Finding Pseudorandom Colorings of Pseudorandom Graphs *

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Abstract

We consider the problem of recovering a planted *pseudorandom* 3-coloring in expanding and low threshold-rank graphs. Alon and Kahale [SICOMP 1997] gave a spectral algorithm to recover the coloring for a random graph with a planted random 3-coloring. We show that their analysis can be adapted to work when coloring is pseudorandom i.e., all color classes are of equal size and the size of the intersection of the neighborhood of a random vertex with each color class has small variance. We also extend our results to partial colorings and low threshold-rank graphs to show the following:

- For graphs on n vertices with threshold-rank r, for which there exists a 3-coloring that is ε-pseudorandom and properly colors the induced subgraph on $(1-\gamma)\cdot n$ vertices, we show how to recover the coloring for $(1-O(\gamma+\varepsilon))\cdot n$ vertices in time $(r\cdot n)^{O(r)}$.
- For expanding graphs on n vertices, which admit a pseudorandom 3-coloring properly coloring all the vertices, we show how to recover such a coloring in polynomial time.

Our results are obtained by combining the method of Alon and Kahale, with eigenspace enumeration methods used for solving constraint satisfaction problems on low threshold-rank graphs.

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1 Introduction

Given an undirected graph $G=(V,E)$, a k-coloring of G is a map $\chi : V \to [k]$ such that for all edges $\{u,v\} \in E$, we have $\chi(u) \neq \chi(v)$. Finding the minimum number of colors with which a graph can be colored, or even finding a coloring which uses few colors for a graph G which is promised to be 3-colorable has been a major open problem in the field of algorithm design. Starting from an early work of Wigderson [17] who showed how to color 3-colorable graphs with $O(\sqrt{n})$ colors, there has been a series of works using novel combinatorial ideas, as well as ideas based on semidefinite programming (SDP), which give algorithms for coloring 3-colorable graphs with fewer colors [6, 13, 7, 4, 9, 5, 14]. The most recent algorithm of

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[14] achieves a coloring with $o(n^{1/5})$ colors. On the hardness side, Dinur, Mossel and Regev [12] showed the hardness of coloring 3-colorable graphs with any constant number of colors, assuming a variant of the Unique Games Conjecture. It is also known [11, 15] to be NP-hard to find an independent set of size $n/9$ (which is a weaker goal than 9-coloring) in a graph G on n vertices, when G is promised to have a $(1-\varepsilon)$-*partial* 3-coloring i.e., a coloring which properly colors the induced subgraph on at least $(1-\varepsilon)\cdot n$ vertices.

There has also been a significant amount of research trying to recover a planted 3-coloring in special families of graphs [8, 1, 10]. Notably, an algorithm by Alon and Kahale [1], shows how to 3-color a random 3-colorable graph G (with high probability over the choice of G). The graphs in their model, denoted $\mathcal{G}_{3,p,n}$, are generated by dividing the vertices in three color classes of size $n/3$ each, and connecting each pair of vertices in distinct color classes independently with probability p. As their main result, [1] are able to recover the color classes in polynomial time with a small probability of failure even when p is as small as $3d/n$ for a constant d. Notice that in expectation a vertex in this graph will have degree $2d$ and will have d neighbors in each color class different from its own.

A generalization of the above result is to consider models with limited amount of randomness (semi-random) or arbitrary graphs with random-like properties (pseudorandom graphs). While both models seek to capture the minimal assumptions needed for the methods developed for random graphs, the study of pseudorandomness properties is also motivated by developing decompositions of worst-case objects into structured and pseudorandom parts [16]. The works by Blum and Spencer [8] already considered semi-random models for $p = n^{-(1-\delta)}$. Motivated by the notion of pseudorandomness and decompositions and pseudorandomness used in the sub-exponential algorithms for Unique Games [2, 3], Arora and Ge [5] showed how to find a large independent set in 3-colorable graphs with small threshold rank. A recent work of David and Feige [10] shows that even when the graph is pseudorandom (an expander) and the 3-coloring is arbitrary, one can recover the coloring on most vertices of the graph. They also show that when the coloring is random, it can be recovered for *all* vertices in the graph.

Our Results

In this work, we focus on showing that the first 2 steps in [1]'s proof can be adapted to (almost) 3-color a special family of pseudorandom graphs with a pseudorandom coloring (see Subsection 1.1 for a comparison with [10]). To state our results, we first define the relevant notions of pseudorandomness. Note that in the definitions below, we take the average degree of the graph to be $2d$, since we think of the degree in each color class as being d.

▶ **Definition 1** (Pseudorandom colorings). Let $G = (V, E)$ be any graph with $|V| = n$ and $|E| = d \cdot n$. Let $\chi\colon V \to \{1, 2, 3\}$ be a (proper or improper) coloring of vertices with 3 colors. χ is considered $(2d, \varepsilon)$-pseudorandom if

1. χ is balanced i.e., all color classes $(1, 2, 3$ above) have the same size, $\frac{n}{3}$.
2. The coloring of G in the above step is a *low variance* coloring i.e., for all $i, j \in \{1, 2, 3\}$, we have $\mathrm{Var}_{v\in \mathrm{COL}_i}\, d_{ij}(v) \le \varepsilon \cdot d^2$.

The first family of pseudorandom graphs is low threshold-rank graphs, which (in the context of coloring) are defined as having a small number of negative eigenvalues with a large magnitude. This notion of pseudorandomness (with a different notion of the threshold) was also considered in the work of Arora and Ge [5].

▶ **Definition 2** (Threshold Rank). A graph $G = (V, E)$ with $|E| = d \cdot n$. The threshold rank of G is defined to be the number of eigenvalues of the adjacency matrix that are smaller than $-9d/10$.

We will also require the following notion of a partial coloring.

