Two-Planar Graphs Are Quasiplanar

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Abstract

It is shown that every 2-planar graph is quasiplanar, that is, if a simple graph admits a drawing in the plane such that every edge is crossed at most twice, then it also admits a drawing in which no three edges pairwise cross. We further show that quasiplanarity is witnessed by a simple topological drawing, that is, any two edges cross at most once and adjacent edges do not cross.

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1 Introduction

For $k \in \mathbb{N}$, a graph $G = (V, E)$ is called *k-planar* if it admits a drawing in the plane such that every edge is crossed at most k times (such a drawing is called a *k-plane* drawing of G). Similarly, G is called *k-quasiplanar* if it admits a drawing in which no k edges pairwise cross each other (a *k-quasiplane* drawing). A planar graph is 0-planar and 2-quasiplanar by definition. A 3-quasiplanar graph is also called *quasiplanar*, for short. The relation between k-planarity and ℓ-quasiplanarity has been studied only recently. Angelini et al. [6] proved that for $k \geq 3$, every k-planar graph is $(k+1)$-quasiplanar. However, the case $k = 2$ was left open. In this note, we show that the result extends to $k = 2$, and prove the following.

▶ **Theorem 1.** *Every 2-planar graph is quasiplanar.*

The inclusion is proper because there exists a family of (simple) quasiplanar graphs on n vertices with $6.5n - O(1)$ edges [3], whereas every 2-planar graph on $n \geq 3$ vertices has at most $5n - 10$ edges [21]. Our proof is constructive, and allows transforming a 2-plane drawing of an n-vertex graph into a quasiplane drawing in time polynomial in n.

Simple topological drawings. The concept of k-planarity and k-quasiplanarity assumes that the drawings are *topological graphs* where the edges are represented by Jordan arcs, edges may cross each other multiple times, and adjacent edges may cross. In a *simple topological graph*, any two edges cross at most once, and no two adjacent edges cross. Excluding the crossings between adjacent edges is a nontrivial condition [14]. For example, Brandenburg et al. [8] showed that every graph that admits a 1-plane simple topological drawing also admits a 1-plane *straight-line drawing* in which crossing edges meet at a right angle.

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Angelini et al. [6] proved that for $k \geq 3$, every k-planar graph admits a $(k+1)$-quasiplane simple topological drawing. A careful analysis of our redrawing algorithm, which transforms a 2-plane drawing of a graph into a quasiplane drawing, reveals that it produces a quasiplane simple topological drawing. Thereby we obtain the following strengthening of Theorem 1.

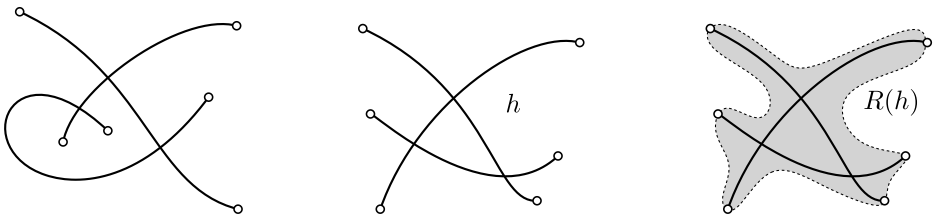
▶ **Theorem 2.** *Every 2-planar graph admits a quasiplane simple topological drawing.*

Related work. Graph planarity is a fundamental concept and a plethora of results has been obtained for planar graphs. The quest for generalizations has motivated the graph minor theory [17]. In the same vein, various notions of *near-planarity* have been studied [19]. The proximity of a graph to planarity may be measured by global parameters, such as the crossing number [22] or graph thickness and their variations [9, 10], or local parameters such as the minimum $k \in \mathbb{N}_0$ for which the graph is k-planar or k-quasiplanar. The concept of *k-planarity* plays a crucial role in proving the current best constants for the classic *Crossing Lemma* [2, 5, 18], and k-quasiplanarity is closely related to Ramsey-type properties of the intersection graph of Jordan arcs in the plane [4]. However, relations between the latter two graph classes have been studied only recently [6].

k-planarity. Planar and 1-planar graphs are fairly well-understood [16]. The Crossing Lemma implies that a k-planar graph on n vertices has at most $4.1\sqrt{k} \cdot n$ edges, and this bound is the best possible apart from constant factors [21]. Tight upper bounds of $4n - 8$, $5n - 10$, and $5.5n - 11$ edges are known for $k = 1$, 2, and 3, respectively [18, 21], and an upper bound of $6n - 12$ edges is known for $k = 4$ [2]. For $k = 1, 2, 3$, so-called *optimal* k-planar graphs (which have the maximum number of edges on n vertices) have recently been completely characterized [7], however they have special properties that in general are not shared by edge-maximal k-planar graphs.

k-quasiplanarity. Pach, Shahrokhi, and Szegedy [20] conjectured that for every $k \in \mathbb{N}$, an n-vertex k-quasiplanar graph has $O(n)$ edges, where the constant of proportionality depends on k. The conjecture has been verified for $k \leq 4$ [1]. The current best upper bound that holds for all $k \in \mathbb{N}$ is $n(\log n)^{O(\log k)}$ due to Fox and Pach [11]. Improvements are known in several important special cases. Suk and Walczak [23] prove that every n-vertex k-quasiplanar graph has $O(2^{\alpha(n)^c} n \log n)$ edges, where $\alpha(n)$ denotes the inverse Ackermann function and c depends only on k, if any two edges intersect in $O(1)$ points. They also show that every n-vertex k-quasiplanar graph has at most $O(n \log n)$ edges if every two edges intersect at most once. These bounds improve earlier work by Fox et al. [12, 13].

Organization. We prove Theorems 1 and 2 in Section 2: We describe a redrawing algorithm in Section 2.1, parameterized by two functions, f and g, that are defined on pairwise crossing triples of edges. In Section 2.3 we analyze local configurations that may produce a triple of pairwise crossing edges after redrawing. In Sections 2.4 and 2.5, we choose suitable functions f and g, and show that our rerouting algorithm with these parameters produces a quasiplane drawing for a 2-planar graph. In Section 2.6, we extend the analysis of our redrawing algorithm and show that it produces a simple topological quasiplane drawing. We conclude in Section 3 with a review of open problems. Due to space limitations, many proofs are omitted; they can be found in the full paper [15].



(a) tangled **(b)** untangled **(c)** region

Figure 1 Tangled and untangled 3-crossings and their associated regions.

