Brief Announcement: Approximation Schemes for Geometric Coverage Problems

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Abstract

In this announcement, we show that the classical Maximum Coverage problem (MC) admits a PTAS via local search in essentially all cases where the corresponding instances of Set Cover (SC) admit a PTAS via the local search approach by Mustafa and Ray [7]. As a corollary, we answer an open question by Badanidiyuru, Kleinberg, and Lee [1] regarding half-spaces in $\mathbb{R}^3$ thereby settling the existence of PTASs for essentially all natural cases of geometric MC problems. As an intermediate result, we show a color-balanced version of the classical planar subdivision theorem by Frederickson [5]. We believe that some of our ideas may be useful for analyzing local search in other settings involving a hard cardinality constraint.

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Related Version A full version of the paper is available at [2], `https://arxiv.org/abs/1607.06665`.

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1 Introduction and Contribution

Let U be a set of ground elements, $\mathcal{F} \subseteq 2^U$ be a family of subsets of U and k be a positive integer. The MAXIMUM COVERAGE (MC) problem asks for a k-subset $\mathcal{F}'$ of $\mathcal{F}$ such that the number $|\bigcup \mathcal{F}'|$ of ground elements covered by $\mathcal{F}'$ is maximized. In the closely related SET COVER problem (SC), the goal is to cover all ground elements using as few sets as possible. Both problems are among the most fundamental NP-hard optimization problems and their approximability is in general well understood [3, 4].

We examine the approximability of *geometric* MC problems. Badanidiyuru et al. [1] provided fixed-parameter approximation schemes for the large class of MC problems with bounded VC dimension. The running times of these schemes are polynomial in $|U|$ and $|\mathcal{F}|$ but *exponential* in k, in the VC dimension, and in the reciprocal of the error parameter $\epsilon > 0$. Prior to their work [1], Mustafa and Ray [7] had shown that local search yields PTASs for many geometric SC problems where a naturally defined *exchange graph* between a locally optimum solution and a globally optimum solution is *planar*. While the range of settings solvable by the fixed-parameter approximation approach of Badanidiyuru et al. is in principle even broader, the approach of Mustafa and Ray gives PTASs with strictly polynomial running time while still encompassing essentially all of the natural cases of (unweighted) SC that are not known to be APX-hard. It is thus an interesting question if their approach gives PTASs (with strictly polynomial running times) also for the corresponding MC problems.

A difficulty in carrying over this approach from SC to MC lies in the hard cardinality constraint in MC. For SC, Mustafa and Ray used the planar subdivision theorem by Frederickson [5] to subdivide the above-mentioned planar exchange graph into *small* pieces where each one provides a candidate swap. This subdivision may, however, be arbitrarily unbalanced with respect to the two feasible solutions forming the node set of the exchange graph. Hence a direct application of this approach would be in conflict with the cardinality constraint. Another difficulty comes from the different objective functions of MC and SC and that the analysis of Mustafa and Ray exploits that *all* ground elements are covered.

In this announcement we summarize how to overcome these issues (see [2] for the full version). A key step in our proof is that the pieces of an (unbalanced) subdivision obtained Frederickson's theorem [5] can be recombined in a careful way to obtain a *color-balanced* version of that theorem (see Theorem 2). Also the subsequent analysis of the performance guarantee requires some new ideas because of the other above-mentioned difficulties, and because our colored subdivision cannot not achieve perfect but only rough balance.

In a b-local search for MC, we start with any feasible solution. We perform a profitable *swap* of cardinality b as long as there is one. The timing is polynomial for constant b.

2 Our Results

For a graph G, a subset S of $V(G)$ is an α*-balanced separator* when its removal breaks G into two collections of connected components such that each collection contains at most an α fraction of $V(G)$ where $\alpha \in [\frac{1}{2}, 1)$ and α is a constant. The size of a separator S is the number of vertices it contains. For a non-decreasing sublinear function f, a subgraph-closed class of graphs is said to be f*-separable* if there is an $\alpha \in [\frac{1}{2}, 1)$ such that for any $n > 2$, any n-vertex graph in the class has an α-balanced separator of size at most $f(n)$.

