# The Fewest Clues Problem of Picross 3D

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## Abstract

Picross 3D is a popular single-player puzzle video game for the Nintendo DS. It is a 3D variant of Nonogram, which is a popular pencil-and-paper puzzle. While Nonogram provides a rectangular grid of squares that must be filled in to create a picture, Picross 3D presents a rectangular parallelepiped (i.e., rectangular box) made of unit cubes, some of which must be removed to construct an image in three dimensions. Each row or column has at most one integer on it, and the integer indicates how many cubes in the corresponding 1D slice remain when the image is complete. It is shown by Kusano et al. that Picross 3D is NP-complete. We in this paper show that the fewest clues problem of Picross 3D is $\Sigma_2^{\mathrm{P}}$-complete and that the counting version and the another solution problem of Picross 3D are #P-complete and NP-complete, respectively.

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## 1 Introduction

Many pencil-and-paper puzzles have been shown to be NP-complete [7]. For example, Akari (also known as Light-ups) [11], Number Place (also known as Sudoku) [13], Shakashaka [5] are all known to be NP-complete. Different from this line of research, Demaine et al. [4] recently introduced the fewest clues problem (FCP) framework for analyzing computational complexity of designing "good" puzzles. The FCP is, given an instance to a puzzle, to decide the minimum number of clues we must add in order to make the instance uniquely solvable. It is of great interest for puzzle makers to know hardness of such a version since it

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is usually the case that they want to ensure a puzzle instance to have a unique solution. In [4], along with the FCP versions of classical NP-complete problems such as 3-SAT, those of the three common Nikoli puzzles (Akari, Number Place, and Shakashaka) are shown to be $\Sigma_2^{\mathrm{P}}$-complete. Here, $\Sigma_2^{\mathrm{P}}$ is the complexity class that lies on the second level of the polynomial hierarchy and includes the class NP. Hence, $\Sigma_2^{\mathrm{P}}$-complete problems are at least as hard as NP-complete problems. See, e.g., [1] for more details.

We in this paper investigate computational complexity of the FCP of Picross 3D and show that it is $\Sigma_2^{\mathrm{P}}$-complete. Picross 3D is a video-game puzzle developed by HAL Laboratory, published by Nintendo, and was first released in 2009. While 2-dimensional Picross (also known as Nonogram) provides a rectangular grid of squares that must be filled in to create a picture, Picross 3D presents a rectangular parallelepiped (i.e., rectangular box) made of unit cubes, some of which must be removed to construct an image in three dimensions. Each row or column has at most one integer on it, and the integer indicates how many cubes in the corresponding 1D slice remain when the image is complete. If the integer is not circled nor boxed, then the remaining cubes in the 1D slice must form a section (i.e., the cubes must be consecutive). If the integer is circled, then the remaining cubes in the 1D slice must be split up into two sections. If the integer is boxed, then the cubes must be split up into three or more sections. If there are no numbers on a row or column, then there are no rules concerning the number of cubes (or sections) to remain. An instance of Picross 3D is shown in Figure 1(a), and its solution is given in 1(b).

As many other puzzles, Picross 3D is shown to be NP-complete via a reduction from 3-SAT [10]. To show the $\Sigma_2^{\mathrm{P}}$-completeness of the FCP of Picross 3D, we reduce to it the FCP of positive 1-in-3 SAT, which is known to be $\Sigma_2^{\mathrm{P}}$-complete [4][3]. We note that those Nikoli puzzles were chosen in [4] because their NP-hardness reductions mostly preserve clue structure and their FCP versions were shown $\Sigma_2^{\mathrm{P}}$-complete by using the same reductions or slightly modified ones. On the other hand, we cannot do the same for the FCP of Picross 3D using the NP-hardness reduction of [10]; we instead modify it to devise a parsimonious reduction from positive 1-in-3 SAT to Picross 3D. Here, a reduction is called *parsimonious* if, for each instance, there exists a one-to-one correspondence between the solution sets of the original instance and the reduced one. Intuitively, since a parsimonious reduction preserves the number of solutions, it helps to provide a reduction that preserves the number of clues. Moreover, it follows from the above parsimonious reduction that (i) the counting version of Picross 3D is #P-complete since so is the counting version of positive 1-in-3 SAT [2, 3][4], and (ii) the another solution problem (ASP) of Picross 3D is NP-complete[5] since so is ASP positive 1-in-3 SAT [12, 13]. Here, ASP Picross 3D is, given an instance of Picross 3D and a solution to it, to determine if there exists another solution to the instance.

We now discuss related work. Picross 3D can be seen as a variant of problems that have been studied in the field of *(3D) discrete tomography.* Discrete tomography deals with problems of determining shape of a discrete object from a set of projections. These problems have applications in, e.g., physical chemistry, medicine, and data coding, and have strong connections with combinatorics and geometry; see [8] for details. Especially, Picross 3D without the consecutiveness conditions on solutions is a basic problem in discrete tomography and intensively studied from an algorithmic point of view; the problem is known to be NP-,

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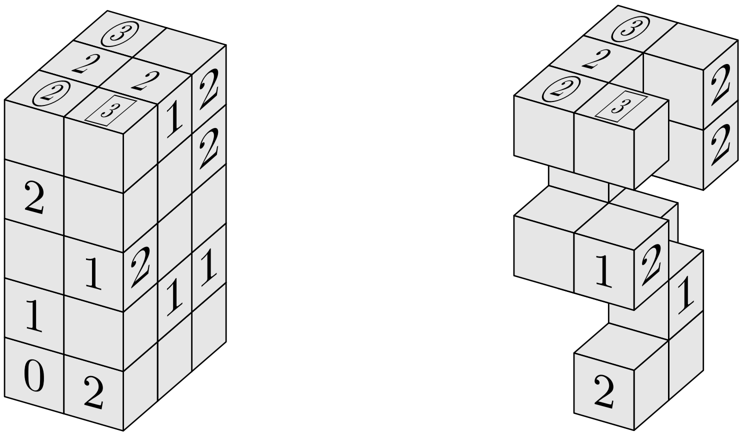
[3] More precisely, the FCP of 1-in-3 SAT is shown $\Sigma_2^{\mathrm{P}}$-complete in [4]. However, the proof in [4] also shows the $\Sigma_2^{\mathrm{P}}$-completeness of the FCP of *positive* 1-in-3 SAT.

[4] More precisely, the counting version of monotone 1-in-3 SAT (i.e., each clause has only positive literals or only negative literals) is shown #P-complete in [2]. However, it is not difficult to modify the proof in [2] to show the #P-completeness of the counting version of positive 1-in-3 SAT.

[5] This fact was pointed out by a reviewer.



**(a)** **(b)**

**Figure 1** An instance of Picross 3D and its solution

ASP-, and #P-complete [9, 6]. Moreover, the 2D version of the problem, which corresponds to Nonogram, can be solved in polynomial time [8]. We note that the reductions in [9, 6] cannot be used to show our results.

## 2 Preliminaries

In this section, we first introduce a formal definition of the puzzle Picross 3D and then introduce the fewest clues problems (FCPs) of Picross 3D and positive 1-in-3 SAT.