2 Proof of Theorem 1

Let $G = (V, E)$ be a 2-planar graph. Assume without loss of generality that G is connected. We need to show that G admits a quasiplane drawing. Note that this quasiplane drawing to be constructed need not—and in general will not—be 2-plane. We may assume, without loss of generality, that G is edge-maximal, in the sense that no new edge can be added (to the abstract graph) without violating 2-planarity. Since G is 2-planar, it admits a 2-plane drawing. We show that it also admits a simple topological 2-plane drawing.

▶ **Lemma 3.** *Every 2-planar graph admits a 2-plane simple topological drawing. Specifically, a 2-plane drawing of a graph G with the minimum number of crossings (among all 2-plane drawings of G) is a simple topological graph.*

Note that a 2-plane drawing may contain a **3-crossing**, that is, a triple of pairwise crossing edges. A 3-crossing in a drawing is **untangled** if the six endpoints of the edges lie on the same face of the arrangement formed by the three edges; otherwise the 3-crossing is **tangled**, see Figure 1a and 1b for an example. Angelini et al. showed [6, Lemma 2] that every 2-planar graph admits a 2-plane drawing in which every 3-crossing is untangled. Their proof starts from a 2-plane drawing and rearranges tangled 3-crossings without introducing any new edge crossings. Therefore, in combination with our Lemma 3 we may start from a 2-plane drawing D of G with the following properties: (i) every 3-crossing is untangled, (ii) no two edges cross more than once, and (iii) no two adjacent edges cross.

If there is no 3-crossing in D, then G is quasiplanar by definition. Otherwise we construct a quasiplane drawing D' of G as described below.

Every 3-crossing in D spans a (topological) **hexagon** in the following sense. Let $\mathcal{H}$ be the set of unordered triples of edges in E that form a 3-crossing in D. In every triple $h \in \mathcal{H}$, each edge crosses both other edges of the triple, and so it cannot cross any edge in $E \setminus h$. Consequently, the triples in $\mathcal{H}$ are pairwise disjoint [6, Observation 1]. For each triple $h \in \mathcal{H}$, let $\mathrm{V}(h)$ denote the set that consists of the six endpoints of the three edges in h. Since h is untangled in D, all six vertices of $\mathrm{V}(h)$ lie on a face f_h of the arrangement induced by the edges of h as drawn in D. Any two vertices of $\mathrm{V}(h)$ that are consecutive along the boundary of f_h can be connected by a Jordan arc that closely follows the boundary of f_h and does not cross any edges in D; see Figure 1c. Together these arcs form a closed Jordan curve, which partitions the plane into two closed regions: let $R(h)$ denote the region that contains the edges of h, and let $\partial R(h)$ denote the boundary of $R(h)$. We think of $\partial R(h)$ as both a closed Jordan curve and as a graph that is a 6-cycle. As the triples in $\mathcal{H}$ are pairwise disjoint, we may assume that the regions $R(h)$, $h \in \mathcal{H}$, have pairwise disjoint interiors.

▶ **Observation 4.** *For every $h \in \mathcal{H}$, every pair of consecutive vertices of the 6-cycle $\partial R(h)$ are connected by an edge in G, and this edge is crossing-free in D.*

Proof. Let $u, v \in V$ be two consecutive vertices of a 6-cycle $\partial R(h)$ for some $h \in \mathcal{H}$.

We show that uv is an edge in G. Indeed, if uv is not an edge of G, then we can augment G with the edge $e = uv$, and insert it into the drawing D as a crossing-free Jordan arc along $\partial R(h)$ to obtain a 2-plane drawing D' of $G \cup \{e\}$. This contradicts our assumption that G is edge-maximal and no edge can be added to G without violating 2-planarity.

We then show that e is crossing free in D. Indeed, if e crosses any other edge in D, we can redraw e as a Jordan arc along $\partial R(h)$, which is crossing-free. The resulting drawing D' of G is 2-plane and has fewer crossings than D. This contradicts our assumption that D has a minimum number of crossings among all 2-plane drawings of G. ◀

By Observation 4 any two consecutive vertices along $\partial R(h)$ of a hexagon $h \in \mathcal{H}$ are connected by an edge e in G. Note that this does not necessarily imply that e is drawn along $\partial R(h)$ in D. It is possible that the cycle formed by the edge e in D and the copy of e drawn along $\partial R(h)$ (which is not part of D) contains other parts of the graph.

▶ **Observation 5.** *(a) Two distinct hexagons in $\mathcal{H}$ share at most five vertices; and (b) three distinct hexagons in $\mathcal{H}$ share at most two vertices.*

Angelini et al. proved [6, Lemma 3 and 4] that there exists an injective map $f : \mathcal{H} \to V$ that maps every hexagon $h \in \mathcal{H}$ to a vertex $v \in \mathrm{V}(h)$. For each hexagon $h \in \mathcal{H}$, exactly one edge in h is incident to the vertex $f(h)$. Let $g(h)$ be one of the two edges in h not incident to $f(h)$. Then for any such choice $g : \mathcal{H} \to E$ is an injective function (because the triples in $\mathcal{H}$ are pairwise disjoint). We complete the construction using a rerouting algorithm that for each hexagon $h \in \mathcal{H}$, reroutes the edge $g(h)$ "around" the vertex $f(h)$. The algorithm—described in detail below—is very similar to the one of Angelini et al., but with a few subtle changes to make it work for 2-planar graphs, rather than k-planar graphs, for $k \geq 3$.

2.1 Rerouting algorithm

We are given a 2-planar graph $G = (V, E)$, and a 2-plane drawing D of G with properties (i)–(iii), as described above. Let the functions $f : \mathcal{H} \to V$ and $g : \mathcal{H} \to E$ be given. (We will determine suitable choices for f and g later.) The algorithm consists of two phases.

Phase 1. For each hexagon $h \in \mathcal{H}$, we perform the following changes in D. Let $h = \{a, b, c\}$ such that the edge a is incident to $f(h)$ and $b = g(h) = uv$, where u is adjacent to $f(h)$ along $\partial R(h)$. Keep the original drawing of the edges a and c. Then arrange (possibly redraw) the edge b inside $R(h)$ so that the oriented Jordan arc uv crosses a before c. Finally, redraw the edge $b = g(h) = uv$ to go around vertex $f(h)$ as follows. See Figure 2.