▶ **Definition 1.** A class $\mathcal{C}$ of instances of MC is called f*-separable* if for any two disjoint feasible solutions $\mathcal{F}$ and $\mathcal{F}'$ of any instance in $\mathcal{C}$ there exists an f-separable graph G with node set $\mathcal{F} \cup \mathcal{F}'$ with the following *exchange* property. If there is a ground element $u \in U$

that is covered both by $\mathcal{F}$ and $\mathcal{F}'$ then there exists an edge (S, S') in G with $S \in \mathcal{F}$ and $S' \in \mathcal{F}'$ with $u \in S \cap S'$.

Note that a class of MC instances where each instance admits planar exchange graphs is $O(\sqrt{n})$-separable.

▶ **Theorem 2.** *Let $\mathcal{G}$ be a subgraph-closed f-separable graph class and G be a 2-colored n-vertex graph in $\mathcal{G}$ with color classes Γ_1, Γ_2 such that $|\Gamma_2| \geq |\Gamma_1|$. For any q and $r \ll n$ where r is suitably large, there is an integer $t \in \Theta(\frac{n}{q \cdot r})$ such that V can be partitioned into $t+1$ sets $\mathcal{X}, V_1, \dots, V_t$ where c_1, c_2 are constants (depending only on f) and there is an integer $q' \in [q, 2q-1]$ satisfying the following properties.*

- **(i)** $N(V_i) \cap V_j = \emptyset$ *for each* $i \neq j$ *and* $\mathcal{X} = \bigcup_i N(V_i)$,
- **(ii)** $|V_i| \geq \frac{q' \cdot r}{2}$ *and* $|V_i| \leq 2 \cdot (q'+1) \cdot r$ *for each* i,
- **(iii)** $|N(V_i)| \leq c_1 \cdot q \cdot f(r)$ *for each* i *(thus,* $|\mathcal{X}| \leq \sum_{i=1}^{t} |N(V_i)| \leq \frac{c_2 \cdot f(r) \cdot n}{r}$*),*
- **(iv)** $\left| |V_i \cap \Gamma_1| - \frac{|\Gamma_1|}{|\Gamma_2|} \cdot |V_i \cap \Gamma_2| \right| \leq 4 \cdot r$ *for each* i.

▶ **Theorem 3.** *For any non-decreasing strictly sublinear function f, every f-separable class of MC instances (closed under removing elements and sets) admits a PTAS via local search.*

The following theorem describes several cases of MC that can be solved by our approach. Therein, we refer to several maximization versions of classical minimum covering problems (such as Vertex Cover). For example, in Maximum Vertex Cover, we are given a graph G and number k and we want to find a k vertices which *cover* as many edges as possible. The others are defined analogously. For a definition of 1.5D Terrain Guarding we refer to Krohn et al. [6].

▶ **Theorem 4.** *Local search gives a PTAS for:*

(V) *the* Maximum Vertex Cover *problem on f-separable and subgraph-closed graph classes,*

(T) *the* Maximum 1.5D Terrain Guarding *problem.*

and the following classes of MC problems:

(C_1) *the set of ground elements is a set of points in $\mathbb{R}^3$, and the family of subsets is induced by a set of halfspaces in $\mathbb{R}^3$.*

(C_2) *the set of ground elements is a set of points in $\mathbb{R}^2$, and the family of subsets is induced by a set of convex pseudodisks (a set of convex objects where any two objects can have at most two intersections in their boundary).*

and the following Maximum Hitting Set *problems:*

(H_1) *the set of ground elements is a set of points in $\mathbb{R}^2$, and the set of ranges is induced by a set of r-admissible regions (this includes pseudodisks, same-height axis-parallel rectangles, circular disks, translates of convex objects).*

(H_2) *the set of ground elements is a set of points in $\mathbb{R}^3$, and the set of ranges is induced by a set of halfspaces in $\mathbb{R}^3$.*

and Maximum Dominating Set *problems in each of the following graph classes:*

(D_1) *intersection graphs of homothetic copies of convex objects (which includes arbitrary squares, regular k-gons, translated and scaled copies of a convex object).*

(D_2) *non-trivial minor-closed graph classes.*

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