### 2.1 Picross 3D

In Picross 3D, we are given a rectangular parallelepiped of height $h$, width $w$, and depth $d$. Each unit square in the front, side and top faces have at most one nonnegative integer that indicates how many cubes the row or column should contain when the image is complete. These integers are conveniently represented by three matrices: an $h \times w$ matrix $F = (f_{i,j})$ called the *front constraint matrix*, an $h \times d$ matrix $S = (s_{i,k})$ called the *side constraint matrix*, and a $d \times w$ matrix $T = (t_{j,k})$ called the *top constraint matrix*. Each element of these matrices is either an integer, a circled integer (e.g., ①), a boxed integer (e.g., $\boxed{1}$), or $\varepsilon$. Here, $\varepsilon$ indicates that there is no constraint concerning the remaining cubes in the corresponding row or column. We denote by $I = (h, w, d, F, S, T)$ an instance of Picross 3D.

For the sake of clarity, we index these matrices as follows.

$$F = \begin{pmatrix} f_{1,1} & f_{1,2} & \cdots & f_{1,w} \\ f_{2,1} & f_{2,2} & \cdots & f_{2,w} \\ \vdots & \vdots & \ddots & \vdots \\ f_{h,1} & f_{h,2} & \cdots & f_{h,w} \end{pmatrix}, S = \begin{pmatrix} s_{1,1} & s_{1,2} & \cdots & s_{1,d} \\ s_{2,1} & s_{2,2} & \cdots & s_{2,d} \\ \vdots & \vdots & \ddots & \vdots \\ s_{h,1} & s_{h,2} & \cdots & s_{h,d} \end{pmatrix},$$

$$T = \begin{pmatrix} t_{1,d} & t_{2,d} & \cdots & t_{w,d} \\ t_{1,d-1} & t_{2,d-1} & \cdots & t_{w,d-1} \\ \vdots & \vdots & \ddots & \vdots \\ t_{1,1} & t_{2,1} & \cdots & t_{w,1} \end{pmatrix}.$$

Note that $j$ is the index of the columns and $k$ is the index (from bottom to top) of the rows of $T = (t_{j,k})$.

A solution to an instance $I = (h, w, d, F, S, T)$ of Picross 3D is a three dimensional matrix $P = (p_{i,j,k}) \in \{0,1\}^{h \times w \times d}$ that satisfies the following conditions: each integer (even if circled or boxed) indicates the number of 1's in the column or row where the integer is written. Namely, we must have $\sum_{k=1}^{d} p_{i,j,k} = f_{i,j}$ for each $i$ and $j$, $\sum_{j=1}^{w} p_{i,j,k} = s_{i,k}$ for each $i$ and $k$, and $\sum_{i=1}^{h} p_{i,j,k} = t_{j,k}$ for each $j$ and $k$. Moreover, (i) if the integer is not circled nor boxed, then all the 1's in the row or column must be consecutive, (ii) if the integer is circled, then the row or column must contain exactly two sections that consecutively consist of only 1's, and (iii) if the integer is boxed, then the row or column must contain more than two sections that consecutively consist of only 1's. We describe a solution to Picross 3D as a sequence of matrices as follows.

$$P = \left( \begin{pmatrix} p_{1,1,1} & p_{1,2,1} & \cdots & p_{1,w,1} \\ p_{2,1,1} & p_{2,2,1} & \cdots & p_{2,w,1} \\ \vdots & \vdots & \ddots & \vdots \\ p_{h,1,1} & p_{h,2,1} & \cdots & p_{h,w,1} \end{pmatrix}, \begin{pmatrix} p_{1,1,2} & p_{1,2,2} & \cdots & p_{1,w,2} \\ p_{2,1,2} & p_{2,2,2} & \cdots & p_{2,w,2} \\ \vdots & \vdots & \ddots & \vdots \\ p_{h,1,2} & p_{h,2,2} & \cdots & p_{h,w,2} \end{pmatrix}, \dots, \begin{pmatrix} p_{1,1,d} & p_{1,2,d} & \cdots & p_{1,w,d} \\ p_{2,1,d} & p_{2,2,d} & \cdots & p_{2,w,d} \\ \vdots & \vdots & \ddots & \vdots \\ p_{h,1,d} & p_{h,2,d} & \cdots & p_{h,w,d} \end{pmatrix} \right).$$

▶ **Example 1.** Let $h = 5, w = 2$, and $d = 3$, and define the constraint matrices as follows.

$$F = \begin{pmatrix} \varepsilon & \varepsilon \\ 2 & \varepsilon \\ \varepsilon & 1 \\ 1 & \varepsilon \\ 0 & 2 \end{pmatrix}, S = \begin{pmatrix} \varepsilon & 1 & 2 \\ \varepsilon & \varepsilon & 2 \\ 2 & \varepsilon & \varepsilon \\ \varepsilon & 1 & 1 \\ \varepsilon & \varepsilon & \varepsilon \end{pmatrix}, T = \begin{pmatrix} \textcircled{3} & \varepsilon \\ 2 & 2 \\ \textcircled{2} & \boxed{3} \end{pmatrix}.$$

Then, $I = (h, w, d, F, S, T)$ represents the instance given in Figure 1(a).

Let

$$P = \left( \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \right).$$

Then $P$ is the solution to $I$ that is depicted in Figure 1(b).

## 2.2 Fewest Clues Problem

We here define the FCP of Picross 3D and positive 1-in-3 SAT. A *positive CNF* is a CNF where every literal occurring in it is positive.

**FCP Picross 3D** Given an instance $I$ of Picross 3D and an integer $\ell$, does there exists a partial assignment of at most $\ell$ variables such that there exists a unique solution to $I$ extending the partial assignment?

**FCP positive 1-in-3 SAT** Given a positive 3-CNF $\varphi$ and an integer $\ell$, does there exists a partial assignment of at most $\ell$ variables such that there exists a unique solution to $\varphi$, where every clause has exactly one true literal, extending the partial assignment?

## 3 Parsimonious Reduction from positive 1-in-3 SAT to Picross 3D

In this section, we provide a parsimonious reduction from positive 1-in-3 SAT to Picross 3D. Recall that a reduction is parsimonious if, for each instance, there exists a one-to-one correspondence between the solution sets of the original instance and the reduced one. The reduction will be used to show the $\Sigma_2^P$-completeness of FCP Picross 3D in the next section. We note that from the reduction it follows that the counting version and the another solution problem (ASP) of Picross 3D are respectively #P-complete and NP-complete, since so are these variants of positive 1-in-3 SAT [2, 3, 12, 13]. We also note that our reduction is similar to the reduction from 3-SAT to Picross 3D in [10]; indeed, the variables of a given 3-CNF are represented in the same way. However, the reduction in [10] is not parsimonious and seems hard to be used for showing the $\Sigma_2^P$-completeness of FCP Picross 3D.

▶ **Proposition 2.** *There exists a parsimonious reduction from positive 1-in-3 SAT to Picross 3D.*

**Proof.** Let $\varphi$ be an instance of positive 1-in-3 SAT, where $\varphi = \bigwedge_{j=1}^{m} C_j$ is a positive 3-CNF with $n$ variables and $m$ clauses, and $C_j = (x_{j_1} \vee x_{j_2} \vee x_{j_3})$ for $j = 1, \dots, m$. Here, $1 \leq j_1, j_2, j_3 \leq n$ and $j_\ell$'s are distinct for $j = 1, \dots, m$. We construct an instance $I_\varphi = (h, w, d, F, S, T)$ of Picross 3D as follows.