1. Erase the portion of b in a small neighborhood of the crossing $a \cap b$ to split b into two Jordan arcs: an arc γ_v from v to a point x close to $a \cap b$, and another arc γ_u from x to u.
2. Keep γ_v as part of the new arc representing b, but discard γ_u and replace it by a new Jordan arc from x to u. This arc first closely follows the edge a towards $f(h)$, then goes around the endpoint $f(h)$ of a until it reaches the edge $f(h)u$ (which exists by Observation 4 and is crossing-free in D). The arc then closely follows the edge $f(h)u$ without crossing it to reach u.

As a result, edges a and b no longer cross and the 3-crossing induced by h is eliminated. However, the rerouting may create new crossings between $g(h)$ and edges incident to $f(h)$ (but not a and $uf(h)$). These new crossings are of no consequence, unless they create a 3-crossing. Hence we have to analyze under which circumstances 3-crossings can arise as a

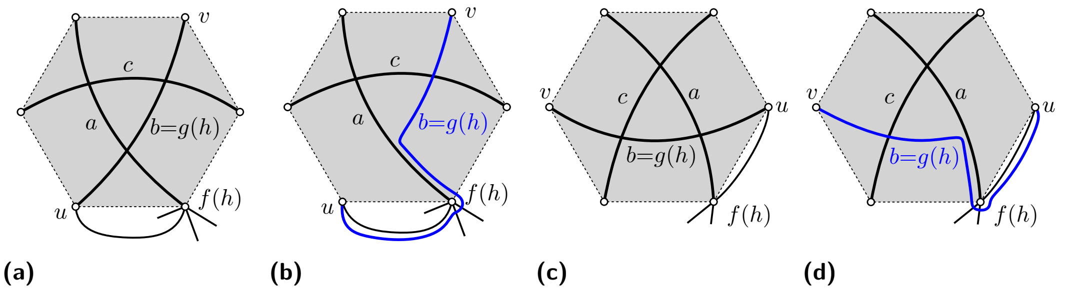


Figure 2 Rerouting $g(h)$ around $f(h)$; $g(h)$ can be either of the two edges not incident to $f(h)$.

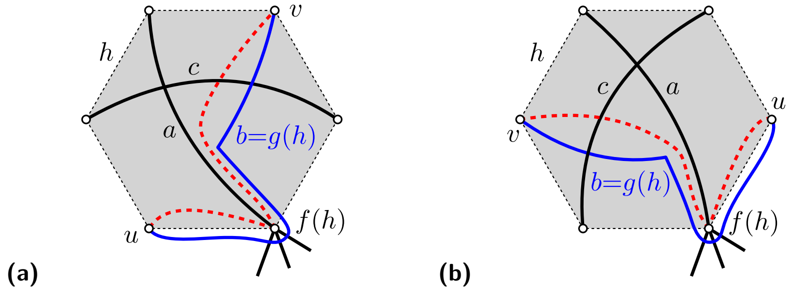


Figure 3 The hexagon h is a home for the two edges that are shown by a dashed red arc. These edges (if present in G) can be safely drawn inside $R(h)$.

result of the reroutings. But first we eliminate some potentially troublesome edge crossings in a second phase of the algorithm.

For an edge $e \in E$ a hexagon $h \in \mathcal{H}$ is a **home** for e if e is both incident to $f(h)$ and adjacent to $g(h)$. If h is a home for e, then e can be drawn inside $R(h)$ so that it has at most one crossing, with the edge $c \in h$ (see Figure 3).

Phase 2. As long as there exists an edge $e \in E$ so that (1) e has a home $h \in \mathcal{H}$, (2) there is no home $h' \in \mathcal{H} \setminus \{h\}$ of e so that e is drawn inside $R(h')$, and (3) e has at least one crossing in the current drawing, we reroute e to be drawn inside $R(h)$.

Note that each $h \in \mathcal{H}$ is a home for at most two edges and conversely an edge can have at most two homes (one for each endpoint because f is injective). Also note that an edge may be rerouted in both Phase 1 and Phase 2. This completes the description of the rerouting algorithm. Let $D(f, g)$ denote the drawing that results from applying both phases of the rerouting algorithm to the original drawing D of G.

2.2 Properties of $D(f, g)$

The edges of G fall into three groups, depending on how they are represented in $D(f, g)$ with respect to D: (1) **nonrerouted** edges have not been rerouted in either phase and remain the same as in D; (2) edges that have been rerouted in Phase 2 we call **safe** (regardless of whether or not they have also been rerouted in Phase 1); and (3) edges that have been rerouted in Phase 1 but not in Phase 2 we call **critical**. An edge is **rerouted** if it is either safe or critical. Let us start by classifying the *new* crossings that are introduced by the rerouting algorithm. Without loss of generality we may assume that in every hexagon $h \in \mathcal{H}$

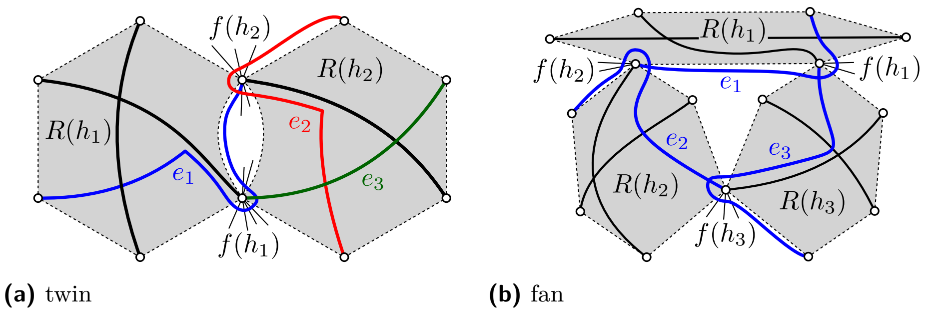


Figure 4 The redrawing may produce 3-crossings in form of twins or fans.

of D the edge $g(h)$ intersects the other two edges of h in the order described in the first paragraph of Phase 1 above. (If not, then redraw the edge $g(h)$ within $R(h)$ accordingly.)