▶ **Definition 3** (Partial Coloring). For a graph $G = (V, E)$, a function $\chi : V \to [3] \cup \{\perp\}$ is said to be a $(1-\gamma)$-partial 3-coloring if

- $|\chi^{-1}(\perp)| \leq \gamma \cdot n$.
- χ is a proper 3-coloring for the induced subgraph on $\chi^{-1}([3])$.

Note that there exists a $(1-\gamma)$-partial 3-coloring χ of G if and only if there exits a *total* (but not necesarily proper) coloring $\chi' : V \to [3]$ such that χ' can be made a $(1-\gamma)$-partial 3-coloring by replacing the colors of γ fraction of vertices by $\perp$. Since our pseudorandomness properties are defined only for total colorings, we will abuse notation to say that there is a pseudorandom $(1-\gamma)$-partial 3-coloring if there exists a pseudorandom χ' which agrees with a $(1-\gamma)$-partial 3-coloring χ on $\chi^{-1}([3])$. We will refer to such a coloring as $(2d, \varepsilon)$-pseudorandom $(1-\gamma)$-partial 3-coloring. We now state the first theorem we prove.

▶ **Theorem 4.** *There exists an algorithm which, given $G = (V, E)$ such that*

- *$|E| = d \cdot n$ and $\mathsf{th}(G) = r$, and*
- *there exists χ, which is $(2d, \varepsilon)$-pseudorandom and forms a $(1-\gamma)$-partial 3-coloring of G,*

runs in time $\left(\frac{\sqrt{r \cdot n}}{\varepsilon}\right)^r \cdot poly(n)$ and w.h.p. returns a $(1 - O(\varepsilon + \gamma))$-partial 3-coloring of G.

We also consider the futher specialized pseudorandom family, where the graphs will also be required to be expanding. Note that the graphs below capture the random family considered by Alon and Kahale [1]. For graphs in this family, we will recover all the color classes exactly.

▶ **Definition 5** (Expanding 3-colorable graphs). A 3-colorable graph is said to be expanding if for some small positive constant $\delta < 1$, $|\lambda_i| \leq \delta d$ holds $\forall\ 2 \leq i \leq n-2$ i.e., if all eigenvalues other than the leading eigenvalue and the last two are small in magnitude.

For graphs coming from this family, we have the following theorem.

▶ **Theorem 6.** *There exists a polynomial-time algorithm which, given $G = (V, E)$ such that*

- *G is an expanding 3-colorable graph with $|E| = d \cdot n$, and*
- *there exists $\chi : V \to [3]$, which is $(2d, \varepsilon)$-pseudorandom and a proper 3-coloring of G,*

recovers χ.

1.1 Related work

There is a huge amount of literature devoted to find a complete or partial (legal) 3-coloring of an input graph under some assumptions on the graph and/or some assumptions on some 3-coloring of the given graph. Here we briefly examine works related to the current work.

1.1.1 The algorithm of Alon and Kahale

[1] described how to 3-color a random graph $G \sim G_{3,d/n,n}$. Thus, they have a random graph with a planted 3-coloring which they 3-color using a three phase algorithm. Their main result shows that graphs coming from $G_{3,d/n,n}$ have nice spectral properties which, in the first phase, can be exploited by a spectral clustering approach to find a good candidate coloring, χ_1, which misclassfies very few vertices. This candidate coloring is later refined to obtain a coloring χ_2 in the next phase. The second phase is a local search which locally improves the coloring on the set $H \subseteq V$ of vertices which have close to d neighbors in each color class according to the current coloring. This is then followed up by a cleanup phase which

recolors vertices in $V \setminus H$ (all the components on the subgraph induced on these vertices have logarithmic sizes and can be brute forced upon).

We show how to construct χ_1 in low threshold-rank graphs which admit a pseudorandom partial coloring. When the graphs are also expanding admit a full coloring, we also show how to complete the second phase.

1.1.2 The results of David and Feige

[10] considered extensions of the problem of coloring random 3-colorable graphs in several directions. In particular, they tried to relax the randomness assumption on the graph and the planted coloring inherent in [1]. To this end, they considered 4 models. They take as input an approximately d-regular (spectrally) expanding graph which can be either *adversarial* or *random* and a balanced planted coloring which again can be *adversarial or random.*

Our results are somewhat incomparable with theirs. Perhaps the most directly related setting from their work is the one where the graph is an arbitrary expander with an arbitrary (balanced) 3-coloring. In this case, they recover a $(1-\gamma)$ partial coloring. In this work, we start with additional pseudorandomness assumptions on the coloring (beyond balance), and can recover a partial coloring without assuming expansion in the input graph G, but making a different assumption about the negative eigenvalues. When G is also expanding in addition to these properties, we can recover the coloring completely.

Notation

Our notations are standard. We will write vectors and matrices in boldface (like $\mathbf{u}$ and $\mathbf{A}$ respectievely). For a graph G we will denote its adjacency matrix by $\mathbf{A} = \mathbf{A}(G)$. We denote a unit vector along the direction of $\mathbf{u}$ with $\hat{\mathbf{u}}$. And the transpose of a vector $\mathbf{u}$ is denoted $\mathbf{u}^T$. Also, $\mathbf{1}$ will denote the vector which is 1 in every coordinate. The stationaty distribution of random walks will be denoted $\boldsymbol{\mu}$. We denote the degree of a vertex $i \in V$ by $deg(i)$. The set of edges with one end point in a set S and the other in T is denoted $E(S,T)$.

2 Partially 3-coloring partially 3-colorable graphs

As a first step, we use lemma 7, to get a full coloring which only miscolors an $O(\varepsilon+\gamma)$ fraction of the vertices. In the next step, we will uncolor the incorrectly colored vertices. The first step is summarized by the following lemma.