We set $h = 4$, $w = 2(m + n - 1) + 1$, and $d = 3n$. Let a function div be defined as

$$\mathrm{div}(i) = \begin{cases} i & \text{if } i = 0, 1 \\ \textcircled{\scriptsize 2} & \text{if } i = 2 \\ \boxed{i} & \text{if } i \geq 3. \end{cases}$$

The front constraint matrix $F$, which is an $h \times w (= 4 \times (2m + 2n - 1))$ matrix, is defined as

$$\text{(i)} \begin{cases} f_{1,j} = f_{4,j} = f_{1,m+n+j} = f_{4,m+n+j} = \mathrm{div}(n), \\ f_{2,j} = 1, \\ f_{3,j} = \mathrm{div}(2), \end{cases}$$

for $1 \leq j \leq m$, (ii) $f_{1,j} = f_{4,j} = f_{1,m+n+j} = f_{4,m+n+j} = \mathrm{div}(m + n - j)$ for $m + 1 \leq j \leq m + n - 1$, (iii) $f_{2,j} = f_{3,j} = 0$ for $m + 1 \leq j \leq w$, and (iv) $f_{1,n+m} = f_{4,n+m} = 0$. Namely,

$$F = \left(\begin{array}{ccccccc}
\multicolumn{3}{c}{\overbrace{\hphantom{\mathrm{div}(n)\quad\dots\quad\mathrm{div}(n)}}^{m}} & \multicolumn{3}{c}{\overbrace{\hphantom{\mathrm{div}(n-1)\quad\dots\quad\mathrm{div}(1)}}^{n-1}} & \\
\mathrm{div}(n) & \dots & \mathrm{div}(n) & \mathrm{div}(n-1) & \dots & \mathrm{div}(1) & 0 \\
1 & \dots & 1 & 0 & \dots & 0 & 0 \\
\mathrm{div}(2) & \dots & \mathrm{div}(2) & 0 & \dots & 0 & 0 \\
\mathrm{div}(n) & \dots & \mathrm{div}(n) & \mathrm{div}(n-1) & \dots & \mathrm{div}(1) & 0
\end{array}\right.$$

$$\left.\begin{array}{cccccc}
\multicolumn{3}{c}{\overbrace{\hphantom{\mathrm{div}(n)\quad\dots\quad\mathrm{div}(n)}}^{m}} & \multicolumn{3}{c}{\overbrace{\hphantom{\mathrm{div}(n-1)\quad\dots\quad\mathrm{div}(1)}}^{n-1}} \\
\mathrm{div}(n) & \dots & \mathrm{div}(n) & \mathrm{div}(n-1) & \dots & \mathrm{div}(1) \\
0 & \dots & 0 & 0 & \dots & 0 \\
0 & \dots & 0 & 0 & \dots & 0 \\
\mathrm{div}(n) & \dots & \mathrm{div}(n) & \mathrm{div}(n-1) & \dots & \mathrm{div}(1)
\end{array}\right).$$

The side constraint matrix $S$, which is an $h \times d (= 4 \times 3n)$ matrix, is defined as

$$\begin{cases} s_{1,3k-2} = s_{4,3k-1} = m + n - k, \\ s_{2,3k-2} = s_{3,3k-1} = \varepsilon, \\ s_{1,3k} = s_{2,3k} = s_{3,3k} = s_{4,3k} = 0, \end{cases}$$

for $1 \leq k \leq n$. Namely,

$$S = \overbrace{\begin{pmatrix} m+n-1 & m+n-1 & 0 & m+n-2 & m+n-2 & 0 & \dots & m & m & 0 \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & 0 & \dots & \varepsilon & \varepsilon & 0 \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & 0 & \dots & \varepsilon & \varepsilon & 0 \\ m+n-1 & m+n-1 & 0 & m+n-2 & m+n-2 & 0 & \dots & m & m & 0 \end{pmatrix}}^{3n}.$$

The top constraint matrix $T$, which is $d \times w (= 3n \times (2m+2n-1))$ matrix, is defined as follows.

$$\text{(i) } t_{j,3k-2} = \begin{cases} 2 & \text{if } x_k \in \{x_{j_1}, x_{j_2}, x_{j_3}\}, \\ 1 & \text{otherwise,} \end{cases}$$

for $1 \leq j \leq m$ and $1 \leq k \leq n$, (ii) $t_{j,3k-2} = t_{j,3k-1} = 1$ for $m+1 \leq j \leq m+n-k$ and $1 \leq k \leq n-1$, (iii) $t_{j,3k-2} = t_{j,3k-1} = 1$ for $m+n+1 \leq j \leq 2m+2n-k$ and $1 \leq k \leq n-1$, and (iv) $t_{jk} = 0$ for the remaining entries. Hence,

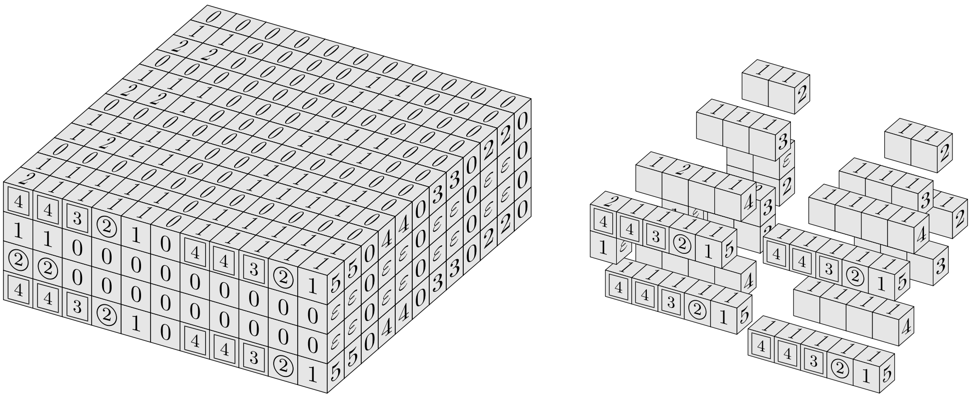
$$T = \left.\begin{pmatrix} 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 1 & \dots & 1 & 0 & 0 & \dots & 0 & 0 & 1 & \dots & 1 & 0 & 0 & \dots & 0 \\ 1 \text{ or } 2 & \dots & 1 \text{ or } 2 & 0 & 0 & \dots & 0 & 0 & 1 & \dots & 1 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 1 & \dots & 1 & 1 & 0 & \dots & 0 & 0 & 1 & \dots & 1 & 1 & 0 & \dots & 0 \\ 1 \text{ or } 2 & \dots & 1 \text{ or } 2 & 1 & 0 & \dots & 0 & 0 & 1 & \dots & 1 & 1 & 0 & \dots & 0 \\ & \vdots & & & & \vdots & & 0 & & \vdots & & & \vdots & & \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 1 & \dots & 1 & 1 & \dots & 1 & 0 & 0 & 1 & \dots & 1 & 1 & \dots & 1 & 0 \\ 1 \text{ or } 2 & \dots & 1 \text{ or } 2 & 1 & \dots & 1 & 0 & 0 & 1 & \dots & 1 & 1 & \dots & 1 & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 1 & \dots & 1 & 1 & \dots & 1 & 1 & 0 & 1 & \dots & 1 & 1 & \dots & 1 & 1 \\ 1 \text{ or } 2 & \dots & 1 \text{ or } 2 & 1 & \dots & 1 & 1 & 0 & 1 & \dots & 1 & 1 & \dots & 1 & 1 \\ \multicolumn{3}{c}{\underbrace{\hphantom{1 \text{ or } 2 \quad \dots \quad 1 \text{ or } 2}}_{m}} & \multicolumn{4}{c}{\underbrace{\hphantom{1 \quad \dots \quad 1 \quad 1}}_{n-1}} & & \multicolumn{3}{c}{\underbrace{\hphantom{1 \quad \dots \quad 1}}_{m}} & \multicolumn{4}{c}{\underbrace{\hphantom{1 \quad \dots \quad 1 \quad 1}}_{n-1}} \end{pmatrix}\right\} 3n.$$

▶ **Example 3.** For $\varphi = (x_1 \vee x_3 \vee x_4)(x_2 \vee x_3 \vee x_4)$, $I_\varphi$ is depicted as Figure 2(a), and its solution is given in 2(b).