▶ **Lemma 6.** *Consider a crossing c of two edges e_1 and e_2 in $D(f,g)$ that is not a crossing in D. After possibly exchanging the roles of e_1 and e_2, the crossing c is of exactly one of the following two types: (a) e_1 is safe and drawn in $R(h)$ for a home $h \in \mathcal{H}$ with $e_2 \in h$ nonrerouted; or (b) e_1 is critical and rerouted around an endpoint of e_2.*

▶ **Lemma 7.** *Consider a safe edge e in $D(f,g)$, and let $h \in \mathcal{H}$ denote the home of e so that $e = f(h)z$, for $z \in \mathrm{V}(h)$, is drawn inside $R(h)$. Then*

(i) *e is not part of a 3-crossing;*
(ii) *e does not cross any edge more than once; and*
(iii) *e crosses an adjacent edge e' only if e' is critical, incident to $f(h)$, and rerouted around $z = f(h')$, for some hexagon $h' \in \mathcal{H} \setminus \{h\}$, with $g(h') = e'$.*

▶ **Lemma 8.** *No two adjacent critical edges cross in $D(f,g)$.*

We are ready to completely characterize the 3-crossings in $D(f,g)$. The characterization allows us to then eliminate these 3-crossings by selecting the functions f and g suitably.

▶ **Definition 9.** Let $D(f,g)$ be a drawing of a graph $G = (V,E)$ with functions $f : \mathcal{H} \to V$ and $g : \mathcal{H} \to E$ as defined above. Three edges $e_1, e_2, e_3 \in E$ form a ...

- **twin** configuration in $D(f,g)$ if they are in two distinct hexagons $h_1, h_2 \in \mathcal{H}$, where $e_1 = g(h_1)$, $e_2 = g(h_2)$ and $e_3 \in h_2 \setminus \{e_2\}$, such that edge e_1 is incident to $f(h_2)$, edge e_3 is incident to $f(h_1)$ but not to $f(h_2)$, and e_3 is drawn inside $R(h_2)$. See Figure 4a.
- **fan** configuration in $D(f,g)$ if they are in three pairwise distinct hexagons $h_1, h_2, h_3 \in \mathcal{H}$, where $e_1 = g(h_1)$, $e_2 = g(h_2)$, and $e_3 = g(h_3)$, such that edge e_1 is incident to $f(h_2)$, edge e_2 is incident to $f(h_3)$, and edge e_3 is incident to $f(h_1)$. See Figure 4b.

▶ **Lemma 10.** *Every 3-crossing in $D(f,g)$ forms a twin or a fan configuration.*

Theorem 1 is an immediate corollary of the following lemma, which we prove in Section 2.5.

▶ **Lemma 11.** *There exist functions $f : \mathcal{H} \to V$ and $g : \mathcal{H} \to E$ for which $D(f,g)$ is a quasiplane drawing of G.*

2.3 Conflict digraph

We define a plane digraph $K = (V, A)$ that represents the interactions between the hexagons in $\mathcal{H}$. The conflict graph depends on G, on the initial drawing D, and on the function

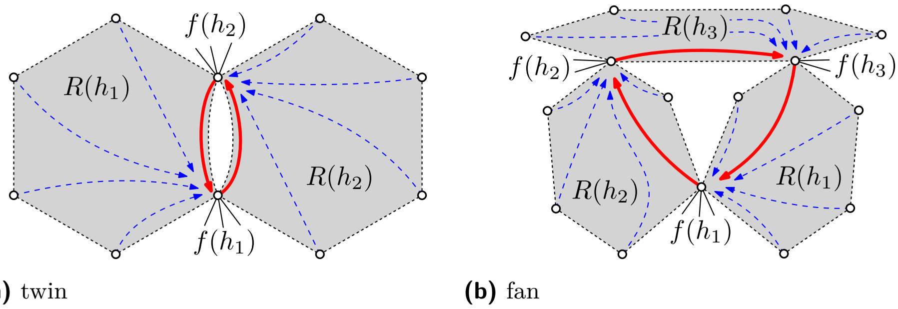


Figure 5 Twin and fan configurations induce cycles in the conflict graph.

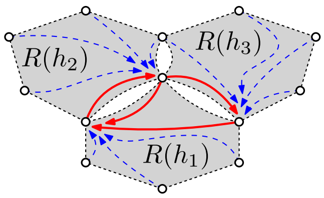
$f : \mathcal{H} \to V$, but it does not depend on the function g. For every hexagon $h \in \mathcal{H}$, we create five directed edges that are all directed towards $f(h)$ and drawn inside $R(h)$. These edges start from the five vertices on $\partial R(h)$ other than $f(h)$; see Figure 5. Note that two vertices in V may be connected by two edges with opposite orientations lying in two different hexagons (for instance, in a twin configuration as shown in Figure 5a). However, K contains neither loops nor parallel edges with the same orientation because f is injective and so every vertex can have incoming edges from at most one hexagon.

▶ **Observation 12.** *Let K be the conflict graph for $G = (V, E)$ and the drawing $D(f, g)$.*
- **(i)** *K is a directed plane graph.*
- **(ii)** *At every vertex $v \in V$, the incoming edges in K are consecutive in the rotation order of incident edges around v.*
- **(iii)** *If $e_1 = v_1v_2$, $e_2 = v_2v_3$, and $e_3 = v_3v_1$ form a fan configuration in $D(f, g)$, then the conflict digraph contains a 3-cycle (v_1, v_2, v_3).*
- **(iv)** *If $e_1 = g(h_1)$, $e_2 = g(h_2)$, and $e_3 \in h_2$ form a twin configuration in $D(f, g)$, then the conflict digraph contains a 2-cycle $(f(h_1), f(h_2))$.*

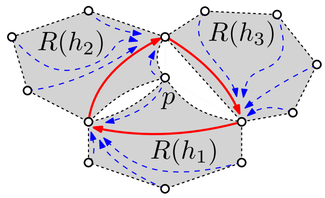
Proof. (i) The edges of K lie in the regions $R(h)$, $h \in \mathcal{H}$. Since these regions are interior-disjoint, edges from different regions do not cross. All edges in the same region $R(h)$, $h \in \mathcal{H}$, are incident to $f(h)$; so they do not cross, either. (ii) For each vertex $v \in V$, there is at most one $h \in \mathcal{H}$ such that $v = f(h)$. All incoming edges of v lie in the region $R(h)$, and all edges lying in $R(h)$ are directed towards $v = f(h)$ by construction. (iii–iv) Both claims follow directly from the definition of fan and twin configurations and the definition of K. ◀

Relations between cycles in K. We observed that K is a plane digraph, where every twin configuration induces a 2-cycle and every fan configuration induces a 3-cycle. So in order to control the appearance of twin and fan configurations in the drawing $D(f, g)$, we need to understand the structure of 2- and 3-cycles in the conflict digraph K. In the following paragraphs we introduce some terminology and prove some structural statements about cycles in K.