▶ **Lemma 7.** *There exists an algorithm which given a graph $G = (V,E)$ such that*

- $|E| = d \cdot n$ *and* $\mathsf{th}(G) = r$
- *there exists a coloring χ which is a $(2d,\varepsilon)$-pseudorandom coloring, and forms a $(1-\gamma)$-partial-3-coloring of G*

runs in time $O\left(\frac{\sqrt{r \cdot n}}{\varepsilon}\right)^r$ and returns a coloring which has $(1 - O(\varepsilon+\gamma))$ fraction of the vertices colored correctly, i.e., the graph induced on them has no monochromatic edges.

Following [1], we consider two special vectors which we call $\mathbf{x}$ and $\mathbf{y}$ as defined below.

$$\mathbf{x}(v) = \begin{cases} 2 & \text{if } v \in \mathrm{COL}_1, \\ -1 & \text{if } v \in \mathrm{COL}_2, \\ -1 & \text{if } v \in \mathrm{COL}_3, \end{cases} \qquad \mathbf{y}(v) = \begin{cases} 0 & \text{if } v \in \mathrm{COL}_1, \\ 1 & \text{if } v \in \mathrm{COL}_2, \\ -1 & \text{if } v \in \mathrm{COL}_3, \end{cases} \tag{1}$$

The point of these vectors is that they are both constant on all color classes and so are their linear combinations. Simlar to [1], we will try to find a vector which is close enough to some linear combination of $\mathbf{x}$ and $\mathbf{y}$ and use it to obtain a coloring of the kind Lemma 7 seeks. Now let us detail our algorithm.

Algorithm 1 Find Coloring

Require: Graph G with $\mathsf{th}(G) = r$ and the eigenvectors $\{\mathbf{v}_{n-r+1}, \mathbf{v}_{n-r+2}, \cdots \mathbf{v}_n\}$

Let $\mathbf{t} \leftarrow$ Subspace enumeration($\sqrt{\frac{\varepsilon}{r}}$)

Let $\mathrm{COL}_1 = \{\, i \in V : \mathbf{t}_i > 1/2 \,\}$, $\mathrm{COL}_2 = \{\, i \in V : \mathbf{t}_i < -1/2 \,\}$ and $\mathrm{COL}_3 = V \setminus (\mathrm{COL}_1 \cup \mathrm{COL}_2)$

This algorithm relies on a procedure called Subspace Enumeration which is described below.

Algorithm 2 Subspace enumeration(τ)

Require: Graph G with $\mathsf{th}(G) = r$ and the eigenvectors $\{\mathbf{v}_{n-r+1}, \mathbf{v}_{n-r+2}, \cdots \mathbf{v}_n\}$

Let $B_r \leftarrow [-100\sqrt{n}, 100\sqrt{n}]^r$

▷ B_r denotes a bounding box in the space of r eigenvectors above

Partition B_r into grid cells. Each cell has length τ in all dimensions.

▷ The number of cells produced is $O\left(\frac{200\sqrt{n}}{\tau}\right)^r$

Let P_r denote the set of all corners of any grid cell.

▷ Thus, $P_r = \{\, \mathbf{p} \in B_r : \tau$ divides all r coordinates in $\mathbf{p} \,\}$

Find in P_r a point $\mathbf{t}$ which has

- $med(\mathbf{t}) = 0$ and $\|\mathbf{t}'\|_2 = \Theta(\sqrt{n})$
- distance at most $\leq O\left(\sqrt{\varepsilon n + \gamma n + \tau^2 r}\right)$ from $span(\mathbf{x}, \mathbf{y})$.

▷ We later show, in Claim 10 that such a vector exists.

return t

To put this plan in motion, we make the following claims which are generalization of the corresponding claims in [1].

▶ **Claim 8.** $\|\mathbf{A}\mathbf{x} + d(1-\gamma)\mathbf{x}\|_2^2, \|\mathbf{A}\mathbf{y} + d(1-\gamma)\mathbf{y}\|_2^2 \leq O(\varepsilon n d^2)$.

▶ **Claim 9.** *There exist small shift vectors $\mathbf{s}_n$ and $\mathbf{s}_{n-1}$ with $\|\mathbf{s}_n\|_2^2, \|\mathbf{s}_{n-1}\|_2^2 \leq O(\varepsilon n)$ such that both $\mathbf{x} - \mathbf{s}_n$ and $\mathbf{y} - \mathbf{s}_{n-1}$ are the linear combinations of last r eigenvectors of $\mathbf{A}$.*

▶ **Claim 10.** *Algorithm 2 finds a vector $\mathbf{t}$ in the span of $\{\mathbf{v}_{n-r+1}, \mathbf{v}_{n-r+2}, \ldots \mathbf{v}_n\}$ in time $O\left(\frac{\sqrt{n}}{\tau}\right)^r$ such that*
- *$med(\mathbf{t}) = 0$ and $\|\mathbf{t}'\|_2 = \Theta(\sqrt{n})$*
- *$\|\mathbf{t} - \mathbf{f}\|_2 \leq O(\sqrt{\varepsilon n + \gamma n})$ where $\mathbf{f}$ is some vector that lies in $span(\mathbf{x}, \mathbf{y})$.*

Now, using these claims we will sketch how to establish Lemma 7. Taking cue from [1], we find a vector in the span of the last r eigenvectors, $span(\mathbf{v}_{n-r+1}, \mathbf{v}_{n-r+2}, \ldots, \mathbf{v}_n)$, a vector $\mathbf{t}$ which is close to a vector $\mathbf{f} \in span(\mathbf{x}, \mathbf{y})$ has length $\Theta(\sqrt{n})$ and median zero. In particular, the intuition is to have $\mathbf{t}$ (which the algorithm finds) be close to a vector $\mathbf{f}$ which has large positive entries indicating the first color class, large negative entries for the second color class and 0 entries for the last color class. In $\mathbf{t}$ hopefully the large positive entries and negative entries of $\mathbf{f}$ will remain away from zero and maintain their sign and the zero entries in $\mathbf{f}$ will hopefully remain close to 0. The remaining details of the proof are given in Appendix A. The proof follows [1].