We now show that the above reduction is parsimonious. We first show several auxiliary claims.

▶ **Claim 3.1.** *Let $P = (p_{i,j,k})$ be a solution to $I_\varphi$. Then, for $1 \leq k \leq n$, we have either*

$$\text{(i)} \begin{cases} p_{1,1,3k-2} = \cdots = p_{1,m+n-k,3k-2} = 1 \\ p_{1,1,3k-1} = \cdots = p_{1,m+n-k,3k-1} = 0 \\ p_{4,1,3k-2} = \cdots = p_{4,m+n-k,3k-2} = 0 \\ p_{4,1,3k-1} = \cdots = p_{4,m+n-k,3k-1} = 1 \\ p_{1,m+n+1,3k-2} = \cdots = p_{1,2m+2n-k,3k-2} = 0 \\ p_{1,m+n+1,3k-1} = \cdots = p_{1,2m+2n-k,3k-1} = 1 \\ p_{4,m+n+1,3k-2} = \cdots = p_{4,2m+2n-k,3k-2} = 1 \\ p_{4,m+n+1,3k-1} = \cdots = p_{4,2m+2n-k,3k-1} = 0, \end{cases}$$



**(a)** **(b)**

**Figure 2** The instance $I_\varphi$ of Picross 3D and its solution for $\varphi = (x_1 \vee x_3 \vee x_4)(x_2 \vee x_3 \vee x_4)$

*or*

$$\text{(ii)} \begin{cases} p_{1,1,3k-2} = \cdots = p_{1,m+n-k,3k-2} = 0 \\ p_{1,1,3k-1} = \cdots = p_{1,m+n-k,3k-1} = 1 \\ p_{4,1,3k-2} = \cdots = p_{4,m+n-k,3k-2} = 1 \\ p_{4,1,3k-1} = \cdots = p_{4,m+n-k,3k-1} = 0 \\ p_{1,m+n+1,3k-2} = \cdots = p_{1,2m+2n-k,3k-2} = 1 \\ p_{1,m+n+1,3k-1} = \cdots = p_{1,2m+2n-k,3k-1} = 0 \\ p_{4,m+n+1,3k-2} = \cdots = p_{4,2m+2n-k,3k-2} = 0 \\ p_{4,m+n+1,3k-1} = \cdots = p_{4,2m+2n-k,3k-1} = 1. \end{cases}$$

**Proof.** We show the claim by induction on $k$. Assume that $k = 1$. We first show that either ($p_{i,1,\ell} = \cdots = p_{i,m+n-1,\ell} = 1$ and $p_{i,m+n+1,\ell} = \cdots = p_{i,2m+2n-1,\ell} = 0$) or ($p_{i,1,\ell} = \cdots = p_{i,m+n-1,\ell} = 0$ and $p_{i,m+n+1,\ell} = \cdots = p_{i,2m+2n-1,\ell} = 1$) hold for $(i,\ell) \in \{1,4\} \times \{1,2\}$. For $(i,\ell) = (1,1)$, since $s_{1,1} = m+n-1$, we have to consecutively set $p_{1,a,1} = \cdots = p_{1,a+m+n-2,1} = 1$ for some $a \geq 1$. On the other hand, since $f_{1,m+n} = 0$, we have $p_{1,m+n,1} = 0$. Therefore, we must have either ($p_{1,1,1} = \cdots = p_{1,m+n-1,1} = 1$ and $p_{1,m+n+1,1} = \cdots = p_{1,2m+2n-1,1} = 0$) or ($p_{1,1,1} = \cdots = p_{1,m+n-1,1} = 0$ and $p_{1,m+n+1,1} = \cdots = p_{1,2m+2n-1,1} = 1$). Similarly, for other $(i,\ell)$, from $s_{i,\ell} = m+n-1$ and $f_{1,m+n} = 0$, we have either ($p_{i,1,\ell} = \cdots = p_{i,m+n-1,\ell} = 1$ and $p_{i,m+n+1,\ell} = \cdots = p_{i,2m+2n-1,\ell} = 0$) or ($p_{i,1,\ell} = \cdots = p_{i,m+n-1,\ell} = 0$ and $p_{i,m+n+1,\ell} = \cdots = p_{i,2m+2n-1,\ell} = 1$). We next show that $p_{1,m+n-1,1} + p_{1,m+n-1,2} = p_{4,m+n-1,1} + p_{4,m+n-1,2} = p_{1,m+n-1,1} + p_{4,m+n-1,1} = 1$ holds. From $f_{1,m+n-1} = \mathrm{div}(1) = 1$, we have $\sum_{\ell=1}^{3n} p_{1,m+n-1,\ell} = 1$. On the other hand, $t_{m+n-1,\ell} = 0$ implies that $p_{1,m+n-1,\ell} = 0$ for $\ell = 3,4,\ldots,3n$. Therefore, we have $p_{1,m+n-1,1} + p_{1,m+n-1,2} = 1$. Similarly, from $f_{4,m+n-1} = \mathrm{div}(1) = 1$ and $t_{m+n-1,\ell} = 0$ for $\ell = 3,4,\ldots,3n$, we obtain $p_{4,m+n-1,1} + p_{4,m+n-1,2} = 1$. Moreover, from $t_{m+n-1,1} = 1$, we have $\sum_{i=1}^{4} p_{i,m+n-1,1} = 1$. On the other hand, $f_{i,m+n-1} = 0$ implies that $p_{i,m+n-1,1} = 0$ for $i = 1,2$. Therefore, we have $p_{1,m+n-1,1} + p_{4,m+n-1,1} = 1$. Combining the above equations, we obtain the claim for $k = 1$.

For $k \geq 2$, assume that the claim holds for $1,\ldots,k-1$. The proof is similar to the one for $k = 1$. We first show that either ($p_{i,1,3k-3+\ell} = \cdots = p_{i,m+n-k,3k-3+\ell} = 1$ and $p_{i,m+n+1,3k-3+\ell} = \cdots = p_{i,2m+2n-k,3k-3+\ell} = 0$) or ($p_{i,1,3k-3+\ell} = \cdots = p_{i,m+n-k,3k-3+\ell} = 0$ and $p_{i,m+n+1,3k-3+\ell} = \cdots = p_{i,2m+2n-k,3k-3+\ell} = 1$) hold for $(i,\ell) \in \{1,4\} \times \{1,2\}$. For $(i,\ell) = (1,1)$, since $s_{1,3k-2} = m+n-k$, we have to consecutively set $p_{1,a,1} = \cdots = p_{1,a+m+n-k-1,1} = 1$ for some $a \geq 1$. On the other hand, since $f_{1,m+n} = 0$, we have $p_{1,m+n,1} =$

