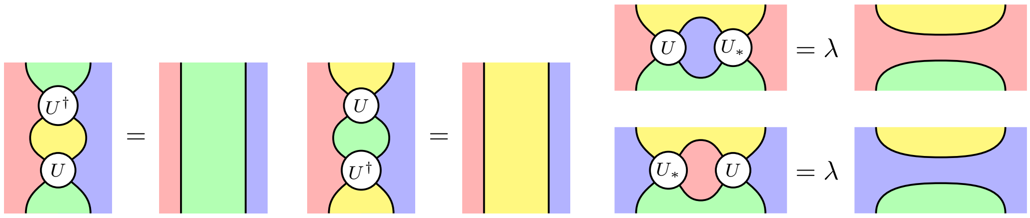
For us, diagrams of this sort represent simple linear algebra data: regions are labelled by indexing sets, and wires and vertices are labelled by indexed families of finite-dimensional Hilbert spaces and linear maps, respectively.[1] Blank regions correspond to the trivial indexing set. In concrete terms, a biunitary therefore comprises a family of linear maps satisfying some algebraic properties.

The *type* of a biunitary is the shading pattern which surrounds the vertex. We show in Section 2 that a variety of structures in quantum information theory correspond exactly

* An extended version of this paper can be found at [44], https://arxiv.org/abs/1609.07775.

1 Formally this is a common generalization of the tensor [23, Example 2.6] and spin model [23, Example 2.8] planar algebras, corresponding to a fragment of the monoidal 2-category **2Hilb** [4]. However, our exposition will be elementary, and we will not assume knowledge of these ideas.

7th Conference on Algebra and Coalgebra in Computer Science (CALCO 2017).
Editors: Filippo Bonchi and Barbara König; Article No. 20; pp. 20:1–20:20
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany



(a) Vertical unitarity **(b)** Horizontal unitarity

Figure 1 The biunitarity equations.

(a) Hadamard matrix (Had) **(b)** Unitary error basis (UEB) **(c)** Quantum Latin square (QLS) **(d)** Controlled family of Hadamards (Had*)

Figure 2 Biunitary types for a variety of quantum structures.

to biunitaries of particular types. Some important examples are given in Figure 2.[2] In the rightmost image, we see that the notation is 3-dimensional, with the blue sheet lying beneath the yellow sheet. Rotations by a quarter-turn, and reflections about the horizontal or vertical axes, preserve the given interpretations in terms of quantum structures.

Some of these characterizations are already known: *Hadamard matrices*[3] were characterized by Jones as biunitaries with alternating shaded and unshaded regions [23], and *unitary error bases*[4] were characterized by the second author as biunitaries with one shaded and three unshaded regions [53, 54]. Here we show that *quantum Latin squares*[5] can be characterized as biunitaries with two adjacent shaded regions and two adjacent unshaded regions. We also show that controlled families can be described by adding an additional shaded region in a certain way; in Figure 2(d), we illustrate one application of this idea to give a biunitary characterization of a controlled family of Hadamard matrices.

Our main results are based on the simple fact that the *diagonal* composite of two biunitaries is again biunitary. We show in Section 3 that, given the description of quantum combinatorial structures in terms of biunitaries as summarized above, one can immediately write down a large number of schemes for the construction of certain quantum structures from others. We give some examples in Figure 3; note that the biunitaries are connected diagonally in each case, as required.

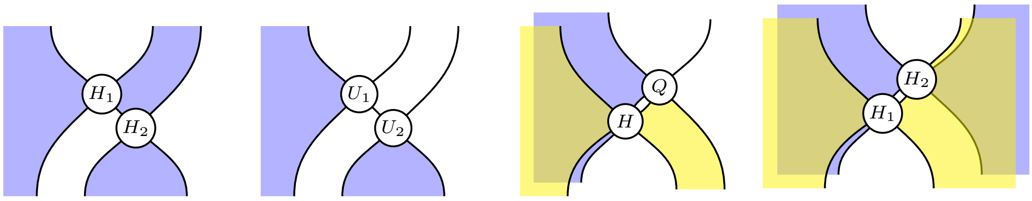
---

[2] Note that some of the inputs or outputs of the biunitary may in general be composite wires. For example, in Figure 2(c) the first input is composite, and Figure 2(d) the first input and second output are composite.

[3] A *Hadamard matrix* is a square complex matrix with entries of modulus 1, which is proportional to a unitary matrix. Fundamental structures in quantum information, they are central in the theories of mutually unbiased bases, quantum key distribution, and many other phenomena [18]. See Definition 6.

[4] A *unitary error basis* is a basis of unitary operators on a finite-dimensional Hilbert space, orthogonal with respect to the trace inner product. They provide the basic data for quantum teleportation, dense coding and error correction procedures [31, 49, 55]. See Definition 8.

[5] A *quantum Latin square* [36] is a square grid of vectors in a finite-dimensional Hilbert space, such that every row and every column is an orthonormal basis. They generalize Latin squares. See Definition 10.



(a) Had + Had ⇝ QLS **(b)** UEB + UEB ⇝ QLS **(c)** Had* + QLS ⇝ UEB **(d)** Had* + Had* ⇝ Had

Figure 3 Some biunitary composites of arity 2.

We explain each of these constructions briefly. Figure 3(a) gives a way to combine two Hadamard matrices to produce a quantum Latin square, generalizing a known construction.[6] (Note that the wires terminating near the upper-right of Figure 3(a) are interpreted as a *single* composite wire for the purpose of identifying it as having the basic quantum Latin square type of Figure 2(c), a method that we use repeatedly, and motivate formally with bracketings in Theorem 16.) Figure 3(b) gives a procedure for combining two unitary error bases to yield a quantum Latin square, a construction we believe to be new. Figure 3(c) combines a controlled family of Hadamard matrices and a quantum Latin square to give a unitary error basis, recovering the quantum shift-and-multiply construction [36, Definition 18]. In Figure 3(d), two families of Hadamard matrices are combined to produce a single Hadamard matrix, recovering a construction of Hosoya and Suzuki [21, Section 1] which generalizes a construction of Diţă [17, Section 4]. These constructions can of course be iterated; for example, combining the constructions of Figures 3(a) and 3(c) gives a way to combine a controlled family of Hadamard matrices and two further Hadamard matrices to produce a single unitary error basis, again a new construction.

In all these cases, correctness of the construction follows immediately from the type-theoretic structure (that is, the shading pattern) of the diagram, relying only on diagonality of the composition; no further details need to be checked. Our approach therefore offers advantages even for those constructions that are already known, since the traditional proofs of correctness are nontrivial. To emphasize this point we compare our graphical techniques to traditional methods, in which constructions are defined using tensor notation. For example, the construction of Figure 3(c) would traditionally be written as follows [36, Definition 18], where $U_{ab,c,d}$ is the (c,d)th matrix entry of the (a,b)th element of the unitary error basis, $Q_{b,d,c}$ is the coefficent of $|c\rangle$ in the (b,d)th position of the quantum Latin square, and $H^b_{a,d}$ is the (a,d)th coefficient of the bth Hadamard matrix:

$$U_{ab,c,d} := H^b_{a,d}\, Q_{b,d,c} \tag{1}$$

It is not trivial to write down correct expressions of this form, and to show that this indeed defines a unitary error basis requires a calculation of several lines [36, Theorem 20] that invokes the distinct algebraic properties of the tensors $Q_{b,d,c}$ and $H^b_{a,d}$. In contrast, in our new approach, it would be easy to discover this construction by considering all ways the basic components can be diagonally composed; correctness is immediate, and all algebraic properties are subsumed by the single concept of biunitarity. Nonetheless, expression (1) can be immediately read-off from the form of the biunitary composite.

[6] When both Hadamard matrices are the same, this agrees with a known construction of a quantum Latin square from a single Hadamard matrix [36, Definition 10].

(a) Had + Had + Had ⇝ UEB

(b) Had^{**} + QLS + QLS + UEB ⇝ UEB

Figure 4 Some biunitary composites of arities 3 and 4.