With the proof of Lemma 7, let us see how to prove Theorem 4.

Proof. (Of Theorem 4) Let $E_{bad} \subseteq E$ be the set of edges which have both endpoints with the same color in the coloring derived using the first step described in the proof of Lemma 7. Let $U_{bad} \subseteq V$ be the set of endpoints of these edges and consider the graph $G[U_{bad}]$ induced on these vertices. We note that the set U of vertices misclassified in Lemma 7 also forms a vertex cover in $G[U_{bad}]$. By the standard 2-approximation algorithm for vertex cover we can find a vertex cover $C \subseteq U_{bad}$ where $|C| \leq 2|U|$. And on "uncoloring" the vertices in the set C we obtain a partial coloring which omits only a $O(|U|)$ of the vertices – which is a $(1 - O(\varepsilon + \gamma))$-partial 3-coloring. ◀

Now, let us prove Claim 8 and Claim 9.

Proof. (Of Claim 8) Let us prove this for the vector $\mathbf{x}$. The proof with vector $\mathbf{y}$ is similar. Let $\mathbf{u} = \mathbf{A}\mathbf{x} + d\mathbf{x}$. Let $\mathbf{a}_i^T$ denote the i^{th} row in $\mathbf{A}$. Thus, $\mathbf{u}_i = \mathbf{a}_i^T\mathbf{x} + d\mathbf{x}_i$. Let us say that in the vector $\mathbf{x}$, $\mathbf{x}_i = 2$ for $i \in \mathrm{COL}_1$, and it is -1 otherwise. Note that for $i \in \mathrm{COL}_1$, we get $\mathbf{u}_i^2 = (\mathbf{a}_i^T\mathbf{x} + d\mathbf{x})^2$

$$= (-d_{12}(i) - d_{13}(i) + 2d_{11}(i) + 2d(1-\gamma))^2 \leq O(M_1(i))$$

where $M_1(i) \stackrel{\text{def}}{=} (d_{12} - d)^2 + (d_{13} - d)^2 + (d_{11} - \gamma d)^2$. In a similar fashion, we see that for $i \in \mathrm{COL}_2$ $\mathbf{u}_i^2 \leq O(M_2(i))$ where $M_2(i) = (d_{21}(i) - d)^2 + (d_{23}(i) - d)^2 + (d_{22} - \gamma d)^2$. And an analogous upper bound holds for $\mathbf{u}_i^2$ when $i \in \mathrm{COL}_3$. Thus,

$$\|\mathbf{u}\|_2^2 = \sum \mathbf{u}_i^2 \leq \sum_{i \in \mathrm{COL}_1} M_1(i) + \sum_{i \in \mathrm{COL}_2} M_2(i) + \sum_{i \in \mathrm{COL}_3} M_3(i) \leq O\big(\varepsilon n d^2\big)$$

as the $\mathrm{Var}_{v \in \mathrm{COL}_i}\, d_{ii}(v) \leq \varepsilon d^2$ as well. The proof for upperbound on $\|\mathbf{A}\mathbf{y} + d\mathbf{y}\|_2^2$ is similar. ◀

And next, we prove Claim 9 as well. Here is a quick adaptation of [1]'s proof.

Proof. (Of Claim 9) Write $\mathbf{x} = \sum c_i \mathbf{v}_i$ in the eigenbasis. Again let $\mathbf{u} = \mathbf{A}\mathbf{x} + d\mathbf{x}$.

$$\|\mathbf{u}\|_2^2 = \sum_{i=1}^{n} c_i^2(\lambda_i + d(1-\gamma))^2 \geq \sum_{i=1}^{n-r} c_i^2(\lambda_i + d(1-\gamma))^2 \geq \Omega(d^2)\sum_{i=1}^{n-r} c_i^2$$

The last step above follows because the first $n - r$ eigenvalues are all at least $-9d/10$. And together with the fact that $\|\mathbf{u}\|_2^2 \leq O\big(\varepsilon n d^2\big)$, we get $\sum_{i=1}^{i=n-r} c_i^2 \leq O(\varepsilon n)$. And this is the shift vector $\mathbf{s}_n$ we seek with the desired bound on its length. A similar argument can be made to find $\mathbf{s}_{n-1}$ and its length using the vector $\mathbf{y}$. ◀

Proof of Claim 10. Let I denote the set of indices that are left uncolored in the hidden planted pseudorandom coloring. Let us define a vector $\mathbf{z}$ as $\mathbf{z}(i) = -1$ for $i \in \mathrm{COL}_1$, $\mathbf{z}(i) = 0$ for $i \in I \cup \mathrm{COL}_2$ and $\mathbf{z}(i) = +1$ for $i \in \mathrm{COL}_3$. Note that this there exists a vector (say $\mathbf{f}$) in $span(\mathbf{x}, \mathbf{y})$ such that $\mathbf{z}$ and $\mathbf{f}$ are pretty close – the distance is at most $\sqrt{\gamma n}$. Further, there also exists a vector $\mathbf{t} \in span(\mathbf{v}_{n-r+1}, \mathbf{v}_{n-r+2}, \cdots, \mathbf{v}_n)$ which is pretty close to $\mathbf{z}$. This distance is easily bounded by arguments similar to Claim 8 and Claim 9. In particular, it follows that the distance of $\mathbf{z}$ and $\mathbf{t}$ considered above is at most $O(\sqrt{\varepsilon n + \gamma n})$. And therefore, the distance $\|\mathbf{t} - \mathbf{f}\|_2 \leq O(\sqrt{\varepsilon n + \gamma n})$. And finally, we also note that $med(\mathbf{t}) = 0$ and $\|\mathbf{t}\|_2 = \Theta(\sqrt{n})$.