0. Moreover, $f_{1,m+n-a} = \mathrm{div}(a)$ implies that $\sum_{b=1}^{3n} p_{1,m+n-a,b} = a$ for $a = 1,\dots,k-1$. On the other hand, by the inductive hypothesis, we have $p_{1,m+n-a,3b-2} + p_{1,m+n-a,3b-1} = 1$ for $a = 1,\dots,k-1$ and $b = 1,\dots,a$. Therefore, we obtain that $p_{1,m+n-a,3k-2} = 0$ for $a = 1,\dots,k-1$. These imply that either ($p_{1,1,3k-2} = \cdots = p_{1,m+n-k,3k-2} = 1$ and $p_{1,m+n+1,3k-2} = \cdots = p_{1,2m+2n-k,3k-2} = 0$) or ($p_{1,1,3k-2} = \cdots = p_{1,m+n-k,3k-2} = 0$ and $p_{1,m+n+1,3k-2} = \cdots = p_{1,2m+2n-k,3k-2} = 1$) hold. Similarly, for other $(i,\ell)$, from $s_{i,\ell} = m+n-k$ and $f_{1,m+n-a} = \mathrm{div}(a)$ for $a = 0,\dots,k-1$, we have either ($p_{i,1,3k-3+\ell} = \cdots = p_{i,m+n-k,3k-3+\ell} = 1$ and $p_{i,m+n+1,3k-3+\ell} = \cdots = p_{i,2m+2n-k,3k-3+\ell} = 0$) or ($p_{i,1,3k-3+\ell} = \cdots = p_{i,m+n-k,\ell} = 0$ and $p_{i,m+n+1,3k-3+\ell} = \cdots = p_{i,2m+2n-k,3k-3+\ell} = 1$). We next show that $p_{1,m+n-k,3k-2} + p_{1,m+n-k,3k-1} = p_{4,m+n-k,3k-2} + p_{4,m+n-k,3k-1} = p_{1,m+n-k,3k-2} + p_{4,m+n-k,3k-2} = 1$ holds. From $f_{1,m+n-k} = \mathrm{div}(k)$, we have $\sum_{\ell=1}^{3n} p_{1,m+n-k,\ell} = k$. By the inductive hypothesis, we have $p_{1,m+n-k,3\ell-2} + p_{1,m+n-k,3\ell-1} = 1$ for $\ell = 1,\dots,k-1$. We also have $p_{1,m+n-k,3\ell} = 0$ for $\ell = 1,\dots,k-1$, since $t_{m+n-k,3\ell} = 0$ for $\ell = 1,\dots,k-1$. Hence, we have $\sum_{\ell=3k-2}^{3n} p_{1,m+n-k,\ell} = 1$. On the other hand, $t_{m+n-k,\ell} = 0$ implies that $p_{1,m+n-1,\ell} = 0$ for $\ell = 3k, 3k+1,\dots,3n$. Therefore, we have $p_{1,m+n-k,3k-2} + p_{1,m+n-k,3k-1} = 1$. Similarly, from $f_{4,m+n-k} = \mathrm{div}(k)$ and $t_{m+n-k,\ell} = 0$ for $\ell = 3,6,\dots,3k-3$ and $\ell = 3k, 3k+1,\dots,3n$, we obtain $p_{4,m+n-k,3k-2} + p_{4,m+n-k,3k-1} = 1$. Moreover, from $t_{m+n-1,k} = 1$, we have $\sum_{i=1}^{4} p_{i,m+n-k,1} = 1$. On the other hand, $f_{i,m+n-k} = 0$ implies that $p_{i,m+n-k,3k-2} = 0$ for $i = 1,2$. Therefore, we have $p_{1,m+n-k,3k-2} + p_{4,m+n-k,3k-2} = 1$. Combining the above equations, we obtain the claim for $k$. This completes the proof. ◀

Intuitively, for $1 \leq k \leq n$, $x_k = 1$ if and only if (i) in Claim 3.1 holds. We also need the following claim.

▶ **Claim 3.2.** *Let $P = (p_{i,j,k})$ be a solution to $I_\varphi$. Then we have $p_{i,j,3k} = 0$ for $i = 1,2,3,4$, $j = 1,\dots,2m+2n-1$, and $k = 1,\dots,n$. Moreover, we have $p_{i,j,k} = 0$ for $i = 2,3$, $j = m+1,\dots,2m+2n-1$, and $k = 1,2,\dots,3n$. Furthermore, we have $p_{i,j,3k-2} = p_{i,j,3k-1} = 0$ for $i = 2,3$, $k = 1,\dots,n$, and $j = m+n-k+1,\dots,m+n-1, 2m+2n-k+1,\dots,2m+2n-1$.*

**Proof.** For $j = 1,\dots,2m+2n+1$ and $k = 1,\dots,n$, we have $t_{j,3k} = 0$, implying that $p_{i,j,3k} = 0$ holds for $i = 1,2,3,4$.

For $i = 2,3$ and $j = m+1,\dots,2m+2n+1$, we have $f_{i,j} = 0$, implying that $p_{i,j,k} = 0$ holds for $k = 1,2\dots,3n$.

For $k = 1,\dots,n$ and $j = m+n-k+1,\dots,m+n-1, 2m+2n-k+1,\dots,2m+2n-1$, we have $t_{j,3k-2} = t_{j,3k-1} = 1$ and $p_{1,j,3k-2} + p_{4,j,3k-2} = p_{1,j,3k-1} + p_{4,j,3k-1} = 1$ by Claim 3.1. Therefore, we have $p_{i,j,3k-2} = p_{i,j,3k-1} = 0$ for $i = 2,3$. ◀

The following claim indicates which variable is true in each clause.

▶ **Claim 3.3.** *Let $P = (p_{i,j,k})$ be a solution to $I_\varphi$. Then, for $j = 1,\dots,m$ and $k = 1,\dots,n$, we have (i) in Claim 3.1 and $t_{j,3k-2} = 2$ if and only if $p_{2,j,3k-2} = 1$ holds.*

**Proof.** Fix $j \in \{1,\dots,m\}$ and $k \in \{1,\dots,n\}$. Assume that (i) in Claim 3.1 and $t_{j,3k-2} = 2$ hold. From Claim 3.1, we have $p_{1,j,3k-2} = 1$. Moreover, since $t_{j,3k-2} = 2$ implies that we have to consecutively set $p_{i,j,3k-2} = p_{i+1,j,3k-2}$ for some $i \geq 1$, we have $p_{2,j,3k-2} = 1$.

Conversely, assume that $p_{2,j,3k-2} = 1$ holds. By Claim 3.1, we have either $p_{1,j,3k-2} = 1$ or $p_{4,j,3k-2} = 1$. Hence, we have $\sum_{i=1}^{4} p_{i,j,3k-2} \geq 2$. Since $t_{j,3k-2} = 1$ or 2 by definition, it follows that $t_{j,3k-2} = 2$. This implies that we have to consecutively set $p_{i,j,3k-2} = p_{i+1,j,3k-2}$ for some $i \geq 1$. Together with $p_{2,j,3k-2} = 1$ and either $p_{1,j,3k-2} = 1$ or $p_{4,j,3k-2} = 1$, we have $p_{1,j,3k-2} = 1$. Hence, from Claim 3.1, we have (i) in Claim 3.1. This completes the proof. ◀

Intuitively, for $j = 1, \dots, m$ and $k = 1, \dots, n$, clause $C_j$ contains $x_k$ and $x_k$ is the unique variable that is true in $C_j$ if and only if $p_{2,j,3k-2} = 1$ holds.