Higher-arity constructions can also be described, such as those given in Figure 4. Both of these we believe to be new. In Figure 4(a), arising as a consequence of the constructions of Figures 3(a) and 3(c), three Hadamard matrices combine to produce a unitary error basis, an elegant construction which we believe to be new.[7] In Figure 4(b), which does not arise as a consequence of lower-arity constructions, we combine a double-controlled family of Hadamard matrices (H), two quantum Latin squares (Q_1, Q_2) and a unitary error basis (U) to produce a new unitary error basis. While the first example is simple and elegant, the second example is indicative of the more complex constructions our technique can produce. Further complex examples are given in Figures 10.

For unitary error bases, we illustrate all constructions of arities 2 and 3 that arise from our methods, and we give examples of constructions of arities 4 and 8. Furthermore, in an extended version of this paper [44], we show that our methods give rise to an infinite family of logically independent constructions, none of which factor through any simpler construction between Hadamard matrices, unitary error bases, quantum Latin squares and controlled families thereof.

Finally, in the extended version we use the 4-fold composite of Figure 4(b) to produce a unitary error basis on an 8-dimensional Hilbert space, which we show cannot be produced by the two known UEB construction methods—algebraic, and quantum shift-and-multiply—even up to equivalence. This is a proof of principle that the biunitary methods we propose can give rise to genuinely new quantum structures.

Significance. Hadamard matrices and unitary error bases provide the mathematical foundation for an extremely rich variety of quantum computational phenomena, amongst them the study of mutually unbiased bases, quantum key distribution, quantum teleportation, dense coding and quantum error correction [18, 29, 31, 49, 55]. Nevertheless their general structure is notoriously difficult to understand; in dimension n, Hadamard matrices have only been classified up to $n = 5$ [50, 52], and the general structure of unitary error bases is virtually unknown for $n > 2$. Quantum Latin squares have been introduced much more recently [7, 35, 36], generalizing classical Latin squares which have a wide range of applications in classical and quantum information [10, 33, 48].

By unifying these quantum structures as special cases of the single notion of biunitary, and providing simple type-theoretical tools to understand the intricate interplay between them, we unify several already-known and seemingly-unrelated constructions [7,17,21,36,55],

[7] When all three Hadamard matrices are the same, this agrees with a known construction of a unitary error basis from a single Hadamard matrix [36, Definition 33] which we believe to be folklore.

uncover an infinite number of new constructions, and produce novel, concrete examples. These new tools may lead to further progress in questions of classification and applications of Hadamard matrices, unitary error bases and quantum Latin squares, and perhaps move us closer to full classification results for these important structures.

On the other hand, biunitaries are central tools in the study and classification of subfactors [23,24,34,39,43], a highly significant activity in the theory of von Neumann algebras. We hope that our work leads to the development of further connections between subfactor theory and quantum computation.

1.1 Related work

Quantum constructions. As well as producing a number of new constructions, our methods encompass and unify several constructions from the literature.

- The *Hadamard method* [36, Definition 33], believed to be folklore, which produces a unitary error basis from a single Hadamard matrix. In Figure 4(a) we give a new generalization, in which three Hadamard matrices produce a unitary error basis.
- The method given in [7, Definition 2.3] and [36, Definition 10], which produces a quantum Latin square from a single Hadamard matrix. In Figure 9(a) we give a new generalization, in which two Hadamard matrices produce a quantum Latin square.
- Werner's *shift-and-multiply construction* [55] which produces a unitary error basis from a family of Hadamard matrices and a Latin square. This is a special case of the quantum shift-and-multiply construction discussed below.
- The *quantum shift-and-multiply construction* due to Musto and the second author [36, Definition 18] which produces a unitary error basis from a family of Hadamard matrices and a quantum Latin square. We give a biunitary description in Figure 9(f).
- *Diţă's construction* [17, Section 4], which produces a Hadamard matrix from a Hadamard matrix and a family of Hadamard matrices, and is widely used [17,32,38,52]. We give a biunitary description in Figure 9(d).
- *Hosoya's and Suzuki's construction* [21, Section 1], which produces a Hadamard matrix from two families of Hadamard matrices. We give a biunitary description in Figure 9(c).

However, there are many known constructions which are beyond our methods. For unitary error bases, we do not know a biunitary characterization of Knill's algebraic construction [30], and for Hadamard matrices we cannot account for many of the varied construction methods [20, 32, 35, 42, 50, 51, 56] which make use of non-compositional structure that is out of reach of the biunitary approach.

Biunitary connections and planar algebras. Biunitary connections were introduced by Ocneanu in 1989 in terms of *paragroups* [39] in an attempt to better understand the combinatorial structures arising in subfactor theory, a branch of the theory of von Neumann algebras. In 1999, Jones introduced the theory of planar algebras [23] and with it the modern graphical formulation of biunitarity used in this paper.

The relation between Hadamard matrices and von Neumann algebras predates the notion of biunitarity and can be traced back to Popa's commuting squares [43] (later shown to be equivalent to biunitarity [26]) and the statistical-mechanical spin models of Jones [23, 25]; there is a significant literature on the interplay between Hadamard matrices, planar algebras and subfactors [20, 37, 38]. Quantum Latin squares appear under the name *magic bases* in a Hopf algebraic approach to subfactor theory by Banica and others [5–7]; however, their biunitary characterization does not seem to have been written down. Unitary error bases were characterized in terms of biunitaries by the second author [53, 54].

Categorical quantum mechanics. This work builds on the programme of categorical quantum mechanics, initiated by Abramsky and Coecke [1] and developed by them and others [2, 3, 11–15, 28, 46], which uses monoidal categories with duals to provide a high-level syntax for quantum information flow. It was shown by the second author that these ideas can be extended to a higher categorical setting [53, 54], developing the work of Baez on a categorified notion of Hilbert space [4], a perspective we use here. The key advantage of this approach is that the notion of Frobenius algebra, used in the monoidal category setting to describe classical information, is no longer needed. While our results could in principle be translated back into the language of Frobenius algebras, they would lose their simplicity and power. In this sense, the current work serves as an advertisement for the essential role that higher category theory can play in quantum information theory.

1.2 Notations and conventions

We denote the n-element set $\{1, \ldots, n\}$ by $[n]$. The letters $a, b, d, e, f, g, h, i, j, k, r, s$ are used to denote indices, the letters n, m, p, q are used to denote dimensions. We use the following shorthand notations to refer to sets of quantum structures:

- UEB_n is the set of n-dimensional unitary error bases;
- QLS_n is the set of n-dimensional quantum Latin squares;
- Had_n is the set of n-dimensional Hadamard matrices;
- For $\mathrm{X} \in \{\mathrm{UEB}_n, \mathrm{QLS}_n, \mathrm{Had}_n\}$, $\mathrm{X}^{p_1,\ldots,p_k}$ is the set of lists of quantum structures of type X controlled by indices in $[p_1]$, $[p_2]$, $\ldots, [p_k]$.

For example, $\mathrm{UEB}^{n,p}_{n^2m}$ is the set of lists of n^2m-dimensional unitary error bases, controlled by indices valued in $[n]$ and $[p]$.

2 Biunitarity

In Section 2.1 we introduce our formalism, and give the definition of biunitarity. In Section 2.2 we recall the biunitary characterizations of Hadamard matrices and unitary error bases, and give new biunitary characterizations of quantum Latin squares and controlled families.

2.1 Mathematical foundations

The graphical calculus for describing composition of multilinear maps was proposed by Penrose [41], and is today widely used in a range of areas [2, 11, 27, 40, 47]. In this scheme, wires represent Hilbert spaces and vertices represent linear maps between them, with wiring diagrams representing composite linear maps. For example, given linear maps $A : W \otimes H \otimes J \to L \otimes M \otimes R$ and $B : V \to H \otimes J$, we can describe a composite linear map $V \otimes W \to L \otimes M \otimes R$ graphically as shown in Figure 7(a).