This only proves that such a vector exists. We need to algorithmically *find* one. To this end, we use the *subspace enumeration* procedure described earlier. Given any vector $\mathbf{p} \in B_r$ (which, recall, was defined in Algorithm 2), this procedure finds a vector $\mathbf{t}$ which is within a distance $\tau\sqrt{r}$ of $\mathbf{p}$. In particular, this also holds for $\mathbf{t}'$. Moreover, $\mathbf{t}'$ being close to $\mathbf{z}$ has several 0 entries one of which is the median. And the vector $\mathbf{t}$ being $\tau\sqrt{r}$ close to $\mathbf{t}'$ also has $med(\mathbf{t}) = 0$, length $\Theta(\sqrt{n})$ and has $\|\mathbf{t} - \mathbf{f}\|_2 \leq O\left(\sqrt{\varepsilon n + \gamma n + \tau^2 r}\right)$ as can be easily seen by triangle inequality. ◀

3 Coloring expanding-3-colorable

In the case of $(2d, \varepsilon)$-expanding-3-colorable, $\gamma = 0$. So, the coloring obtained by the first step of the algorithm returns a $(1 - O(\varepsilon))$ partial coloring. We briefly review what [1] do in the second step for 3-coloring a random 3-colorable graph $G \in G_{3,d/n,3n}$. Their algorithm receives as input a 3-colored graph G and a set U of *bad* vertices (with $|U| = O(n/d)$) that have been incorrectly colored. This bad 3 coloring of G is improved via an iterative process. Each step in the iteration reduces the number of bad vertices by a constant factor. The algorithm is given below, and we use the same algorithm. However, our analysis is different.

Algorithm 3 Improve Coloring

Require: A $(1 - O(\varepsilon))$partial 3-coloring of G vertices.

```
Let the current color classes be denoted V_1^0, V_2^0 and V_3^0
Add the uncolored vertices arbitrarily to the partitions in any order.
for i = 1: log n do
    For j ∈ {1,2,3}, put v ∈ V_j^i only if |N(v) ∩ V_j^{i-1}| ≤ |N(v) ∩ V_l^{i-1}| for all l ≠ j
    ▷ i.e.,, put v in the least popular color class among its neighbors from previous iteration
    ▷ V_1^i, V_2^i, V_3^i denotes the color classes in the i^th iteration
end for
```

The above algorithm derives from the following intuition. Given a 3-colored graph $G \in G_{3,d/n,3n}$ with a small set U of bad vertices, a "local search procedure" can recolor those vertices in V which do not have way too many neighbors in U. Thus, another way to express the intuition is to say that a bad vertex at the beginning of iteration i remains bad after iteration i finishes only if it is surrounded by many bad vertices. To make this formal, let us consider the set

$$W = \{\, v \in V \,:\, deg_U(v) \geq d/4 \,\}\,.$$

W will be referred to as being *U-rich*. The main step in the argument is to show that the size of U-rich set is at most $\frac{|U|}{2}$ and that at the end of every iteration the set which remains incorrectly colored is only a subset of the U-rich set. We will now use this argument in the case of expanding graphs. This is done in the following lemma.

▶ **Lemma 11.** *There exists a polynomial time algorithm which returns, given $(2d, \varepsilon)$-expanding-3-colorable graph $G = (V, E)$ (recall that this means for some absolute positive constant $\delta < 1$, $|\lambda_i| \leq \delta d$ for $2 \leq i \leq n - 2$) and a $(1 - O(\varepsilon))$ partial 3-coloring of V, a proper 3 coloring of V.*

We will prove Lemma 11 by using the expander mixing lemma. The key here would be to again show that $|U|$-rich sets have small size if $|U|$ is small. This is done in the following claim.

▶ **Claim 12.** *Let $U \subseteq V$ be a set with size at most $O(\varepsilon n)$. Let*

$$W = \{\, v \in V \,:\, deg_U(v) \geq 2d/5 \,\}\,.$$

Then $|W| \leq 0.99|U|$.

Proof of Claim 12. We begin by finding upper bounds and lower bounds for $|E(U,W)|$. We see that

$$|E(U,W)| \geq \frac{1}{2}\cdot\frac{2d|W|}{5} = \frac{d|W|}{5}.$$

The inequality here comes because there are $\frac{2d}{5}$ edges incident on every vertex in $|W|$ with at least one endpoint in U. The factor of $1/2$ compensates for the fact that both the endpoints of the edge may belong to U.

Now, we need to upper bound $|E(U,W)|$. To do this, we use expander mixing lemma. Let the indicator vector for U be denoted $\mathbb{1}_{\mathbf{U}}$ and the indicator vector for W be denoted $\mathbb{1}_{\mathbf{W}}$. Let us write $\mathbb{1}_{\mathbf{U}} = \sum \alpha_i \mathbf{v}_i$ and $\mathbb{1}_{\mathbf{W}} = \sum \beta_i \mathbf{v}_i$. We know

$$|E(U,W)| = \mathbb{1}_{\mathbf{U}}{}^T \mathbf{A} \mathbb{1}_{\mathbf{W}} = |\left(\sum \alpha_i \mathbf{v}_i^T\right)\mathbf{A}\left(\sum \beta_j \mathbf{v}_j\right)| = |\sum(\alpha_i\beta_i\lambda_i)|$$

And it is immediately seen that the following holds

$$|E(U,W)| \leq |\alpha_1\beta_1\lambda_1| + |\alpha_n\beta_n|\cdot|\lambda_n| + |\alpha_{n-1}\beta_{n-1}|\cdot|\lambda_{n-1}| + \sum_{i=2}^{n-2}|\alpha_i\beta_i\lambda_i| \tag{2}$$

Now, we upper bound the RHS of Equation 2. We show how to bound each of the summands seprately.