We now construct a bijection between the solution sets of $\varphi$ and $I_\varphi$. We first construct a mapping from the solution set of $\varphi$ to that of $I_\varphi$. Let $x$ be a solution to $\varphi$. Then define an assignment $P$ to $I_\varphi$ as follows. For $1 \le k \le n$, if $x_k = 1$ then set

$$\begin{cases} p_{1,1,3k-2} = \cdots = p_{1,m+n-k,3k-2} = 1 \\ p_{1,1,3k-1} = \cdots = p_{1,m+n-k,3k-1} = 0 \\ p_{4,1,3k-2} = \cdots = p_{4,m+n-k,3k-2} = 0 \\ p_{4,1,3k-1} = \cdots = p_{4,m+n-k,3k-1} = 1 \\ p_{1,m+n+1,3k-2} = \cdots = p_{1,2m+2n-k,3k-2} = 0 \\ p_{1,m+n+1,3k-1} = \cdots = p_{1,2m+2n-k,3k-1} = 1 \\ p_{4,m+n+1,3k-2} = \cdots = p_{4,2m+2n-k,3k-2} = 1 \\ p_{4,m+n+1,3k-1} = \cdots = p_{4,2m+2n-k,3k-1} = 0, \end{cases} \tag{1}$$

and if $x_k = 0$ then set

$$\begin{cases} p_{1,1,3k-2} = \cdots = p_{1,m+n-k,3k-2} = 0 \\ p_{1,1,3k-1} = \cdots = p_{1,m+n-k,3k-1} = 1 \\ p_{4,1,3k-2} = \cdots = p_{4,m+n-k,3k-2} = 1 \\ p_{4,1,3k-1} = \cdots = p_{4,m+n-k,3k-1} = 0 \\ p_{1,m+n+1,3k-2} = \cdots = p_{1,2m+2n-k,3k-2} = 1 \\ p_{1,m+n+1,3k-1} = \cdots = p_{1,2m+2n-k,3k-1} = 0 \\ p_{4,m+n+1,3k-2} = \cdots = p_{4,2m+2n-k,3k-2} = 0 \\ p_{4,m+n+1,3k-1} = \cdots = p_{4,2m+2n-k,3k-1} = 1. \end{cases} \tag{2}$$

For $j = 1, \dots, m$, if clause $C_j$ contains $x_k$ and $x_k = 1$, then set $p_{2,j,3k-2} = 1$ and $p_{3,j,3k-2} = 0$. For $j = 1, \dots, m$, if clause $C_j$ contains $x_k$ and $x_k = 0$, then set $p_{2,j,3k-2} = 0$ and $p_{3,j,3k-2} = 1$. For all the remaining $p_{i,j,k}$, set $p_{i,j,k} = 0$. We show that the assignment $P$ constructed from $x$ as above is a solution to $I_\varphi$. We show this by showing that each constraint is satisfied by $P$.

We first examine the constraints for matrix $F$. For $j = 1, \dots, m$, we have $\sum_{k=1}^{3n} p_{1,j,k} = n$ from (1) and (2). Therefore, the constraint $f_{1,j} = \mathrm{div}(n)$ is satisfied for $j = 1, \dots, m$. Similarly, the constraint $f_{4,j} = \mathrm{div}(n)$ is satisfied for $j = 1, \dots, m$, since $\sum_{k=1}^{3n} p_{4,j,k} = n$ holds. For $j = 1, \dots, m$, we have $\sum_{k=1}^{3n} p_{2,j,k} = 1$ if and only if there exists exactly one $x_k$ in $C_j$ such that $x_k = 1$, since $x_k = 1$ implies that $p_{2,j,3k-2} = 1$. Because $x$ is a solution to $\varphi$, we have $\sum_{k=1}^{3n} p_{2,j,k} = 1$ and the constraint $f_{2,j} = 1$ is satisfied for $j = 1, \dots, m$. Similarly, the constraint $f_{3,j} = 2$ is satisfied for $j = 1, \dots, m$, since $\sum_{k=1}^{3n} p_{3,j,k} = 2$ holds if and only if there exist exactly two $x_k's$ in $C_j$ such that $x_k = 0$.

We then focus on the constraints for matrix $S$. For $i = 1, 4$ and $k = 1, \dots, n$, we have $\sum_{j=1}^{2m+2n-1} p_{i,j,3k-2} = m+n-k$ from (1) and (2). Therefore the constraint $s_{i,3k-2} = m+n-k$ is satisfied for $i = 1, 4$ and $k = 1, \dots, n$. Similarly, the constraint $s_{i,3k-1} = m + n - k$ is satisfied for $i = 1, 4$ and $k = 1, \dots, n$. Furthermore, since $p_{i,j,3k} = 0$ for $i = 1, 2, 3, 4$, $j = 1, \dots, 2m + 2n - 1$, and $k = 1, \dots, n$, the constraint $s_{i,3k} = 0$ is satisfied for $i = 1, 2, 3, 4$ and $k = 1, \dots, n$.

Finally, we examine the constraints for matrix $T$. Firstly, for $j = 1, \dots, m$ and $k = 1, \dots, n$, we have $t_{j,3k-2} = 1$ or 2 by definition. Assume first that $t_{j,3k-2} = 1$ holds. Then, by definition, clause $C_j$ does not contain variable $x_k$. Hence, we have $p_{2,j,3k-2} = p_{3,j,3k-2} = 0$. Moreover, from assignment (1) and (2), we have $p_{1,j,3k-2} + p_{4,j,3k-2} = 1$. Therefore, the constraint $t_{j,3k-2} = 1$ is satisfied. Assume next that $t_{j,3k-2} = 2$ holds. Then, by definition, clause

$C_j$ contains variable $x_k$. Hence, we have $p_{2,j,3k-2} + p_{3,j,3k-2} = 1$. Furthermore, we have $p_{1,j,3k-2} + p_{4,j,3k-2} = 1$ from (1) and (2). Therefore, $\sum_{i=1}^{4} p_{i,j,3k-2} = 2$ holds. Moreover, from Claim 3.3, we have $p_{1,j,3k-2} = 1$ if and only if $p_{2,j,3k-2} = 1$ holds. Hence, we must have either ($p_{1,j,3k-2} = 1$ and $p_{2,j,3k-2} = 1$) or ($p_{3,j,3k-2} = 1$ and $p_{4,j,3k-2} = 1$). In either case, we have two consecutive 1's. Therefore, the constraint $t_{j,3k-2} = 2$ is satisfied. Secondly, for $j = 1, \dots, m$ and $k = 1, \dots, n$, we have $t_{j,3k-1} = 1$ by definition. Since $p_{1,j,3k-1}+p_{4,j,3k-1} = 1$ by (1) and (2), and $p_{1,j,3k-1} = p_{4,j,3k-1} = 0$, the constraint $t_{j,3k-1} = 1$ is satisfied. Thirdly, for $j = m+1, \dots, m+n-1$ and $k = 1, \dots, n$, we have $t_{j,3k-2} = 1$ if $j \leq m+n-k$ and $t_{j,3k-2} = 0$ if $j \geq m+n-k+1$ by definition. If $j \leq m+n-k$, then we have $p_{1,j,3k-2} + p_{1,j,3k-2} = 1$ from (1) and (2), and $p_{2,j,3k-1} = p_{3,j,3k-1} = 0$. Thus, we have $\sum_{i=1}^{4} p_{i,j,3k-2} = 1$ and the constraint $t_{j,3k-2} = 1$ is satisfied. If $j \geq m+n-k+1$, then we have $p_{i,j,3k-2} = 0$ for $i = 1, 2, 3, 4$. Thus, we have $\sum_{i=1}^{4} p_{i,j,3k-2} = 0$ and the constraint $t_{j,3k-2} = 0$ is satisfied. Similarly, for $j = m+1, \dots, m+n-1$ and $k = 1, \dots, n$, the constraint $t_{j,3k-1} = 0$ is satisfied. Fourthly, for $j = 1, \dots, 2m+2n-1$ and $k = 1, \dots, n$, we have $t_{j,3k} = 0$ by definition. Since $p_{i,j,3k} = 0$ holds for $i = 1, 2, 3, 4$, the constraint $t_{j,3k} = 0$ is satisfied. Finally, for $j = m+n$ and $k = 1, 2 \dots, 3n$, we have $t_{j,k} = 0$ by definition. Since $p_{i,j,k} = 0$ holds for $i = 1, 2, 3, 4$, the constraint $t_{j,k} = 0$ is satisfied.