In this article we use a generalized calculus that involves *regions*, as well as wires and vertices. This is an instance of the graphical calculus for monoidal 2-categories [8, 9, 22, 45] applied to the 2-category[8] of finite-dimensional 2-Hilbert spaces [4]. The 2-category of 2-Hilbert spaces can be described as follows [19, 53]:

- objects are natural numbers $n, m, \ldots$;
- 1-morphisms $n \to m$ are $m \times n$-matrices of finite-dimensional Hilbert spaces (Figure 5(a));
- 2-morphisms are matrices of linear maps (Figure 5(b)).

$$\begin{pmatrix} H_{11} & \cdots & H_{1n} \\ \vdots & \ddots & \vdots \\ H_{m1} & \cdots & H_{mn} \end{pmatrix}$$

(a) A 1-morphism $H : n \to m$.

$$\begin{pmatrix} H_{11} \xrightarrow{\phi_{11}} H'_{11} & \dots & H_{1n} \xrightarrow{\phi_{1n}} H'_{1n} \\ \vdots & \ddots & \vdots \\ H_{m1} \xrightarrow{\phi_{m1}} H'_{m1} & \dots & H_{mn} \xrightarrow{\phi_{mn}} H'_{mn} \end{pmatrix}$$

(b) A 2-morphism $\phi : H \Rightarrow H'$.

Figure 5 The 1- and 2-morphisms in the 2-category of 2-Hilbert spaces.

$$\phi : \begin{pmatrix} J_1 \\ J_2 \end{pmatrix} \Rightarrow \begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \end{pmatrix} \circ \begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix}$$

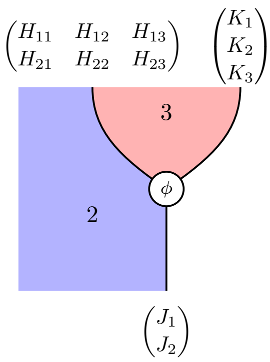
(a) A 2-morphism ϕ.

$$\begin{pmatrix} J_1 \xrightarrow{\phi_1} (H_{11} \otimes K_1) \oplus (H_{12} \otimes K_2) \oplus (H_{13} \otimes K_3) \\ J_2 \xrightarrow{\phi_2} (H_{21} \otimes K_1) \oplus (H_{22} \otimes K_2) \oplus (H_{23} \otimes K_3) \end{pmatrix}$$

(b) The 2-morphism ϕ as a matrix of linear maps.

$$\phi_{i,j} : J_i \to H_{i,j} \otimes K_j \quad \text{for } i \in [2] \text{ and } j \in [3]$$

(c) The 2-morphism ϕ as a family of linear maps indexed by its adjacent regions.



(d) Graphical representation of the 2-morphism ϕ.

Figure 6 Translating between equivalent expressions for 2-morphisms.

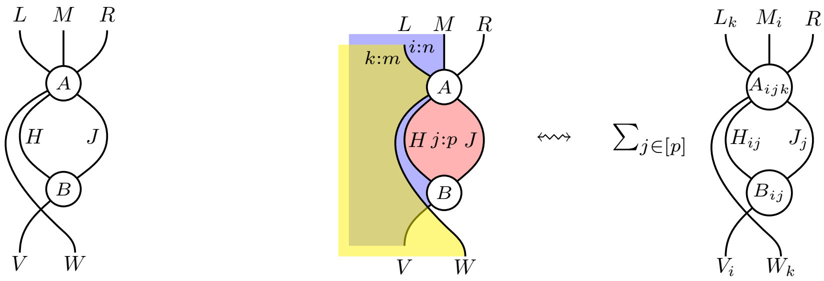
Composition of 1-morphisms is given by 'matrix multiplication' of matrices of Hilbert spaces, with addition and multiplication of complex numbers replaced by direct sum and tensor product, respectively. Composition of 2-morphisms is given by componentwise composition of linear maps.

In the graphical calculus, regions, wires and vertices represent objects, 1-morphisms and 2-morphisms, respectively. The 2-category has a monoidal structure, acting on objects as multiplication, and on 1- and 2-morphisms as the Kronecker product of matrices of Hilbert spaces and linear maps, respectively; this is represented graphically by 'layering' one diagram above another.

Elementary description. While these structures are now well-understood in higher representation theory, they are not yet prevalent in the computer science community. To help the reader understand these new concepts, we also give a direct account of the formalism in elementary terms, that can be used without reference to the higher categorical technology. In Figure 6 we indicate how to translate between the categorical language presented above and the more elementary language used here.

In this direct perspective, shaded regions are labelled by *finite sets*, indexed by a parameter; we write $i{:}n$ to indicate that the parameter i varies over the set $[n]$.[9] Wires and vertices now represent *families* of Hilbert spaces and linear maps respectively, indexed by the parameters of all adjoining regions. A composite surface diagram represents a family of

[8] Here and throughout, we use the term '2-category' to refer to the fully weak structure, which is sometimes called 'bicategory'.
[9] For simplicity we will often omit these labels.



(a) An ordinary tensor diagram. **(b)** A shaded tensor diagram.

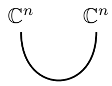
Figure 7 The graphical calculus.

composite linear maps, indexed by the parameters of all open regions, with closed regions being summed over.

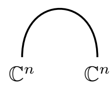
We give an example in Figure 7(b). The coloured diagram on the left represents an entire family of composite linear maps. The maps which comprise this family are given by the right-hand diagrams for different values of k and i, which index the open regions. The closed region labelled $j : p$ is summed over.

Given this interpretation of diagrams D as families of linear maps D_i, we define two diagrams D, D' to be equal when all the corresponding linear maps D_i, D_i' are equal, and the scalar product λD as the family of linear maps λD_i.

Duality. We define the linear maps $\eta : \mathbb{C} \to \mathbb{C}^n \otimes \mathbb{C}^n$ and $\epsilon : \mathbb{C}^n \otimes \mathbb{C}^n \to \mathbb{C}$ as follows:



$\eta : 1 \mapsto \sum_i |i\rangle \otimes |i\rangle$



$\epsilon : |i\rangle \otimes |j\rangle \mapsto \delta_{ij}$ (2)

Assuming for simplicity that all Hilbert spaces are chosen to be of the form $\mathbb{C}^n$ for some $n \in \mathbb{N}$, we introduce the following notation for families of linear maps of the form (2):


(3)

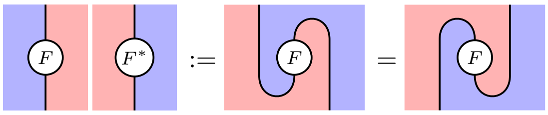
The notation is justified, since the following equations can be demonstrated:


(4)

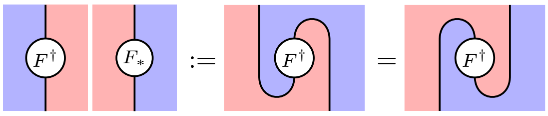
Dagger structure. Given a family of linear maps, we define its *adjoint* (or *dagger*) to be the family consisting of the adjoints of the linear maps. Graphically, we can think of the adjoint as a reflection about a horizontal axis. This is justified, since the following holds:


(5)

In total, every vertex appears in four variants:



(6)



(7)

The equations on the right-hand sides can be shown to follow from the definitions (2).

A dagger structure gives rise to a general notion of unitarity.

▶ **Definition 1.** A vertex U is *unitary* when it satisfies the following equations:

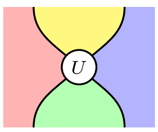
(8)

Standard boundaries. We only make use of a restricted portion of this calculus: wires which bound only one shaded region correspond to the 1-dimensional Hilbert space $\mathbb{C}$ for any value of the controlling parameter. (Hilbert spaces that do not bound regions may be of any finite dimension.) In particular, since they are 1-dimensional, the Hilbert spaces arising from these boundaries are not depicted in the corresponding family of tensor diagrams:

(9)

Biunitaries. Having defined our graphical calculus, we introduce our main algebraic entity.

▶ **Definition 2.** A *biunitary* is a vertex



(10)

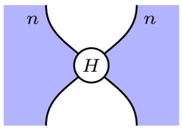
which satisfies the equations of Figure 1 for some scalar $\lambda \in \mathbb{C}$.