For a cycle c in K, let $\mathsf{int}(c)$ denote the **interior** of c, let $\mathsf{ext}(c)$ denote the **exterior** of c, let $R(c)$ denote the closed bounded **region** bounded by c, and let $\mathrm{V}(c)$ denote the **vertex set** of c. We use the notation $i \oplus 1 := 1 + (i \bmod k)$ and $i \ominus 1 := 1 + ((k + i - 2) \bmod k)$ to denote successors and predecessors, respectively, in a circular sequence of length k that is indexed $1, \ldots, k$. Let c_1 and c_2 be two cycles in the conflict graph K. We say that c_1 and c_2 are **interior-disjoint** if $\mathsf{int}(c_1) \cap \mathsf{int}(c_2) = \emptyset$. We say that c_1 **contains** c_2 if $R(c_2) \subseteq R(c_1)$. See Figure 6a for an example. In both cases, c_1 and c_2 may share vertices and edges, but they may also be vertex-disjoint.



(a) A smooth 3-cycle contains a smooth 2-cycle.



(b) A nonsmooth 3-cycle.

Figure 6 Examples: smooth cycles and containment.

▶ **Lemma 13.** *If a vertex $v \in V$ is incident to two interior-disjoint cycles in K, then these cycles have opposite orientations (clockwise vs. counterclockwise). Consequently, every vertex $v \in V$ is incident to at most two interior-disjoint cycles in K.*

Ghosts. A cycle in the conflict digraph K is **short** if it has length two or three. We say that a 3-cycle in K is a **ghost** if two of its vertices induce a 2-cycle in K. Let $\mathcal{C}$ be the set of all short cycles in K that are not ghosts. Intuitively, we do not worry about a ghost cycle c so much. It will turn out later that by taking care of the 2-cycle c' that makes c a ghost, we also take care of c at the same time.

▶ **Lemma 14.** *A short cycle in K is uniquely determined by its vertex set.*

▶ **Lemma 15.** *Let $c_1, c_2 \in \mathcal{C}$. If $\mathrm{V}(c_1) \cap \mathsf{int}(c_2) \neq \emptyset$, then c_2 contains c_1.*

Proof. Suppose to the contrary that there exist short cycles $c_1, c_2 \in \mathcal{C}$ such that $v_1 \in \mathrm{V}(c_1) \cap \mathsf{int}(c_2)$ but c_2 does not contain c_1. Then some point along c_1 lies in $\mathsf{ext}(c_2)$. Since K is a plane graph, an entire edge of c_1 must lie in $\mathsf{ext}(c_2)$. Denote this edge by (v_2, v_3). Recall that c_1 is short (that is, it has at most three vertices), consequently, $c_1 = (v_1, v_2, v_3)$. Since c_1 has points in both $\mathsf{int}(c_2)$ and $\mathsf{ext}(c_2)$, the two cycles intersect in at least two points. In a plane graph, the intersection of two cycles consists of vertices and edges. Consequently $\mathrm{V}(c_1) \cap \mathrm{V}(c_2) = \{v_2, v_3\}$. Recall that c_2 is also short, and so it has a directed edge between any two of its vertices. However, (v_2, v_3) lies in $\mathsf{ext}(c_2)$, so the reverse edge (v_3, v_2) is present in c_2. That is, $\{v_2, v_3\}$ induces a 2-cycle in K. Hence both c_1 and c_2 are ghosts, contrary to our assumption. ◀

Smooth cycles. In order to avoid twin and fan configurations in $D(f, g)$, we would like to choose an injective function $f : \mathcal{H} \to V$, with $f(h) \in \mathrm{V}(h)$, that avoids short cycles in K, except for a special type of cycles (called *smooth*) to be defined next.

▶ **Definition 16.** Let $c = (v_1, \ldots, v_k)$ be a simple short cycle in the conflict graph K. Recall that every edge in K lies in a region $R(h)$, $h \in \mathcal{H}$, and is directed to $f(h)$. So the cycle c corresponds to a cycle of hexagons $(h_1, \ldots, h_k)$, such that the vertex $v_i = f(h_i)$ lies on the boundary of hexagons h_i and $h_{i \oplus 1}$, for $i = 1, \ldots, k$. We say that the hexagons $h_1, \ldots, h_k$ are **associated** with c. The cycle c is **smooth** if none of the associated hexagons has a vertex in $\mathsf{int}(c)$. (For example, the cycles in Figure 6a are smooth, but the 3-cycle in Figure 6b is not.)

Note that a smooth cycle in K may contain many vertices of various hexagons in its interior; the restrictions apply only to those (two or three) hexagons that are associated with the cycle. For instance, there could be several hexagons in the white regions between the hexagons in Figure 6. Let $\mathcal{C}_\mathrm{s}$ denote the set of all smooth cycles in $\mathcal{C}$, that is, the set of all short smooth nonghost cycles in K. In Section 2.4, we show how to choose f such that all cycles in $\mathcal{C}$ are smooth, that is, $\mathcal{C} = \mathcal{C}_\mathrm{s}$.

Properties of smooth cycles. The following three lemmata formulate some important properties of smooth cycles that hold for any injective function $f : \mathcal{H} \to V$, where $f(h) \in \mathrm{V}(h)$ for all $h \in \mathcal{H}$.

▶ **Lemma 17.** *Let $c \in \mathcal{C}_\mathrm{s}$ and let $u \in \mathsf{int}(c)$ be a vertex of G. Then there is no edge (u, v) in K for any $v \in \mathrm{V}(c)$.*

Proof. Suppose for the sake of a contradiction that (u, v) is an edge of K with $v \in \mathrm{V}(c)$. Let h be the hexagon with $f(h) = v$. All edges towards v are drawn inside h so that, in particular, $u \in \mathrm{V}(h)$. As h is associated with c, this contradicts the assumption that c is smooth. ◀

▶ **Lemma 18.** *Let $c_1, c_2 \in \mathcal{C}_\mathrm{s}$ so that $c_1 \neq c_2$ and c_2 contains c_1. Then $\mathrm{V}(c_1) \cap \mathrm{V}(c_2) = \emptyset$.*

Proof. Suppose to the contrary that there exists a vertex $u \in \mathrm{V}(c_1) \cap \mathrm{V}(c_2)$. We claim that $\mathrm{V}(c_1) \cap \mathsf{int}(c_2) = \emptyset$. To see this, consider a vertex $v \in \mathrm{V}(c_1) \cap \mathsf{int}(c_2)$. Then following c_1 from v to u we find an edge (x, y) of K so that $x \in \mathsf{int}(c_2)$ and $y \in \mathrm{V}(c_2)$. However, such an edge does not exist by Lemma 17. Hence there is no such vertex v and $\mathrm{V}(c_1) \cap \mathsf{int}(c_2) = \emptyset$. Given that c_2 contains c_1, it follows that $\mathrm{V}(c_1) \subseteq \mathrm{V}(c_2)$.