1. Let us begin by bounding the first summand. For intuition sake consider the $\varepsilon = 0$ case. Then in fact we have a $2d$ regular graph and $\mathbf{v}_1 = \mathbf{u}_n$ where $\mathbf{u}_n = \frac{1}{\sqrt{n}}\cdot\mathbf{1}$ denotes the normalized uniform distribution vector.

$$\begin{aligned}|\alpha_1\beta_1\lambda_1| \leq& |\mathbb{1}_{\mathbf{U}}{}^T\mathbf{v}_1|\cdot|\mathbb{1}_{\mathbf{W}}{}^T\mathbf{v}_1|\cdot 2d\\ &= \frac{|U|}{\sqrt{n}}\cdot\frac{|W|}{\sqrt{n}}\cdot 2d = \frac{|U|\cdot|W|\cdot 2d}{n}\end{aligned} \tag{3}$$

In case $\varepsilon > 0$, we will show that something close holds. In particular, we will show that the vector $\mathbf{v}_1$ has a large dot product with $\mathbf{u}_n$. Denote by $\boldsymbol{\mu}$ the stationary distribution vector for random walks on G (and thus, $\boldsymbol{\mu}_i = \frac{deg(i)}{\sum deg(i)}$). Now, note

$$\begin{aligned}\left(\frac{\mathbf{v}_1^T\mathbf{1}}{\sqrt{n}}\right)^2 &= \left(\frac{1}{\sqrt{n}}\cdot\left(\frac{\boldsymbol{\mu}}{\|\boldsymbol{\mu}\|_2}\right)\right)^2 = \frac{1}{n}\cdot\frac{(\sum\boldsymbol{\mu}_i)^2}{\|\boldsymbol{\mu}\|_2^2} = \frac{1}{n}\cdot\frac{1}{\|\boldsymbol{\mu}\|_2^2} = \frac{(\sum deg(i))^2}{n\sum deg(i)^2}\\ &= \frac{(\mathbb{E}[deg(i)])^2}{\mathbb{E}[deg(i)^2]} = \frac{1}{1+\dfrac{\mathrm{Var}_{i\sim V}\, deg(i)}{\mathbb{E}[deg(i)]^2}} \geq \frac{1}{1+4\varepsilon d^2/d^2} = \frac{1}{1+4\varepsilon} \geq 1-8\varepsilon\end{aligned}$$

Here the first inequality follows because of Claim 13 which is proved later in this section.

▶ **Claim 13.** $\mathrm{Var}_{i\sim V}\, deg(i) \leq 4\varepsilon d^2$.

This means that the vector $\mathbf{v}_1$ is very close to the vector $\mathbf{1}/\sqrt{n}$. Now, we know that vector $\mathbb{1}_{\mathbf{U}}{}^T\frac{\mathbf{1}}{\sqrt{n}}$ is small. And we know that $\frac{\mathbf{v}_1^T\mathbf{1}}{\sqrt{n}}$ is large. It follows from this that $\mathbb{1}_{\mathbf{U}}{}^T\mathbf{v}_1$ will also be fairly small. In particular, we get

$$|\alpha_1\beta_1\lambda_1| \leq 5\frac{|U|}{\sqrt{n}}(1+8\varepsilon)\cdot\frac{|W|}{\sqrt{n}}(1+8\varepsilon)\cdot 2d \leq \frac{|U|\cdot|W|\cdot 2d(1+20\varepsilon)}{n} \tag{4}$$

Now, we bound the second summand.

As we will see in Claim 15, we have $\sqrt{n}\mathbf{v}_n$ can be expressed as a linear combination of vectors $\mathbf{x}-\mathbf{s}_n$ and $\mathbf{y}-\mathbf{s}_{n-1}$ with coefficients $O(1)$ in absolute value. So, let us write $\sqrt{n}\mathbf{v}_n = a_1(\mathbf{x}-\mathbf{s}_n) + b_1(\mathbf{y}-\mathbf{s}_{n-1})$ with a_1 and b_1 being constants. A similar expression can be obtained for $\sqrt{n}\mathbf{v}_{n-1}$. Now, we claim

▶ **Claim 14.** $|\alpha_n\beta_n\lambda_n| \leq O\left(d \cdot \left(\frac{|U||W|}{n} + \frac{|U|\sqrt{|W|}\sqrt{\varepsilon}}{\sqrt{n}} + \frac{|W|\sqrt{|U|}\sqrt{\varepsilon}}{\sqrt{n}} + \varepsilon\sqrt{|U||W|}\right)\right)$

Proof of Claim 14.

$$\begin{aligned} |\alpha_n\beta_n\lambda_n| &\leq \frac{1}{\sqrt{n}}|\mathbb{1}_{\mathbf{U}}{}^T\sqrt{n}\mathbf{v}_n| \cdot \frac{1}{\sqrt{n}}|\mathbb{1}_{\mathbf{W}}{}^T\sqrt{n}\mathbf{v}_n| \cdot 2d \\ &\leq \frac{|\mathbb{1}_{\mathbf{U}}{}^T a_1(\mathbf{x}-\mathbf{s}_n) + \mathbb{1}_{\mathbf{U}}{}^T b_1(\mathbf{y}-\mathbf{s}_{n-1})|}{\sqrt{n}} \\ &\quad \times \frac{|\mathbb{1}_{\mathbf{W}}{}^T a_2(\mathbf{x}-\mathbf{s}_n) + \mathbb{1}_{\mathbf{W}}{}^T b_2(\mathbf{y}-\mathbf{s}_{n-1})|}{\sqrt{n}} \cdot 2d \end{aligned} \tag{5}$$

We now need to understand how to bound terms like $\frac{1}{\sqrt{n}}|\mathbb{1}_{\mathbf{U}}{}^T a_1(\mathbf{x}-\mathbf{s}_n)|$. To this end, note that we have the following.

a. $|\mathbb{1}_{\mathbf{U}}{}^T a_1(\mathbf{x}-\mathbf{s}_n)| \leq |\mathbb{1}_{\mathbf{U}}{}^T a_1\mathbf{x}| + |a_1|\|\mathbf{s}_n\|_2\|\mathbb{1}_{\mathbf{U}}\|_2 \leq |\mathbb{1}_{\mathbf{U}}{}^T a_1\mathbf{x}| + |a_1|\sqrt{\varepsilon}\sqrt{n}\sqrt{|U|}$.

b. $|\mathbb{1}_{\mathbf{U}}{}^T\mathbf{x}| = |2U_1 - U_2 - U_3| \leq 2|U|$

Here U_1 refers to number of vertices colored with COL_1 in U. Other terms have analogous meaning.