Note that biunitarity depends on a chosen partition of the input and output wires into two parts, which is extra data. Also, the scalar λ is uniquely determined by the type of U, and necessarily real and positive.

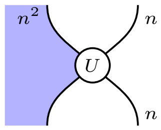
We will usually use the following equivalent formulation of biunitarity.

▶ **Definition 3.** The *clockwise* and *anticlockwise quarter rotation* of a vertex U of type (10) is given by the following composites, respectively:

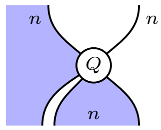
(11)



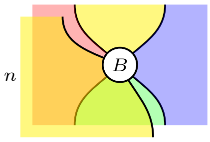
(a) Hadamard matrix.



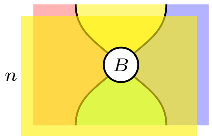
(b) Unitary error basis.



(c) Quantum Latin square.



(d) Half-plane controlled family.



(e) Full-plane controlled family.

Figure 8 Quantum structures and their corresponding biunitary types.

▶ **Proposition 4.** *Given a vertex U of type* (10)*, the following are equivalent:*

1. *U is biunitary;*
2. *U is unitary, and its clockwise quarter rotation is proportional to a unitary;*
3. *U is unitary, and its anticlockwise quarter rotation is proportional to a unitary.*

Furthermore, in cases 2 and 3, the proportionality factor is unique up to a phase and given by a square root of λ.

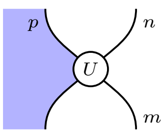
▶ **Corollary 5.** *Given a biunitary, arbitrary quarter rotations, or reflections about horizontal or vertical axes, are again proportional to biunitaries.*

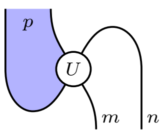
In particular, as soon as we have characterized specific quantum structures in terms of biunitaries of certain types, we know that rotated and reflected versions of this type also correspond to this quantum structure, possibly after multiplication by a scalar.

2.2 Characterizing quantum structures

In this section we recall the biunitary characterizations of Hadamard matrices and unitary error bases, and give new characterizations of quantum Latin squares and controlled families. These results are summarized in Figure 8. In the following diagrams, all wires are either standard boundaries (9) or Hilbert spaces that do not bound any region.

Dimensional constraints. For a linear map $f : H \to J$ to be unitary imposes certain algebraic constraints on the dimensions of H and J; namely, $\dim(H) = \dim(J)$. For a vertex of type (10) to be biunitary similarly induces certain constraints on the allowed labels for the surrounding regions and wires. In all cases, these constraints are easily identified and solved for. For example, consider the following vertex U and its clockwise quarter rotation:





(12)

Here, n, m and p denote the dimensions of the corresponding region or wire, respectively. For the first of these to be unitary requires that $n = m$, while for the second to be unitary requires $p = nm$. By Theorem 4, for U to be biunitary, we therefore require $(n, m, p) = (n, n, n^2)$, and the set of allowed types is parameterized by a single natural number. For the rest of this section, we will label biunitaries by their allowed dimensions.

Hadamard matrices. Hadamard matrices were identified by Jones to be characterized in terms of biunitarity [23]. Complex Hadamard matrices play an important role in mathematical physics and quantum information theory [18]; in particular, they encode the data of a basis which is unbiased with respect to the computational basis.

▶ **Definition 6.** A *Hadamard matrix* is a matrix $H \in \mathrm{Mat}_n(\mathbb{C})$ with the following properties, for $i, j \in [n]$:

$$H_{i,j}\overline{H}_{i,j} = 1 \sum_k H_{i,k}\overline{H}_{j,k} = \delta_{i,j} n \qquad \sum_k \overline{H}_{k,i} H_{k,j} = \delta_{i,j} n \tag{13}$$

The last two equations are equivalent, but we include them both for completeness.

The biunitary characterization of Hadamard matrices is due to Jones in the setting of the spin model planar algebra, which our mathematical setup generalizes. It was shown in [53, Theorem 4.5] that this characterization is equivalent to that of Coecke and Duncan in terms of interacting Frobenius algebras [12].

▶ **Proposition 7** (Jones [23, Section 2.11])**.** *Hadamard matrices of dimension n correspond to biunitaries of the type shown in Figure 8(a).*

Unitary error bases. Originally introduced by Knill [31], unitary error bases are ubiquitous in modern quantum information theory. They lie at the heart of quantum error correction [49] and procedures such as superdense coding and quantum teleportation [55].

▶ **Definition 8** (Knill [31])**.** A *unitary error basis* (UEB) on an n-dimensional Hilbert space H is a collection of unitary matrices $\{U_a \in \mathrm{U}(H) \mid a \in [n^2]\}$, satisfying the following orthogonality property, for $a, b \in [n^2]$:

$$\mathrm{Tr}(U_a^\dagger U_b) = n\,\delta_{a,b} \tag{14}$$

That is, a UEB is an orthogonal basis of the space $\mathrm{End}(H)$ consisting of unitary matrices.

We denote the (i, j)th matrix element of the matrix U_a by $U_{a,i,j} = (U_a)_{i,j} = \langle i|\, U_a\, |j\rangle$.

▶ **Proposition 9** (Vicary [53, Theorem 4.2])**.** *Unitary error bases on an n-dimensional Hilbert space correspond to biunitaries of the type shown in Figure 8(b).*

Quantum Latin squares. Quantum Latin squares were introduced by Musto and the second author [36] as generalizations of classical Latin squares, with applications to the construction of unitary error bases. Related constructions were also introduced independently by Banica and Nicoară [7].

▶ **Definition 10** (Musto & V. [36, Definition 1])**.** A *quantum Latin square* (QLS) on an n-dimensional Hilbert space H is a square grid of vectors $\{|Q_{a,b}\rangle \in H \mid a, b \in [n]\}$ such that each row $\{|Q_{a,b}\rangle \mid b \in [n]\}$ and each column $\{|Q_{a,b}\rangle \mid a \in [n]\}$ form an orthonormal basis of H; for $a, b, c \in [n]$:

$$\langle Q_{a,b}|Q_{a,c}\rangle = \delta_{b,c} \qquad \langle Q_{a,c}|Q_{b,c}\rangle = \delta_{a,b} \tag{15}$$

We denote the ith entry of the vector $|Q_{a,b}\rangle$ by $Q_{a,b,i} = \langle i|Q_{a,b}\rangle$.

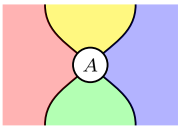
▶ **Proposition 11.** *Quantum Latin squares on an n-dimensional Hilbert space correspond to biunitaries of the type shown in Figure 8(c).*

Controlled families. In quantum information we often want to describe lists of structures, parameterized by a given index. A standard name for such a list is a controlled family.

▶ **Definition 12.** For a given quantum structure X, an *n-controlled family* is an ordered list of n instances of X.

In index notation, we reserve superscript for controlling indices. For example, a controlled family of Hadamard matrices would be written as $H^c_{a,b}$, where c iterates through the controlled family and a and b are the actual indices of the Hadamard matrix H^c.

▶ **Proposition 13.** *An n-controlled family of biunitaries of type*



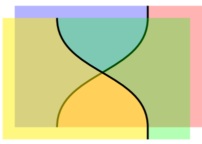
(16)

corresponds to a biunitary of the same type with an additional half-plane or full-plane sheet of dimension n attached, as shown in Figure 8(d) and (e).

By Corollary 5, we could have put the half-plane controlling sheet in one of 4 different orientations. Furthermore, it makes no difference if the controlling sheet goes in front or behind. We therefore have 8 different half-plane controls and 2 different full plane controls. For example. the type shown in Figure 2(d) indicates a controlled family of Hadamards.

In our pseudo-3d graphical notation, it can be hard to see if a rear sheet is actually connected to a vertex. In our diagrams, we will use the convention that all sheets drawn beneath a vertex are connected to it.

Interchangers. The vertex representing the crossing of wires at different depths is called an *interchanger*:



(17)

This is given canonically for all index values as the swap map $H \otimes J \to J \otimes H$.