If c_1 is a 3-cycle, then so is c_2 and Lemma 14 contradicts our assumption $c_1 \neq c_2$. Hence c_1 is a 2-cycle and c_2 is a 3-cycle. But then c_2 is a ghost, in contradiction to $c_2 \in \mathcal{C}_\mathrm{s}$. ◀

▶ **Lemma 19.** *Any two cycles in $\mathcal{C}_\mathrm{s}$ are interior-disjoint or vertex disjoint.*

Proof. Let $c_1, c_2 \in \mathcal{C}_\mathrm{s}$ with $c_1 \neq c_2$. Suppose, to the contrary, that $\mathsf{int}(c_1) \cap \mathsf{int}(c_2) \neq \emptyset$ and $V(c_1) \cap V(c_2) \neq \emptyset$. Without loss of generality, an edge (u_1, u_2) of c_2 lies in the interior of c_1.

We may assume that u_1 and u_2 are common vertices of c_1 and c_2. Indeed, if u_1 and u_2 were not common vertices of the cycles, then a vertex of c_2 would lie in the interior of c_1. Then c_1 contains c_2 by Lemma 15, and $\mathrm{V}(c_1) \cap \mathrm{V}(c_2) = \emptyset$ by Lemma 18.

We may further assume that both c_1 and c_2 are 3-cycles. Indeed, if the vertex set of one of them contains that of the other, then one of them is a 3-cycle and the other is a 2-cycle. Since both c_1 and c_2 are present in $\mathcal{C}$, one of them would be a ghost cycle in $\mathcal{C}$, contradicting the definition of $\mathcal{C}$.

Since (u_1, u_2) is a directed edge of c_2 that lies in the interior of c_1, and c_1 is a 3-cycle that has an edge between any two of its vertices, the edge (u_2, u_1) is present in c_1. This implies that $c_3 = (u_1, u_2)$ is a 2-cycle in K. Therefore $c_3 \in \mathcal{C}$, and both c_1 and c_2 are ghost cycles in $\mathcal{C}$, contradicting the definition of $\mathcal{C}$, $\mathcal{C} \supseteq \mathcal{C}_\mathrm{s}$. This confirms that $c_1, c_2 \in \mathcal{C}_\mathrm{s}$, $c_1 \neq c_2$, are interior-disjoint or vertex disjoint, as claimed. ◀

2.4 Choosing the special vertices $f(h)$

As noted above, Angelini et al. proved [6, Lemmata 3 and 4] that there exists an injective map $f : \mathcal{H} \to V$ that maps every hexagon $h \in \mathcal{H}$ to a vertex $v \in \mathrm{V}(h)$. We review their argument (using Hall's matching theorem), and then strengthen the result to establish some additional properties of the function $f : \mathcal{H} \to V$.

Hall's condition. Let $\mathcal{A} \subseteq \mathcal{H}$ be a subset of hexagons, and let $\mathrm{V}(\mathcal{A}) \subseteq V$ be the set of vertices incident to the hexagons in $\mathcal{A}$. Following Angelini et al. [6, Lemma 4] we obtain Hall's condition via double counting.

▶ **Lemma 20.** *For every subset $\mathcal{A} \subseteq \mathcal{H}$, we have $|\mathrm{V}(\mathcal{A})| \geq 2|\mathcal{A}| + 2$.*

▶ **Corollary 21.** *There exists an injective map $f : \mathcal{H} \to V$ that maps every hexagon $h \in \mathcal{H}$ to a vertex $v \in \mathrm{V}(h)$.*

▶ **Corollary 22.** *For every nonempty subset $\mathcal{A} \subseteq \mathcal{H}$, we have $|\mathrm{V}(\mathcal{A})| \geq |\mathcal{A}| + 5$.*

Proof. If $|\mathcal{A}| = 1$, then $|\mathcal{A}| + 5 = 6$ and a single hexagon has 6 distinct vertices. If $|\mathcal{A}| = 2$, then $|\mathcal{A}|+5 = 7$; and two distinct hexagons have at least 7 distinct vertices by Observation 5a. Otherwise $|\mathcal{A}| \geq 3$, and Lemma 20 yields $|\mathrm{V}(\mathcal{A})| \geq 2|\mathcal{A}| + 2 \geq |\mathcal{A}| + 5$. ◀

▶ **Lemma 23.** *There exists an injective function $f : \mathcal{H} \to V$ such that $f(h) \in \mathrm{V}(h)$, for every $h \in \mathcal{H}$, and every cycle in $\mathcal{C}$ is smooth.*

2.5 Choosing the special edges $g(h)$

Let $f : \mathcal{H} \to V$ be a function as described in Lemma 23. That is, in the following we assume $\mathcal{C} = \mathcal{C}_{\mathrm{s}}$ (all short nonghost cycles in K are smooth). We use Hall's theorem to show that there is a matching of the cycles in $\mathcal{C}$ to the vertices in V such that each cycle is matched to an incident vertex. For a subset $\mathcal{B} \subseteq \mathcal{C}$, let $\mathrm{V}(\mathcal{B})$ denote the set of all vertices incident to some cycle in $\mathcal{B}$.