Putting all of this together and absorbing all constants in the $O(.)$, we get

$$|\alpha_n\beta_n\lambda_n| \leq O\left(d \cdot \left(\frac{|U|}{\sqrt{n}} + \sqrt{\varepsilon}\sqrt{|U|}\right) \cdot \left(\frac{|W|}{\sqrt{n}} + \sqrt{\varepsilon}\sqrt{|W|}\right)\right) \tag{6}$$

◀

2. The $3rd$ item has an analogous bound to the one above. That is, we have

$$|\alpha_{n-1}\beta_{n-1}\lambda_{n-1}| \leq O(|\alpha_n\beta_n\lambda_n|) \tag{7}$$

3. The last item can be simply bounded by using Cauchy Schwartz and the fact that all other eigenvalues are at most δd for some sufficiently small δ. This gives

$$\sum_{i=1}^{n-2} |\alpha_i\beta_i\lambda_i| \leq \delta d \cdot \sqrt{|U| \cdot |W|} \tag{8}$$

Putting all of these bounds Equation 4, Equation 6, Equation 7, Equation 8 together we get and upper bound on $|E(U,W)|$. Also, we already computed a lower bound on $|E(U,W)| \geq d|W|/5$. Chaining the upper bound and the lower bound thus obtained, using the fact that $|U| \leq O(\varepsilon n)$ we get

$$\begin{aligned} \frac{\sqrt{|W|}}{5} &\leq \frac{2|U|\sqrt{|W|}(1+20\varepsilon)}{n} + O\left(\frac{|U|\sqrt{|W|}}{n} + \frac{\sqrt{\varepsilon|U||W|}}{\sqrt{n}} + \frac{\sqrt{\varepsilon}|U|}{\sqrt{n}} + \varepsilon\sqrt{|U|}\right) \\ &\quad + \delta\sqrt{|U|} \\ \implies \frac{\sqrt{|W|}}{10} &\leq O\left(\varepsilon\sqrt{|U|} + \delta\sqrt{|U|}\right) \\ \implies |W| &\leq O(\delta^2 + \varepsilon^2)|U|. \end{aligned}$$

This finishes the proof of Claim 12. ◀

Proof of Lemma 11. Observe that by Claim 12, it follows that the set of badly colored vertices shrinks significantly in each step of the local search. Thus, after $O(\log |U|)$ many steps, we do not have any bad vertices and we get a proper 3-coloring. This finishes the description of the algorithm for adapting the second step of [1] algorithm which finishes the proof of Lemma 11 ◀

Proof of Claim 13. Note that here we are taking variance in the degree of a random vertex over the full graph whereas by definition in $(2d, \varepsilon)$-expanding-3-colorable graph case we only know $\mathrm{Var}_{v \in \mathrm{COL}_j} d_{ij}(v) \le \varepsilon d^2$. So, a little more work is needed. Let us begin by noting that $\mathbb{E}\left[deg(i)^2\right] = \sum_{j \in \{1,2,3\}} \mathbb{E}\left[deg(i)^2 | i \in \mathrm{COL}_j\right]/3$. Also, note $\mathbb{E}[degi | i \in \mathrm{COL}_1] = \mathbb{E}[deg(i) | i \in \mathrm{COL}_2] = \mathbb{E}[deg(i) | i \in \mathrm{COL}_3] = d$. Let $M_1 = \mathbb{E}_{i \in \mathrm{COL}_1}\left[(d_{12}(i) + d_{23})^2\right]$ and $N_1 = \mathbb{E}_{i \in \mathrm{COL}_1}\left[(d_{12}(i)^2 + d_{13}(i)^2)\right]$ and similarly define M_2, N_2 and M_3, N_3. So, writing $\mathrm{Var}_{i \sim V} deg(i) = \mathbb{E}\left[deg(i)^2\right] - \mathbb{E}[deg(i)]^2$ and expanding by using the foregoing expressions, we get

$$\begin{aligned}
\mathrm{Var}_{i \sim V} &= \sum_{j \in \{1,2,3\}} \frac{\mathbb{E}\left[deg(i)^2 | i \in \mathrm{COL}_j\right] - 4d^2}{3} \\
&= \sum_{j \in \{1,2,3\}} \frac{M_j - 4d^2}{3} \\
&\le \sum_{j \in \{1,2,3\}} \frac{2N_j - 4d^2}{3} \\
&= \frac{2(\mathrm{Var}\, d_{12} + \mathrm{Var}\, d_{13} + \ldots \mathrm{Var}\, d_{32} + \mathrm{Var}\, d_{31})}{3} \\
&\le 4\varepsilon d^2
\end{aligned}$$

◀

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A Missing Proofs

Proof of Lemma 7. Obtain vectors $\mathbf{t}$ and $\mathbf{f}$ from Claim 10. These vectors have several useful properties which will be useful for us in what follows. Write $\mathbf{t} = \mathbf{f} + \mathbf{w}$. We have $\|\mathbf{w}\|_2 \leq O\left(\sqrt{\varepsilon n + \gamma n + \tau^2 r}\right)$.