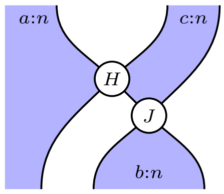
▶ **Proposition 14.** *The interchanger* (17) *is biunitary.*

3 Biunitary composition

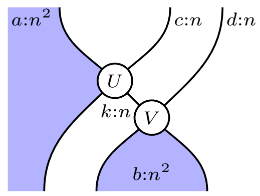
The results of this section are corollaries of the following idea.

▶ **Theorem 15.** *Arbitrary finite diagonal composites of biunitaries are again biunitary.*

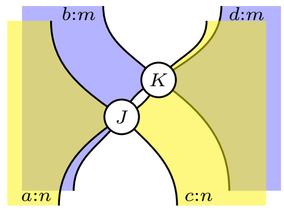
Since we have established in Section 2 that biunitaries of various types correspond to different quantum structures, Theorem 15 suggests the possibility of building new quantum structures from existing ones by diagonal composition. In Section 3.1, we demonstrate that binary diagonal composites of biunitaries are again biunitary. We then consider the problem of diagonally composing the biunitaries corresponding to Hadamards, quantum Latin squares, unitary error bases and controlled families to produce other such structures, investigating binary composites in Section 3.2, ternary composites in Section 3.3, and higher composites in Section 3.4. In an extended version of this paper [44], we argue that our methods gives rise to an infinite number of genuinely distinct constructions.



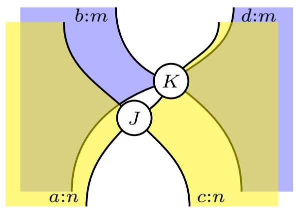
(a) $\mathrm{Had}_n + \mathrm{Had}_n \rightsquigarrow \mathrm{QLS}_n$



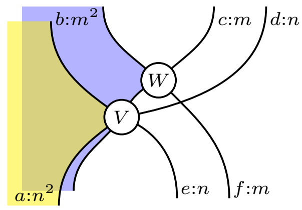
(b) $\mathrm{UEB}_n + \mathrm{UEB}_n \rightsquigarrow \mathrm{QLS}_{n^2}$



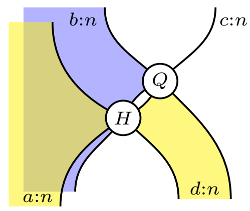
(c) $\mathrm{Had}_n^m + \mathrm{Had}_m^n \rightsquigarrow \mathrm{Had}_{nm}$ (Hosoya-Suzuki)



(d) $\mathrm{Had}_n + \mathrm{Had}_m^n \rightsquigarrow \mathrm{Had}_{nm}$



(e) $\mathrm{UEB}_n^{m^2} + \mathrm{UEB}_m \rightsquigarrow \mathrm{UEB}_{nm}$



(f) $\mathrm{Had}_n^n + \mathrm{QLS}_n \rightsquigarrow \mathrm{UEB}_n$

Figure 9 Binary biunitary constructions.

3.1 Diagonal composition

It is straightforward to see that the diagonal composite of two biunitaries is again biunitary.

▶ **Theorem 16.** *Let U, V and W be biunitaries with the types illustrated below.*

Then the following diagonal composites are biunitary, with respect to the indicated partitions of the input and output wires:

(18)

Except for the pinwheel composite[10] [16], which can be handled separately, this shows that Theorem 15 holds.

3.2 Binary composites

We give a number of quantum constructions in Figure 9, each involving the diagonal composite of two biunitaries. Correctness of all these constructions follows as corollaries from Theorem 16, and the results of Section 2.2 as summarized in Figure 8.

Quantum Latin squares. We begin by presenting two quantum Latin square constructions. The following construction produces a quantum Latin square from two Hadamard matrices, generalizing [7, Definition 2.3] and [36, Definition 10].

[10] The *pinwheel composite* is a way to compose five 2-morphisms in a double category, in a way which cannot be described in terms of repeated binary composites.

▶ **Corollary 17** ($\mathrm{Had}_n + \mathrm{Had}_n \rightsquigarrow \mathrm{QLS}_n$). *The construction of Figure 9(a) produces an n-dimensional quantum Latin square*

$$Q_{a,b,c} = \tfrac{1}{\sqrt{n}} H_{a,c} J_{c,b} \tag{19}$$

from the following data, with $a, b, c \in [n]$:
- $H_{a,c}$ *and* $J_{c,b} \in \mathrm{Had}_n$*, n-dimensional Hadamards.*

The factor $\frac{1}{\sqrt{n}}$ arises as described in Theorem 4, since the biunitary J is of rotated Hadamard type. Such a biunitary is a *unitary* matrix; given an ordinary Hadamard matrix, we need to rescale it by a factor of $\frac{1}{\sqrt{n}}$ to obtain such a unitary. A similar comment applies to several of the constructions below. The next construction, which we believe to be new, produces a quantum Latin square from two unitary error bases.

▶ **Corollary 18** ($\mathrm{UEB}_n + \mathrm{UEB}_n \rightsquigarrow \mathrm{QLS}_{n^2}$). *The construction of Figure 9(b) produces an n^2-dimensional quantum Latin square*

$$Q_{a,b,cd} = \tfrac{1}{\sqrt{n}} \textstyle\sum_{k\in[n]} U_{a,c,k} V_{b,k,d} \tag{20}$$

from the following data, with $a, b \in [n^2]$ and $c, d \in [n]$:
- $U_{a,c,k}$ *and* $V_{b,k,d} \in \mathrm{UEB}_n$*, n-dimensional UEBs.*

Note that we concatenate indices corresponding to tensor products of Hilbert spaces or products of indexing sets; for example, for a QLS on a Hilbert space $V \otimes W$, the coefficient of the basis vector $|i, j\rangle = |i\rangle \otimes |j\rangle$ in the (a, b)th position of the quantum Latin square will be written as $Q_{a,b,ij}$. Similarly, if the indexing set of a UEB is the product of two sets $[n] \times [m]$ we denote its (a, b)th element by U_{ab} with coefficients $U_{ab,i,j}$.

Hadamard matrices. The following construction produces a single Hadamard matrix from two controlled families.

▶ **Corollary 19** ($\mathrm{Had}_n^m + \mathrm{Had}_m^n \rightsquigarrow \mathrm{Had}_{nm}$). *The construction of Figure 9(c) produces an nm-dimensional Hadamard matrix*

$$H_{ab,cd} = J^b_{a,c} K^c_{b,d} \tag{21}$$

from the following data, with $a, c \in [n]$ and $b, d \in [m]$:
- $J^b_{a,c} \in \mathrm{Had}_n^m$*, an m-controlled family of n-dimensional Hadamard matrices;*
- $K^c_{b,d} \in \mathrm{Had}_m^n$*, an n-controlled family of m-dimensional Hadamard matrices.*

This construction was introduced in 2003 by Hosoya and Suzuki [21] under the name *generalized tensor product*. A better known special case, due to Diţă [17], is a central tool in the study and classification of Hadamard matrices; we give it explicitly in Figure 9(d).

Unitary error bases. We now turn our attention to unitary error bases. By a manual combinatorial check, it can be verified that the constructions in Figure 9(e) and (f) are the only possible binary constructions of UEBs using only Hadamard matrices, UEBs or QLSs and controlled families thereof.

▶ **Corollary 20** ($\mathrm{UEB}_n^{m^2} + \mathrm{UEB}_m \rightsquigarrow \mathrm{UEB}_{nm}$). *The construction of Figure 9(e) produces an nm-dimensional unitary error basis*

$$U_{ab,cd,ef} = V^b_{a,d,e} W_{b,c,f} \tag{22}$$

from the following data, with $a \in [n^2]$, $b \in [m^2]$, $c, f \in [m]$ and $d, e \in [n]$:

- $V^b_{a,d,e} \in \mathrm{UEB}^{m^2}_n$, *an* m^2*-controlled family of* n*-dimensional unitary error bases;*
- $W_{b,c,f} \in \mathrm{UEB}_m$, *an* m*-dimensional unitary error basis.*

In Figure 9(e), we have used biunitarity of the interchanger as established in Proposition 14.