▶ **Lemma 24.** *For every set $\mathcal{B}_0 \subseteq \mathcal{C}$ of pairwise interior-disjoint cycles, $|\mathcal{B}_0| \leq |\mathrm{V}(\mathcal{B}_0)|$.*

Proof. We use double counting. Let I be the set of all pairs $(v, c) \in V \times \mathcal{B}_0$ such that v is incident to c. Every cycle is incident to ≥ 2 vertices, hence $|I| \geq 2|\mathcal{B}_0|$. By Lemma 13, every vertex is incident to at most two interior-disjoint cycles. Consequently, $|I| \leq 2|\mathrm{V}(\mathcal{B}_0)|$. The combination of the upper and lower bounds for $|I|$ yields $|\mathcal{B}_0| \leq |\mathrm{V}(\mathcal{B}_0)|$. ◀

▶ **Lemma 25.** *For every set $\mathcal{B} \subseteq \mathcal{C}$ of cycles, we have $|\mathcal{B}| \leq |\mathrm{V}(\mathcal{B})|$.*

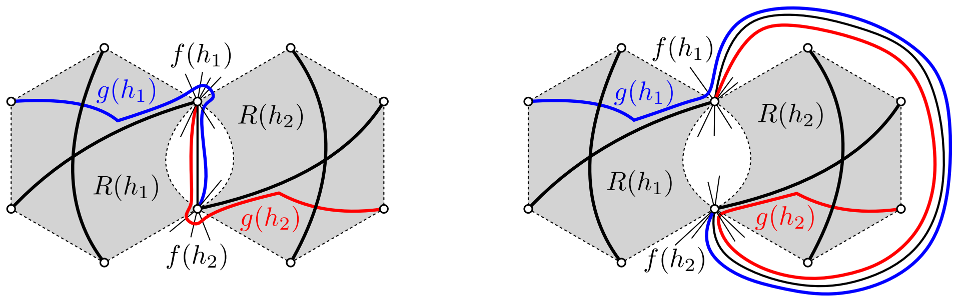
▶ **Lemma 26.** *There exists an injective function $s : \mathcal{C} \to V$ that maps every cycle in $\mathcal{C}$ to one of its vertices.*

We are ready to define the function $g : \mathcal{H} \to E$, that maps every hexagon $h \in \mathcal{H}$ to one of its edges.

▶ **Lemma 27.** *There is a function $g : \mathcal{H} \to E$ such that*
- *for every $h \in \mathcal{H}$, $g(h) \in h$ and $g(h)$ is not incident to $f(h)$;*
- *for every 2-cycle $(f(h_1), f(h_2))$ in K, the edges $g(h_1)$ and $g(h_2)$ do not cross in $D(f, g)$;*
- *for every 3-cycle $(f(h_1), f(h_2), f(h_3))$ in K, at least two of the edges $g(h_1)$, $g(h_2)$, and $g(h_3)$ do not cross in $D(f, g)$.*

Proof. By Lemma 26, there is an injective function $s : \mathcal{C} \to V$ that maps every cycle $c \in \mathcal{C}$ to one of its vertices. For each cycle $c \in \mathcal{C}$, vertex $s(c)$ is the endpoint of some directed edge $(q(c), s(c))$ in the conflict graph. Consequently, there is a hexagon $h \in \mathcal{H}$ such that $s(c) = f(h)$ and $q(c) \in \mathrm{V}(h)$. We say that h is *assigned* to the cycle c. We distinguish two types of hexagons, depending on whether or not they are assigned to a 2-cycle of $\mathcal{C}$.

Hexagons that are not assigned to 2-cycles. For every hexagon h that is not assigned to any cycle, choose $g(h)$ to be an arbitrary edge in h that is not incident to the vertex $f(h)$. For every hexagon h that is assigned to a 3-cycle $c \in \mathcal{C}$, choose $g(h)$ to be the (unique) edge in h that is incident to neither $q(c)$ nor $s(c)$. If $c = (f(h_1), f(h_2), f(h_3))$ and without loss of generality $s(c) = f(h_2)$, then $g(h_2)$ is not incident to $f(h_1) = q(c)$, consequently $g(h_1)$ is disjoint from $g(h_2)$. (Note that $g(h_1)$ is not incident to $f(h_2) = s(c)$ because this would induce a 2-cycle in K, making c a ghost.)



(a) **(b)**

Figure 7 In Case 1 of Lemma 27, the edge $g(h_2)$ is incident to $f(h_1)$. We set $g(h_1)$ so that it is incident to $f(h_2)$. Regardless of how $f(h_1)f(h_2)$ is drawn, the edge separates $g(h_1)$ and $g(h_2)$ and ensures that they are disjoint.

Hexagons assigned to 2-cycles. Consider a 2-cycle $c \in \mathcal{C}$, and let h_1 and h_2 denote the associated hexagons so that without loss of generality $s(c) = f(h_1)$. Suppose without loss of generality that c is oriented clockwise. We distinguish three cases.

Case 1: $g(h_2)$ has already been selected and $g(h_2)$ is incident to $f(h_1)$. Then let $g(h_1)$ be the unique edge in h_1 incident to $f(h_2)$ (Figure 7). We claim that $g(h_1)$ and $g(h_2)$ do not cross in $D(f, g)$. As both edges are critical, by Lemma 6 they can only cross in the neighborhood of $f(h_1)$ or $f(h_2)$. Let a_i be the edge of h_i incident to $f(h_i)$, for $i \in \{1, 2\}$. The edge $g(h_i)$, for $i \in \{1, 2\}$, follows a_i towards the neighborhood of $f(h_i)$ and then crosses the edges incident to $f(h_i)$ following a_i in clockwise order (the orientation of c) until reaching the edge $f(h_1)f(h_2)$. Then $g(h_i)$ follows $f(h_1)f(h_2)$ to its other endpoint, without crossing the edge. Therefore, the path formed by the edges a_1, $f(h_1)f(h_2)$, and a_2 splits the neighborhoods of $f(h_1)$ and $f(h_2)$ into two components so that $g(h_1)$ and $g(h_2)$ are in different components. Thus $g(h_1)$ and $g(h_2)$ do not cross, as claimed.

Case 2: $g(h_2)$ has already been selected and $g(h_2)$ is not incident to $f(h_1)$. Then let $g(h_1)$ be the unique edge in h_1 incident to neither $f(h_1)$ nor $f(h_2)$ (Figure 8a). We claim that $g(h_1)$ and $g(h_2)$ do not cross in $D(f, g)$. As both edges are critical, by Lemma 6 they can only cross in the neighborhood of $f(h_1)$ or $f(h_2)$. But as $g(h_1)$ is not incident to $f(h_2)$, there is a neighborhood of $f(h_2)$ that is disjoint from $g(h_1)$, and so $g(h_1)$ and $g(h_2)$ do not cross there. Similarly, there is a neighborhood of $f(h_1)$ that is disjoint from $g(h_2)$, and so $g(h_1)$ and $g(h_2)$ do not cross there, either. Thus $g(h_1)$ and $g(h_2)$ do not cross in $D(f, g)$.