Now, let $\alpha_i = \mathbf{f}|_{\mathrm{COL}_i}$ for $1 \leq i \leq 3$ – that is, α_i's are scalars and equal the constant value that $\mathbf{f}$ takes over the i^{th} color class. Assume $\alpha_1 \geq \alpha_2 \geq \alpha_3$. Using the fact that $\|\mathbf{w}\|^2 \leq O(\varepsilon n + \gamma n + \tau^2 r)$, it follows that only $O\big(\varepsilon n + \gamma n + \tau^2 r\big)$ coordinates in $\mathbf{w}$ can take on values large in magnitude (at least 0.01). Note that this means $|\alpha_2| \leq 1/4$. This is because, we have $\mathbf{t} - \mathbf{f} = \mathbf{w}$ and only a few entries in $\mathbf{w}$ can be big in magnitude. In more detail, suppose if $\alpha_2 > 1/4$ were to hold then at least $2n - O\big(\varepsilon n + \gamma n + \tau^2 r\big)$ entries in $\mathbf{t}$ will be big too and thus will have the same sign. This contradicts the 0 median assumption in $\mathbf{t}$. An analogous argument prevents $\alpha_2 < -1/4$.
Finally, recall that $\alpha_1 + \alpha_2 + \alpha_3 = 0$ (since $\mathbf{f} \in \mathit{span}(\mathbf{x}, \mathbf{y})$) and

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = \frac{\|\mathbf{f}\|^2}{n/3} \geq \frac{\|\mathbf{t}\|^2 - \|\mathbf{w}\|^2}{n/3} \geq 6 - O\big(\varepsilon + \gamma + \tau^2 r/n\big).$$

Together with the fact that $|\alpha_2| \leq 1/4$, this implies that $\alpha_1 \geq 3/4$ and $\alpha_3 \leq -3/4$. This means that $\mathbf{t}$ which is obtained by corrupting $\mathbf{f}$ on at most $O(\varepsilon n)$ entries coming from the noisy vector is a pretty good coloring. And the number of misclassified vertices in the coloring according to $\mathbf{t}$ is at most $O(\varepsilon n + \gamma n + \tau^2 r)$ – the ℓ_2^2 length of the noisy vector $\mathbf{w}$.

And now note that setting $\tau = \sqrt{\frac{\varepsilon}{r}}$ prodcues $\mathbf{w}$ with $\|w\|_2^2 = O(\varepsilon n + \gamma n)$. And the size of the discrete subspace that we search over is at most $O\left(\frac{\sqrt{r \cdot n}}{\varepsilon}\right)^r$ as claimed. And this finishes the proof. ◀

▶ **Claim 15.** *The vectors* $\sqrt{n}\mathbf{v}_n$ *and* $\sqrt{n}\mathbf{v}_{n-1}$ *can be expressed as a linear combination of vectors* $\mathbf{x}-\mathbf{s}_n$ *and* $\mathbf{y}-\mathbf{s}_{n-1}$ *with coefficients at most a constant in absolute value.*

Proof of Claim 15. Let us first quickly see that $\mathbf{v}_n$ lies in $span(\mathbf{x}-\mathbf{s}_n, \mathbf{y}-\mathbf{s}_{n-1})$. By definition of $(2d,\varepsilon)$-expanding-3-colorable graphs, we have that all eigenvalues other than λ_1, λ_{n-1} and λ_n are small in absolute value. So, there exist tiny shift vectors $\mathbf{s}_n, \mathbf{s}_{n-1}$ (according to Claim 9) such that both $\mathbf{x}-\mathbf{s}_n, \mathbf{y}-\mathbf{s}_{n-1}$ lie in $span(\mathbf{v}_n, \mathbf{v}_{n-1})$. Thus, the space spanned by the last 2 eigenvectors is a tiny perturbation of $span(\mathbf{x}, \mathbf{y})$. So, any vector in $span(\mathbf{v}_n, \mathbf{v}_{n-1})$ also lies in $span(\mathbf{x}-\mathbf{s}_n, \mathbf{y}-\mathbf{s}_{n-1})$. And so, does $\sqrt{n}\mathbf{v}_n$.

Intuitively, the at most constant in absolute value part should follow because $\mathbf{x}-\mathbf{s}_n, \mathbf{y}-\mathbf{s}_{n-1}, \sqrt{n}\mathbf{v}_n, \sqrt{n}\mathbf{v}_{n-1}$ have length $\Theta(\sqrt{n})$ and $\mathbf{x}-\mathbf{s}_n, \mathbf{y}-\mathbf{s}_{n-1}$ are nearly orthogonal. This is because $\mathbf{x}$ and $\mathbf{y}$ are orthogonal and $\mathbf{s}_n$ and $\mathbf{s}_{n-1}$ are very tiny perturbations to these orthogonal vectors. In more detail,

$$\begin{aligned}
&\sqrt{n}\mathbf{v}_n = \alpha(\mathbf{x}-\mathbf{s}_n) + \beta(\mathbf{y}-\mathbf{s}_{n-1}) \\
\implies &\alpha\mathbf{x} + \beta\mathbf{y} = \sqrt{n}\mathbf{v}_n + \alpha\mathbf{s}_n + \beta\mathbf{s}_{n-1} \\
\implies &\|\alpha\mathbf{x} + \beta\mathbf{y}\|_2 \leq \sqrt{n} + |\alpha|\cdot\|\mathbf{s}_n\|_2 + |\beta|\cdot\|\mathbf{s}_{n-1}\|_2 \qquad (9)
\end{aligned}$$

On taking squared ℓ_2 norms on both sides of (9), it follows that

$$\frac{6n\alpha^2}{3} + \frac{2n\beta^2}{3} \leq n + O(\alpha^2+\beta^2)\varepsilon n.$$

On simplifying it is clear that this means both $|\alpha|$ and $|\beta|$ are $O(1)$. ◀