It is also possible to compose biunitaries of different types to obtain unitary error bases, as shown by the following biunitary characterization of an existing construction, the *quantum shift-and-multiply* method [36], which simultaneously generalizes the shift-and-multiply method [55] and the Hadamard method [36, Definition 33].

▶ **Corollary 21** ($\mathrm{Had}^n_n + \mathrm{QLS}_n \rightsquigarrow \mathrm{UEB}_n$). *The construction of Figure 9(f) produces an* n*-dimensional unitary error basis*

$$U_{ab,c,d} = H^b_{a,d}\, Q_{b,d,c} \tag{23}$$

from the following data, with $a, b, c, d \in [n]$*:*
- $H^b_{a,d} \in \mathrm{Had}^n_n$, *an* n*-controlled family of* n*-dimensional Hadamard matrices;*
- $Q_{b,d,c} \in \mathrm{QLS}_n$, *an* n*-dimensional quantum Latin square.*

3.3 Ternary constructions

Here we list all ternary biunitary constructions of unitary error bases from Hadamard matrices, unitary error bases, quantum Latin squares and controlled families thereof, which do not factor through constructions of lower arity. We summarize them in Figure 10(a)–(d). Up to equivalence, we assert that this list is complete, although we do not prove completeness in a formal way. To our knowledge, all constructions in this section are new. As before, all these results are corollaries of Theorem 16, and the results of Section 2.2 as summarized in Figure 8.

The constructions of Figure 10(a) and Figure 10(b) can be seen as slight alterations of constructions that factor through the constructions of Figure 9.

▶ **Corollary 22** ($\mathrm{Had}^{m^2,n}_n + \mathrm{UEB}^{n,n}_m + \mathrm{QLS}_n \rightsquigarrow \mathrm{UEB}_{nm}$). *The construction of Figure 10(a) produces an* nm*-dimensional UEB*

$$U_{abc,de,fg} := H^{b,c}_{a,f}\, V^{c,f}_{b,e,g}\, Q_{c,f,d} \tag{24}$$

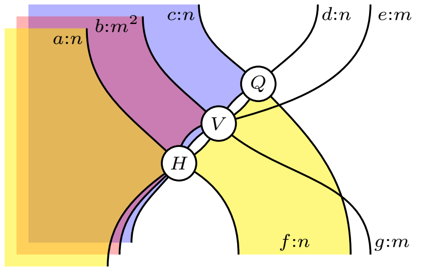
from the following data, with $a, c, d, f \in [n]$, $b \in [m^2]$ *and* $e, g \in [m]$*:*
- $H^{b,c}_{a,f} \in \mathrm{Had}^{m^2,n}_n$, *an* (m^2, n)*-controlled family of* n*-dimensional Hadamard matrices;*
- $V^{c,f}_{b,e,g} \in \mathrm{UEB}^{n,n}_m$, *an* (n, n)*-controlled family of* m*-dimensional unitary error bases;*
- $Q_{c,f,d} \in \mathrm{QLS}_n$, *an* n*-dimensional quantum Latin square.*

▶ **Corollary 23** ($\mathrm{Had}^{n,m}_{nm} + \mathrm{QLS}^{m,m}_n + \mathrm{QLS}_m \rightsquigarrow \mathrm{UEB}_{nm}$). *The construction of Figure 10(b) produces an* nm*-dimensional UEB*

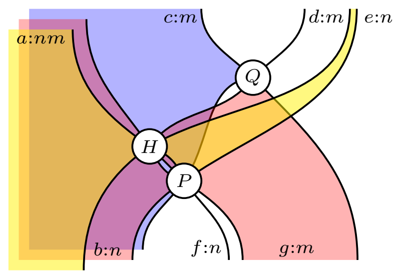
$$U_{abc,de,fg} := H^{b,c}_{a,eg}\, P^{c,g}_{e,b,f}\, Q_{c,g,d} \tag{25}$$

from the following data, with $a \in [nm]$ $b, e, f \in [n]$ *and* $c, d, g \in [m]$*:*
- $H^{b,c}_{a,eg} \in \mathrm{Had}^{n,m}_{nm}$, *an* (n, m)*-controlled family of* nm*-dimensional Hadamard matrices;*
- $P^{c,g}_{e,b,f} \in \mathrm{QLS}^{m,m}_n$, *an* (m, m)*-controlled family of* n*-dimensional quantum Latin squares;*
- $Q_{c,g,d} \in \mathrm{QLS}_m$, *an* m*-dimensional quantum Latin square.*

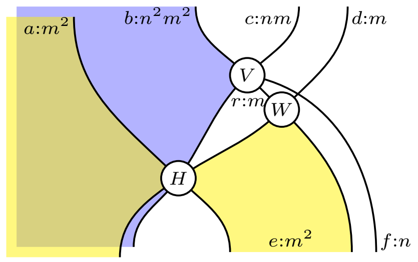
The following is geometrically the simplest of our ternary constructions. It involves a closed wire, so the index expression includes a sum.



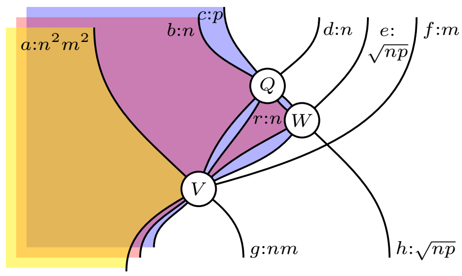
(a) $\mathrm{Had}_n^{m^2,n} + \mathrm{UEB}_m^{n,n} + \mathrm{QLS}_n \rightsquigarrow \mathrm{UEB}_{nm}$



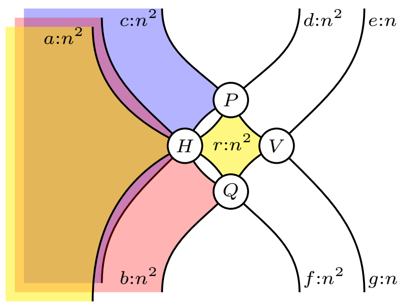
(b) $\mathrm{Had}_{nm}^{n,m} + \mathrm{QLS}_n^{m,m} + \mathrm{QLS}_m \rightsquigarrow \mathrm{UEB}_{nm}$



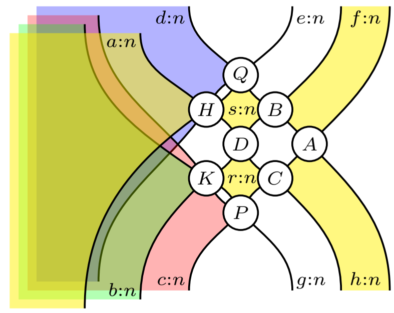
(c) $\mathrm{Had}_{m^2}^{n^2m^2} + \mathrm{UEB}_{nm} + \mathrm{UEB}_m \rightsquigarrow \mathrm{UEB}_{nm^2}$



(d) $\mathrm{UEB}_{nm}^{n,p} + \mathrm{QLS}_n^p + \mathrm{UEB}_{\sqrt{np}} \rightsquigarrow \mathrm{UEB}_{nm\sqrt{np}}$



(e) $\mathrm{Had}_{n^2}^{n^2,n^2} + 2{\times}\mathrm{QLS}_{n^2} + \mathrm{UEB}_n \rightsquigarrow \mathrm{UEB}_{n^3}$



(f) $4{\times}\mathrm{Had}_n + 2{\times}\mathrm{Had}_n^n + 2{\times}\mathrm{QLS}_n \rightsquigarrow \mathrm{UEB}_{n^2}$

Figure 10 Higher-order unitary error basis constructions.