Case 3: no hexagon h_1 is assigned to a 2-cycle so that $g(h_2)$ has already been selected. Then we are left with hexagons that correspond to 2-cycles and form *cycles* $L = (h_1, \ldots, h_k)$ such that $(f(h_i), f(h_{i\oplus 1}))$ is a 2-cycle in $\mathcal{C}$, for $i = 1 \ldots, k$. These cycles are interior-disjoint by Lemma 19, and any two consecutive cycles in L have opposite orientations by Lemma 13. It follows that k is even.

Since every 2-cycle in L is smooth, the three vertices $f(h_{i\ominus 1})$, $f(h_i)$, and $f(v_{i\oplus 1})$ are consecutive along $\partial R(h_i)$. For every odd $i \in \{1, \ldots, k\}$, let $g(h_i)$ be the (unique) edge in h_i incident to $f(h_{i\ominus 1})$ (and incident to neither $f(h_i)$ nor $f(h_{i\oplus 1})$). Similarly, for every even $i \in \{1, \ldots, k\}$, let $g(h_i)$ be the edge in h_i incident to $f(h_{i\oplus 1})$ (and incident to neither $f(h_i)$ nor $f(h_{i\ominus 1})$). Refer to Figure 8b.

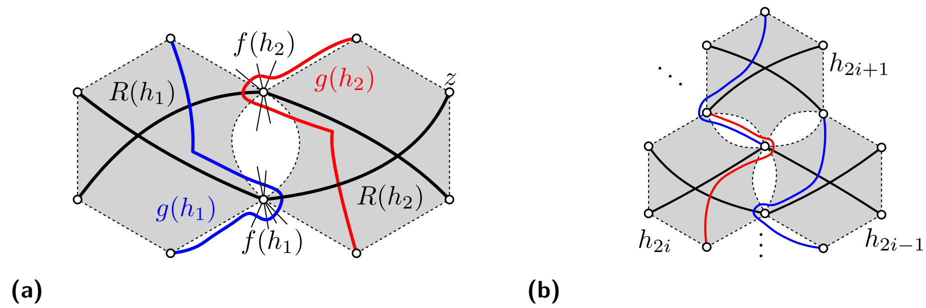


Figure 8 (a) In Case 2 of Lemma 27, the edge $g(h_2)$ is not incident to $f(h_1)$. We set $g(h_1)$ so that it is not incident to $f(h_2)$, to ensure that $g(h_1)$ and $g(h_2)$ are disjoint. (b) In Case 3 we face a cycle of 2-cycles. We consistently select edges to be rerouted in even (red edge) and odd (blue edge) hexagons so that they are pairwise disjoint.

For every odd index $i \in \{1, \ldots, k\}$, the rerouted edges $g(h_i)$ and $g(h_{i\oplus 1})$ are incident to neither $f(h_{i\oplus 1})$ nor $f(h_i)$. Similarly, for every even index $i \in \{1, \ldots, k\}$, the rerouted edges $g(h_i)$ and $g(h_{i\oplus 1})$ are incident to $f(h_{i\oplus 1})$ and $f(h_i)$, respectively. In both cases, the rerouted edges $g(h_i)$ and $g(h_{i\oplus 1})$ are disjoint.

Ghost cycles. It remains to consider ghost cycles. Let c_1 be a ghost cycle in K. Without loss of generality, assume that $c_1 = (v_1, v_2, v_3)$, where $v_1 = f(h_1)$, $v_2 = f(h_2)$, and $v_3 = f(h_3)$, and $c_2 = (v_1, v_2)$ is a 2-cycle in $\mathcal{C}$. Recall that c_2 is smooth (cf. Lemma 23). By construction, $g(h_1)$ and $g(h_2)$ do not cross in $D(f, g)$. Hence at least two of the edges in $\{g(h_1), g(h_2), g(h_3)\}$ do not cross in $D(f, g)$, as required. ◀

The combination of Lemma 10, Lemma 23, and Lemma 27 proves Lemma 11, which completes the proof of Theorem 1.

2.6 Quasiplane simple topological drawings

The redrawing algorithm in Section 2.1 transformed a 2-plane drawing D with properties (i)–(iii), and rerouted some of the edges in two phases to obtain a quasiplane drawing $D(f, g)$. In this section, we show that the algorithm produces a simple topological drawing, that is, any two edges cross at most once, and no two adjacent edges cross.

▶ **Theorem 2.** *Every 2-planar graph admits a quasiplane simple topological drawing.*

3 Conclusions

We have proved that every 2-planar graph is quasiplanar (Theorem 1) by showing that a 2-plane topological graph can be transformed into a quasiplane topological graph, in which no three edges pairwise cross. Theorem 2 strengthens the result to produce a quasiplane *simple* topological graph (any two edges cross at most once and adjacent edges do not cross).

In Section 2.4, we have shown that we can choose one vertex $f(h)$ for each hexagon $h \in \mathcal{H}$ such that all 2- and 3-cycles in the conflict graph K have some special properties. It is unclear, however, whether 2- and 3-cycles can be avoided altogether by a suitable choice of the function f. We formulate an open problem to this effect: Given a set $\mathcal{H}$ of interior-disjoint (topological) hexagons in the plane on a vertex set V, is there an injective function $f : \mathcal{H} \to V$

such that the conflict digraph K contains no 2-cycles (alternatively, neither 2- nor 3-cycles)? Several fundamental problems remain open for k-quasiplanar graphs:

- What is the computational complexity of recognizing k-quasiplanar graphs? Is there a polynomial-time algorithm that decides whether a given graph is quasiplanar (or k-quasiplanar for a given constant k)?
- Is there a constant c_k for every $k \in \mathbb{N}$ such that an n-vertex k-quasiplanar graph has at most $c_k n$ edges [20]? Affirmative answers are known for $k \leq 4$ only [1].
- By Theorem 1 and the main result in [6], every k-planar graph is $(k+1)$-quasiplanar, where $k \in \mathbb{N}$, $k \geq 2$. Angelini et al. [6] ask whether this result can be improved for large k: Denote by $\ell(k) \in \mathbb{N}$ the minimum integer such that every k-planar graph is ℓ-quasiplanar. Prove or disprove that $\ell(k) = o(k)$.

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