We next show that if $I_\varphi$ has a solution, then $\varphi$ has a solution. To show this, we construct a solution $x$ to $\varphi$ from a solution $P$ to $I_\varphi$ as follows. Note that, for each $k = 1, \dots, n$, we have either (i) or (ii) in Claim 3.1, since $P$ is a solution to $I_\varphi$. For each $k$, set $x_k = 1$ if (i) holds and $x_k = 0$ if (ii) holds. We show that $x$ defined as above is a solution to $\varphi$. It suffices to show that for each $j = 1, \dots, m$, clause $C_j$ contains exactly one $x_k$ that is set to 1. Fix $j \in \{1, \dots, m\}$. From $f_{2,j} = 1$, we have $\sum_{k=1}^{3n} p_{2,j,k} = 1$. Moreover, from $t_{j,3k} = 0$, we have $p_{3,j,3k-3} = 0$ for $k = 1, \dots, n$. Furthermore, from $t_{j,3k-1} = 1$ and $p_{1,j,3k-1} + p_{4,j,3k-1} = 1$ by Claim 3.1, we have $p_{2,j,3k-1} = 0$ for $k = 1, \dots, n$. Therefore, we have $\sum_{k=1}^{n} p_{2,j,3k-2} = 1$. From Claim 3.3, $p_{2,j,3k-2} = 1$ holds if and only if $p_{1,1,3k-2} = 1$ and $t_{j,3k-2} = 2$ holds. Therefore, together with $\sum_{k=1}^{n} p_{2,j,3k-2} = 1$, there exists exactly one $k$ such that $p_{1,1,3k-2} = 1$ and $t_{j,3k-2} = 2$ holds. By definition, we have $t_{j,3k-2} = 2$ if and only if $C_j$ contains $x_k$, and $p_{1,1,3k-2} = 1$ if and only if $x_k = 1$. Therefore, $C_j$ contains exactly one $x_k$ that is set to 1. Hence, $x$ is a solution to $\varphi$.

We finally show that the above reduction is parsimonious. To show this, we show that the mappings between the solution sets of $\varphi$ and $I_\varphi$ defined above are inverse to each other. Let $x$ be a solution to $\varphi$ and let $P$ be the solution of $I_\varphi$ corresponding to $x$. Moreover, let $x'$ be the solution constructed from $P$. We show that $x = x'$ holds. Observe first that $x_k = 1$ if and only if $p_{1,1,3k-2} = 1$ from (1) and (2) for $k = 1, \dots, n$. Furthermore, $p_{1,1,3k-2} = 1$ if and only if $x'_k = 1$ from Claim 3.1 for $k = 1, \dots, n$. Hence, $x_k = x'_k$ for $k = 1, \dots, n$ and thus $x = x'$.

Conversely, let $P$ be a solution to $I_\varphi$ and let $x$ be the solution to $\varphi$ constructed from $P$. Moreover, let $P'$ be the solution constructed from $x$. We show that $P = P'$ holds. Firstly, for $k = 1, \dots, n$, $P$ satisfies (i) in Claim 3.1 if and only if $x_k = 1$ holds. Furthermore, $x_k = 1$ holds if and only if $P'$ satisfies (i) in Claim 3.1 for $k = 1, \dots, n$. Hence, $P$ and $P'$ coincide in the indices appearing in Claim 3.1 for $k = 1, \dots, n$. Secondly, from $t_{j,3k-1} = 1$ and $p_{1,k,3k-1} + p_{4,k,3k-1} = p'_{1,k,3k-1} + p'_{4,k,3k-1} = 1$ for $j = 1, \dots, m$ and $k = 1, \dots, n$, we have $p_{2,k,3k-1} = p_{3,k,3k-1} = p'_{2,k,3k-1} = p'_{3,k,3k-1} = 0$ for $j = 1, \dots, m$ and $k = 1, \dots, n$. Thirdly, from the proof of Claim 3.1, we have $p_{i,j,3k-2} = p_{i,j,3k-1} = p'_{i,j,3k-2} = p'_{i,j,3k-1} = 0$ for $i = 1, 4$, $k = 2, \dots, n$, and $j = m+n-k+1, \dots, 2m+2n+1$. Fourthly, from Claim 3.2 we have $p_{i,j,3k} = p'_{i,j,3k} = 0$ for $i = 1, 2, 3, 4$, $j = 1, \dots, 2m+2n+1$, and $k = 1, \dots, n$, and $p_{i,j,k} = p'_{i,j,k} = 0$ for $i = 2, 3$, $j = m+1, \dots, 2m+2n+1$, and $k = 1, 2, \dots, 3n$. Finally, from Claim 3.3, we have $p_{1,1,3k-2} = 1$ and $t_{j,3k-2} = 2$ if and only if $p_{2,j,3k-2} = 1$ holds for $j = 1, \ldots, m$ and $k = 1, \ldots, n$. Since $p_{1,1,3k-2} = p'_{1,1,3k-2}$ holds from the above argument, we have $p_{2,j,3k-2} = p'_{2,j,3k-2}$ for $j = 1, \ldots, m$ and $k = 1, \ldots, n$. Hence, $P = P'$ holds. This completes the proof. ◀

▶ **Corollary 4.** *The counting version of Picross 3D is #P-complete and ASP Picross 3D is NP-complete.*

**Proof.** The former follows from the #P-completeness of the counting version of positive 1-in-3 SAT [2, 3] and Proposition 2. The latter follows from the NP-completeness of ASP positive 1-in-3 SAT [12, 13] and Proposition 2. ◀

## 4 $\Sigma_2^P$-completeness of FCP Picross 3D

In this section, we show the following theorem using the reduction in the previous section.

▶ **Theorem 5.** *FCP Picross 3D is* $\Sigma_2^P$*-complete.*

**Proof.** Since Picross 3D is in NP, FCP Picross 3D is in $\Sigma_2^P$ [4]. We hence show that FCP Picross 3D is $\Sigma_2^P$-hard in the following.

Let $(\varphi, \ell)$ be an instance of FCP positive 1-in-3 SAT. We show that $(\varphi, \ell)$ is a yes instance if and only if $(I_\varphi, \ell)$ is a yes instance, where $I_\varphi$ is defined in the proof of Proposition 2.