▶ **Corollary 24** ($\mathrm{Had}_{m^2}^{n^2m^2} + \mathrm{UEB}_{nm} + \mathrm{UEB}_m \rightsquigarrow \mathrm{UEB}_{nm^2}$). *The construction of Figure 10(c) produces an* nm^2*-dimensional UEB*

$$U_{ab,cd,ef} := \textstyle\sum_{r\in[m]} H_{a,e}^b \, V_{b,c,rf} \, W_{e,r,d} \tag{26}$$

from the following data, with $a, e \in [m^2]$, $b \in [n^2m^2]$, $c \in [nm]$, $d \in [m]$, *and* $f \in [n]$:

- $H_{a,e}^b \in \mathrm{Had}_{m^2}^{n^2m^2}$, *an* n^2m^2*-controlled family of* m^2*-dimensional Hadamard matrices;*
- $V_{b,c,rf} \in \mathrm{UEB}_{nm}$, *an* nm*-dimensional unitary error basis;*
- $W_{e,r,d} \in \mathrm{UEB}_m$, *an* m*-dimensional unitary error basis.*

Our final ternary construction is the first to involve a sum over a closed region, which again gives rise to a summation.

▶ **Corollary 25** ($\mathrm{UEB}_{nm}^{n,p} + \mathrm{QLS}_n^p + \mathrm{UEB}_{\sqrt{np}} \rightsquigarrow \mathrm{UEB}_{nm\sqrt{np}}$). *For* $n, m, p \in \mathbb{N}$ *such that* $\sqrt{np} \in \mathbb{N}$*, the construction of Figure 10(d) produces an* $nm\sqrt{np}$*-dimensional UEB*

$$U_{abc,def,gh} := \textstyle\sum_{r\in[n]} V_{a,rf,g}^{b,c} \, Q_{b,r,d}^c \, W_{rc,e,h} \tag{27}$$

from the following data, with $a \in [n^2m^2]$*,* $b, d \in [n], c \in [p]$*,* $e, h \in [\sqrt{np}]$*,* $f \in [m]$*, and* $g \in [nm]$*:*

- $V_{a,rf,g}^{b,c} \in \mathrm{UEB}_{nm}^{n,p}$*, an* (n,p)*-controlled family of* nm*-dimensional unitary error bases;*
- $Q_{b,r,d}^{c} \in \mathrm{QLS}_n^p$*, an* p*-controlled family of* n*-dimensional quantum Latin squares;*
- $W_{rc,e,h} \in \mathrm{UEB}_{\sqrt{np}}$*, an* $\sqrt{np}$*-dimensional unitary error basis.*

3.4 Higher constructions

Interesting biunitary composites exist at higher arity, and are easy to find by experimentation. We give two examples which seem elegant, and which we believe are new.

▶ **Corollary 26** ($\mathrm{Had}_{n^2}^{n^2,n^2} + 2{\times}\mathrm{QLS}_{n^2} + \mathrm{UEB}_n \rightsquigarrow \mathrm{UEB}_{n^3}$)**.** *The construction in Figure 10(e) produces an* n^3*-dimensional UEB*

$$U_{abc,de,fg} = \textstyle\sum_{r\in[n^2]} H_{a,r}^{b,c}\, P_{c,r,d}\, Q_{r,b,f}\, V_{r,e,g} \tag{28}$$

from the following data, with $a, b, c, d, f \in [n^2]$ *and* $e, g \in [n]$*:*

- $H_{a,r}^{b,c} \in \mathrm{Had}_{n^2}^{n^2,n^2}$*, an* (n^2, n^2)*-controlled family of* n^2*-dimensional Hadamard matrices;*
- $P_{c,r,d}, Q_{r,b,f} \in \mathrm{QLS}_{n^2}$*,* n^2*-dimensional quantum Latin squares;*
- $V_{r,e,g} \in \mathrm{UEB}_n$*, an* n*-dimensional unitary error bases.*

In an extended version of this paper [44], we use this construction to produce a new unitary error basis that cannot be obtained by the most general previously known methods.

Finally, we give the 8-ary construction of Figure 10(f).

▶ **Corollary 27** ($4{\times}\mathrm{Had}_n + 2{\times}\mathrm{Had}_n^n + 2{\times}\mathrm{QLS}_n \rightsquigarrow \mathrm{UEB}_{n^2}$)**.** *The construction in Figure 10(f) produces an* n^2*-dimensional UEB*

$$U_{abcd,ef,gh} = \tfrac{1}{n} \textstyle\sum_{r,s\in[n]} A_{f,h}\, B_{s,f}\, C_{r,h}\, D_{s,r}\, H_{a,s}^{d}\, K_{b,r}^{c}\, Q_{d,s,e}\, P_{r,c,g} \tag{29}$$

from the following data, with $a, b, c, d, e, f, g, h \in [n]$*:*

- $A_{f,h}, B_{s,f}, C_{r,h}, D_{s,r} \in \mathrm{Had}_n$*,* n*-dimensional Hadamard matrices;*
- $H_{a,s}^{d}, K_{b,r}^{c} \in \mathrm{Had}_n^n$*,* n*-controlled families of* n*-dimensional Hadamard matrices;*
- $Q_{d,s,e}, P_{r,c,g} \in \mathrm{QLS}_n$*,* n*-dimensional quantum Latin squares.*

Acknowledgements. We are grateful to Bruce Bartlett, André Henriques, Scott Morrison, Benjamin Musto and Dominic Verdon for useful discussions. We also thank the anonymous referees for valuable suggestions. Early versions of these results were presented at WIP 2016 and QIP 2017, and we are grateful to the organizers of these events.

References

1 Samson Abramsky and Bob Coecke. A categorical semantics of quantum protocols. In *Proceedings of LICS*, 2004. doi:10.1109/lics.2004.1319636.

2 Samson Abramsky and Bob Coecke. Categorical quantum mechanics. In *Handbook of Quantum Logic and Quantum Structures*, pages 261–323. Elsevier, 2009. doi:10.1016/B978-0-444-52869-8.50010-4.

3 Miriam Backens. The ZX-calculus is complete for stabilizer quantum mechanics. *NJP*, 16:093021, 2014. doi:10.1088/1367-2630/16/9/093021.

4 John C. Baez. Higher-dimensional algebra II. 2-Hilbert spaces. *Advances in Mathematics*, 127(2):125–189, 1997. doi:10.1006/aima.1997.1617.

5 Teodor Banica, Julien Bichon, and Benoît Collins. Quantum permutation groups: a survey. In *Noncommutative Harmonic Analysis with Applications to Probability*. Polish Academy of Sciences, 2007. doi:10.4064/bc78-0-1.

6 Teodor Banica, Julien Bichon, and Jean-Marc Schlenker. Representations of quantum permutation algebras. *Journal of Functional Analysis*, 257(9):2864–2910, 2009. doi:10.1016/j.jfa.2009.04.013.

7 Teodor Banica and Remus Nicoară. Quantum groups and Hadamard matrices. *Panamerican Mathematical Journal*, 17:1–24, 2007. arXiv:math/0610529.

8 John W. Barrett, Catherine Meusburger, and Gregor Schaumann. Gray categories with duals and their diagrams. *to appear*, 2012. arXiv:1211.0529.

9 Bruce Bartlett. Quasistrict symmetric monoidal 2-categories via wire diagrams, 2014. arXiv:1409.2148.

10 Tristan Benoist and Ion Nechita. On bipartite unitary matrices generating subalgebra-preserving quantum operations. *Linear Algebra and its Applications*, 521:70–103, 2017. doi:10.1016/j.laa.2017.01.020.

11 Bob Coecke. Kindergarten quantum mechanics: Lecture notes. In *AIP Conference Proceedings*. AIP Publishing, 2006. doi:10.1063/1.2158713.

12 Bob Coecke and Ross Duncan. Interacting quantum observables. In *Automata, Languages and Programming*, volume 5126, pages 298–310. Springer, 2008. doi:10.1007/978-3-540-70583-3_25.

13 Bob Coecke, Chris Heunen, and Aleks Kissinger. Categories of quantum and classical channels. *Quant. Inf. Proc.*, 2014. doi:10.1007/s11128-014-0837-4.

14 Bob Coecke and Aleks Kissinger. *Picturing Quantum Processes*. Cambridge University Press, 2017.

15 Bob Coecke, Dusko Pavlovic, and Jamie Vicary. A new description of orthogonal bases. *Mathematical Structures in Computer Science*, 23(03):555–567, 2012. doi:10.1017/s0960129512000047.

16 Robert Dawson and Robert Paré. Characterizing tileorders. *Order*, 10(2):111–128, 1993. doi:10.1007/bf01111295.

17 Petre Diţă. Some results on the parametrization of complex Hadamard matrices. *Journal of Physics A*, 37(20):5355–5374, 2004. doi:10.1088/0305-4470/37/20/008.

18 Thomas Durt, Berthold-Georg Englert, Ingemar Bengtsson, and Karol Życzkowski. On mutually unbiased bases. *International Journal of Quantum Information*, 08(04):535–640, 2010. doi:10.1142/s0219749910006502.

19 Josep Elgueta. A strict totally coordinatized version of Kapranov and Voevodsky's 2-category 2Vect. *Mathematical Proceedings of the Cambridge Philosophical Society*, 142(03):407, 2007. doi:10.1017/s0305004106009881.

20 Uffe Haagerup. Orthogonal maximal abelian ∗-subalgebras of the n×n matrices and cyclic n-roots. *Institut for Matematik*, 29:296–322, 1996.

21 Rie Hosoya and Hiroshi Suzuki. Type II matrices and their Bose-Mesner algebras. *Journal of Algebraic Combinatorics*, 17:19–37, 2003. doi:10.1023/a:1021960623533.

22 Benjamin Hummon. *Surface diagrams for Gray categories*. PhD thesis, UC San Diego, 2012. URL: http://escholarship.org/uc/item/5b24s9cc.

23 Vaughan F. R. Jones. Planar algebras, I, 1999. arXiv:math/9909027.

24 Vaughan F. R. Jones, Scott Morrison, and Noah Snyder. The classification of subfactors of index at most 5. *Bull. Amer. Math. Soc.*, 51(2):277–327, 2013. doi:10.1090/s0273-0979-2013-01442-3.

25 Vaughan F.R. Jones. On knot invariants related to some statistical mechanical models. *Pacific Journal of Mathematics*, 137:311–334, 1989. doi:10.2140/pjm.1989.137.311.

26 Vaughan F.R. Jones and Vaikalathur S. Sunder. *Introduction to Subfactors*. Cambridge University Press (CUP), 1997. doi:10.1017/cbo9780511566219.

27 André Joyal and Ross Street. The geometry of tensor calculus, I. *Advances in Mathematics*, 88(1):55–112, 1991. doi:10.1016/0001-8708(91)90003-p.

28 Aleks Kissinger and Vladimir Zamdzhiev. Quantomatic: A proof assistant for diagrammatic reasoning. In *Automated Deduction - CADE-25*, pages 326–336. Springer International Publishing, 2015. doi:10.1007/978-3-319-21401-6_22.

29 Andreas Klappenecker and Martin Rötteler. Unitary error bases: Constructions, equivalence, and applications. In *Applied Algebra, Algebraic Algorithms and Error-Correcting Codes*, pages 139–149. Springer, 2003. doi:10.1007/3-540-44828-4_16.

30 Emanuel Knill. Group representations, error bases and quantum codes. *Los Alamos NLR LAUR-96-2807*, 1996. doi:10.2172/378680.

31 Emanuel Knill. Non-binary unitary error bases and quantum codes. *Los Alamos National Laboratory Report LAUR-96-2717*, 1996. doi:10.2172/373768.

32 Máté Matolcsi, Júlia Réffy, and Ferenc Szöllősi. Constructions of complex Hadamard matrices via tiling abelian groups. *Open Systems & Information Dynamics*, 14(03):247–263, 2007. doi:10.1007/s11080-007-9050-6.

33 David A. Meyer and Thomas G. Wong. Connectivity is a poor indicator of fast quantum search. *Physical Review Letters*, 114(11), 2015. doi:10.1103/physrevlett.114.110503.

34 Scott Morrison and Emily Peters. The little desert? Some subfactors with index in the interval $(5, 3+\sqrt{5})$. *International Journal of Mathematics*, 25(08):1450080, 2014. doi:10.1142/s0129167x14500803.

35 Benjamin Musto. Constructing mutually unbiased bases from quantum Latin squares, 2016. arXiv:1605.08919.

36 Benjamin Musto and Jamie Vicary. Quantum Latin squares and unitary error bases. *Quantum Information and Computation*, 2016. to appear. arXiv:1504.02715.

37 Remus Nicoară. A finiteness result for commuting squares of matrix algebras. *Journal of Operator Theory*, 55(2):295–310, 2006. arXiv:math/0404301.

38 Remus Nicoară. Subfactors and Hadamard matrices. *Journal of Operator Theory*, 64(2):453–468, 2010. arXiv:0704.1128.

39 Adrian Ocneanu. Quantized groups, string algebras, and Galois theory for algebras. In *Operator Algebras and Applications*, pages 119–172. CUP, 1989. doi:10.1017/cbo9780511662287.008.

40 Román Orús. A practical introduction to tensor networks: Matrix product states and projected entangled pair states. *Annals of Physics*, 349:117–158, 2014. doi:10.1016/j.aop.2014.06.013.

41 Roger Penrose. Applications of negative-dimensional tensors. In D.J.A. Welsh, editor, *Combinatorial Mathematics and its Applications*, pages 221–244. Academic Press, New York, 1971.

42 Mihai Petrescu. *Existence of Continuous Families of Complex Hadamard Matrices of Certain Prime Dimensions and Related Results*. PhD thesis, University of California, Los Angeles, 1997.

43 Sorin Popa. Orthogonal pairs of $*$-subalgebras in finite von Neumann algebras. *Journal of Operator Theory*, 9:253–268, 1983.

44 David Reutter and Jamie Vicary. Biunitary constructions in quantum information, 2016. arXiv:1609.07775.

45 Christopher Schommer-Pries. *The classification of two-dimensional extended topological field theories*. PhD thesis, Department of Mathematics, University of California, Berkeley, 2009. arXiv:1112.1000.

46 Peter Selinger. Dagger compact closed categories and completely positive maps. *Electronic Notes in Theoretical Computer Science*, 170:139–163, 2007. doi:10.1016/j.entcs.2006.12.018.

47 Peter Selinger. A survey of graphical languages for monoidal categories. In *New Structures for Physics*, pages 289–355. Springer, 2010. doi:10.1007/978-3-642-12821-9_4.

48 Claude E. Shannon. Communication theory of secrecy systems. *Bell System Technical Journal*, 28(4):656–715, 1949. doi:10.1002/j.1538-7305.1949.tb00928.x.

49 Peter W. Shor. Fault-tolerant quantum computation. In *Proceedings of 37th Conference on Foundations of Computer Science*, pages 56–65. IEEE Computer Society Press, 1996. doi:10.1109/sfcs.1996.548464.

50 Ferenc Szöllősi. *Construction, classification and parametrization of complex Hadamard matrices*. PhD thesis, Central European University, Budapest, Hungary, 2011. arXiv:1110.5590.

51 Ferenc Szöllősi. Complex Hadamard matrices of order 6: a four-parameter family. *Journal of the London Mathematical Society*, 85(3):616–632, 2012. doi:10.1112/jlms/jdr052.

52 Wojciech Tadej and Karol Życzkowski. A concise guide to complex Hadamard matrices. *Open Systems & Information Dynamics*, 13(02):133–177, 2006. doi:10.1007/s11080-006-8220-2.

53 Jamie Vicary. Higher quantum theory, 2012. URL: https://arxiv.org/abs/1207.4563.

54 Jamie Vicary. Higher semantics of quantum protocols. In *Proceedings of LICS*, 2012. doi:10.1109/lics.2012.70.

55 Reinhard F. Werner. All teleportation and dense coding schemes. *Journal of Physics A*, 34(35):7081–7094, 2001. doi:10.1088/0305-4470/34/35/332.

56 Pawel Wocjan and Thomas Beth. New construction of mutually unbiased bases in square dimension, 2004. arXiv:quant-ph/0407081.