We first show that if $(\varphi, \ell)$ is a yes instance, then $(I_\varphi, \ell)$ is a yes instance. For simplicity, we identify a partial assignment with a set of single-variable assignments corresponding to it in the following. Let $\{x_k = \varepsilon_k \mid k \in K\}$ be a clue that makes $\varphi$ uniquely solvable, where $K \subseteq \{1, \ldots, n\}$, $|K| \leq \ell$, and $\varepsilon_k$ is either 0 or 1 for $k \in K$. We claim that $\{p_{1,1,3k-2} = \varepsilon_k \mid k \in K\}$ is a clue that makes $I_\varphi$ uniquely solvable. In fact, since $\{x_k = \varepsilon_k \mid k \in K\}$ can be extended to a solution of $\varphi$, $\{p_{1,1,3k-2} = \varepsilon_k \mid k \in K\}$ can also be extended to a solution to $I_\varphi$. Moreover, if there exist two solutions extending $\{p_{1,1,3k-2} = \varepsilon_k \mid k \in K\}$ in $I_\varphi$, then there must be two solutions to $\varphi$ corresponding to these solutions since the reduction is parsimonious. These two solutions to $\varphi$ coincide in the indices in $K$ from the argument in the proof of Proposition 2. This contradicts that $\{x_k = \varepsilon_k \mid k \in K\}$ is a clue that makes $\varphi$ uniquely solvable. Therefore, $\{p_{1,1,3k-2} = \varepsilon_k \mid k \in K\}$ is a clue that makes $I_\varphi$ uniquely solvable. Since $|\{p_{1,1,3k-2} = \varepsilon_k \mid k \in K\}| \leq \ell$, we have that $(I_\varphi, \ell)$ is a yes instance.

We next show that if $(I_\varphi, \ell)$ is a yes instance, then $(\varphi, \ell)$ is a yes instance. Let $c_{\text{pic}} = \{p_{i_v,j_v,k_v} = \varepsilon_v \mid v \in V\}$ be a clue that makes $I_\varphi$ uniquely solvable, where $|V| \leq \ell$, $(i_v, j_v, k_v) \in \{1, \ldots, h\} \times \{1, \ldots, w\} \times \{1, \ldots, d\}$ for $v \in V$, and $\varepsilon_v$ is either 0 or 1 for $v \in V$. We construct a clue $c_{\text{sat}}$ of $\varphi$ as follows. Set $c_{\text{sat}} = \emptyset$. For $k = 1, \ldots, n$, add $x_k = 1$ to $c_{\text{sat}}$ if $c_{\text{pic}}$ contains at least one of the following assignments:

$$\begin{aligned}
&p_{1,1,3k-2} = 1, \ldots, p_{1,m+n-k,3k-2} = 1\\
&p_{1,1,3k-1} = 0, \ldots, p_{1,m+n-k,3k-1} = 0\\
&p_{4,1,3k-2} = 0, \ldots, p_{4,m+n-k,3k-2} = 0\\
&p_{4,1,3k-1} = 1, \ldots, p_{4,m+n-k,3k-1} = 1\\
&p_{1,m+n+1,3k-2} = 0, \ldots, p_{1,2m+2n-k,3k-2} = 0\\
&p_{1,m+n+1,3k-1} = 1, \ldots, p_{1,2m+2n-k,3k-1} = 1\\
&p_{4,m+n+1,3k-2} = 1, \ldots, p_{4,2m+2n-k,3k-2} = 1\\
&p_{4,m+n+1,3k-1} = 0, \ldots, p_{4,2m+2n-k,3k-1} = 0.
\end{aligned} \tag{3}$$

Moreover, for $k = 1, \dots, n$, add $x_k = 0$ to $c_{\text{sat}}$ if $c_{\text{pic}}$ contains at least one of the following assignments:

$$\begin{aligned}
&p_{1,1,3k-2} = 0, \dots, p_{1,m+n-k,3k-2} = 0\\
&p_{1,1,3k-1} = 1, \dots, p_{1,m+n-k,3k-1} = 1\\
&p_{4,1,3k-2} = 1, \dots, p_{4,m+n-k,3k-2} = 1\\
&p_{4,1,3k-1} = 0, \dots, p_{4,m+n-k,3k-1} = 0\\
&p_{1,m+n+1,3k-2} = 1, \dots, p_{1,2m+2n-k,3k-2} = 1\\
&p_{1,m+n+1,3k-1} = 0, \dots, p_{1,2m+2n-k,3k-1} = 0\\
&p_{4,m+n+1,3k-2} = 0, \dots, p_{4,2m+2n-k,3k-2} = 0\\
&p_{4,m+n+1,3k-1} = 1, \dots, p_{4,2m+2n-k,3k-1} = 1.
\end{aligned} \tag{4}$$

Furthermore, for $k = 1, \dots, n$, add $x_k = 1$ to $c_{\text{sat}}$ if $t_{j,3k-2} = 2$ and $c_{\text{pic}}$ contains at least one of the following assignments:

$$\begin{aligned}
&p_{2,1,3k-2} = 1, p_{2,2,3k-2} = 1, \dots, p_{2,m,3k-2} = 1,\\
&p_{3,1,3k-2} = 0, p_{3,2,3k-2} = 0, \dots, p_{3,m,3k-2} = 0.
\end{aligned} \tag{5}$$

Finally, for $k = 1, \dots, n$, add $x_k = 0$ to $c_{\text{sat}}$ if $t_{j,3k-2} = 2$ and $c_{\text{pic}}$ contains at least one of the following assignments:

$$\begin{aligned}
&p_{2,1,3k-2} = 0, p_{2,2,3k-2} = 0, \dots, p_{2,m,3k-2} = 0,\\
&p_{3,1,3k-2} = 1, p_{3,2,3k-2} = 1, \dots, p_{3,m,3k-2} = 1.
\end{aligned} \tag{6}$$

Then clearly $|c_{\text{sat}}| \leq \ell$ holds. We show that $c_{\text{sat}}$ is a clue that makes $\varphi$ uniquely solvable. To show this, we construct a clue $c'_{\text{pic}}$ to $I_\varphi$ from $c_{\text{sat}}$ as follows. Set $c'_{\text{pic}} = \emptyset$. Firstly, for $k = 1, \dots, n$, if $x_k = 1$ is in $c_{\text{sat}}$, then add to $c'_{\text{pic}}$ all the assignments in (3). Secondly, for $k = 1, \dots, n$, if $x_k = 0$ is in $c_{\text{sat}}$, then add to $c'_{\text{pic}}$ all the assignments in (4). Thirdly, for $k = 1, \dots, n$, if $x_k = 1$ is in $c_{\text{sat}}$ and $t_{j,3k-2} = 2$, then add to $c'_{\text{pic}}$ all the assignments in (5). Fourthly, for $k = 1, \dots, n$, if $x_k = 0$ is in $c_{\text{sat}}$ and $t_{j,3k-2} = 2$, then add to $c'_{\text{pic}}$ all the assignments in (6). Finally, as in Claim 3.2, add $p_{i,j,k} = 0$ to $c'_{\text{pic}}$ if $p_{i,j,k} = 0$ holds for any solution to $I_\varphi$. Then clearly $c_{\text{pic}} \subseteq c'_{\text{pic}}$ holds. Since $c_{\text{pic}}$ determines the solution uniquely, so does $c'_{\text{pic}}$. Moreover, for any solution $x$ to $\varphi$ extending $c_{\text{sat}}$, the solution $P'$ to $I_\varphi$ corresponding to $x$ contains $c'_{\text{pic}}$, i.e., $c'_{\text{pic}} \subseteq P'$ holds. Hence, $P'$ is uniquely determined, and so is $x$ from Proposition 2. Therefore, $c_{\text{sat}}$ is a clue that makes $\varphi$ uniquely solvable. This completes the proof. ◀

## 5 Conclusion

We in this paper show that FCP Picross 3D is $\Sigma_2^{\text{P}}$-complete. To show the result, we provide a parsimonious reduction from positive 1-in-3 SAT, where the FCP of it is known to be $\Sigma_2^{\text{P}}$-complete [4]. From the reduction, we also show that the counting version of Picross 3D is #P-complete and ASP Picross 3D is NP-complete.

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