Brief Announcement: Bounded-Degree Cut is Fixed-Parameter Tractable

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Abstract

In the bounded-degree cut problem, we are given a multigraph $G = (V, E)$, two disjoint vertex subsets $A, B \subseteq V$, two functions $\mathrm{u}_A, \mathrm{u}_B : V \to \{0, 1, \ldots, |E|\}$ on V, and an integer $k \geq 0$. The task is to determine whether there is a minimal (A, B)-cut (V_A, V_B) of size at most k such that the degree of each vertex $v \in V_A$ in the induced subgraph $G[V_A]$ is at most $\mathrm{u}_A(v)$ and the degree of each vertex $v \in V_B$ in the induced subgraph $G[V_B]$ is at most $\mathrm{u}_B(v)$. In this paper, we show that the bounded-degree cut problem is fixed-parameter tractable by giving a $2^{18k}|G|^{O(1)}$-time algorithm. This is the first single exponential FPT algorithm for this problem. The core of the algorithm lies two new lemmas based on important cuts, which give some upper bounds on the number of candidates for vertex subsets in one part of a minimal cut satisfying some properties. These lemmas can be used to design fixed-parameter tractable algorithms for more related problems.

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1 Introduction

A *cut* of a graph is a partition of the vertices of the graph into two disjoint subsets. Graph cuts play an important role in combinatorial optimization and graph theory. The classical *minimum cut problem* is well known to be polynomially solvable [13]. Due to the rich application realm of this problem, many variants and extensions have been investigated. Some problems ask to partition the graph into more than two parts to disconnect some vertices such as the *k-way cut problem* (the *k-cut problem*) [14, 15], the *multiterminal cut problem* [9, 21] and the *multicut problem* [5, 19]. Some problems are still going to partition the graph into two parts, but with some additional requirements beyond the disconnectivity. One of the most extensively studied additional requirements is the constraint on the numbers of vertices or edges in each of the two parts. For examples, the *balanced cut problem* [1, 12, 16] and the *minimum bisection problem* [7, 10, 11] require the numbers of vertices in the two parts of the cut as close as possible. The (*balanced*) *judicious bipartition problem* [17] has

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conditions on the numbers of edges in the two parts. Some other well studied additional requirements include conditions on the connectivity of the two parts such as the *2-disjoint connected subgraphs problem* [8], and conditions on the degree of the two parts, such as the series of bipartition problems with degree constraints [2, 3, 4, 20, 22].

In this paper, we study the *bounded-degree cut* problem, which belongs to the latter kind of the extensions: to partition a given graph into two parts with some degree constraints on the induced subgraphs of the two parts. We mainly consider the upper bounds of the degree. An (S,T)-cut (V_1,V_2) is minimal if $E_G(V_1)$ does not contain $E_G(V_1')$ or $E_G(V_2')$ as a subset for any $S \subseteq V_1' \subsetneqq V_1$ and $T \subseteq V_2' \subsetneqq V_2$. Our problem is defined as follows.

BOUNDED-DEGREE CUT (with parameter: k)
Instance: A multigraph $G=(V,E)$, two disjoint nonempty vertex subsets $A,B \subseteq V$, two functions u_A and u_B from V to $\{0,1,\dots,|E|\}$ and an integer $k \geq 0$.
Question: Does there exist a minimal (A,B)-cut (V_A,V_B) such that
the number of edges with one endpoint in V_A and one endpoint in V_B is at most k,
for each vertex $v \in V_A$, the degree of it in the induced graph $G[V_A]$ is at most $\mathrm{u}_A(v)$, and
for each vertex $v \in V_B$, the degree of it in the induced graph $G[V_B]$ is at most $\mathrm{u}_B(v)$?

During the last decade, cut related problems were extensively studied from the viewpoint of parameterized algorithms [12, 18, 7, 17, 15, 19, 21, 6]. In this paper, we will study BOUNDED-DEGREE CUT from the viewpoint of parameterized algorithms. Our main result is the first single-exponential FPT algorithm for BOUNDED-DEGREE CUT, which implies that BOUNDED-DEGREE CUT can be solved in polynomial time for $k = O(\log |G|)$.

▶ **Theorem 1.** BOUNDED-DEGREE CUT *can be solved in* $2^{18k} \cdot |G|^{O(1)}$ *time.*

2 The main idea

The most crucial techniques in this paper are: to use important cuts introduced by Marx [18] to obtain the following two general lemmas for bounded sets related to cuts; and then based on these two lemmas, to construct from a given instance a set of at most 2^{18k} new "easy" instances such that the original instance is feasible if and only if at least one of the "easy" instances is feasible.

▶ **Lemma 2.** *Let $A,B,C \subseteq V$ be non-empty subsets in a graph $G=(V,E)$ and k and ℓ be nonnegative integers. Then one can find in $2^{3(k+\ell)}(n+m)^{O(1)}$ time a family $\mathcal{X}$ of at most $2^{3(k+\ell)}$ subsets of C with a property that $C \cap V_1 \in \mathcal{X}$ for any minimal (A,B)-cut (V_1,V_2) with size at most k such that $|C \cap V_1| \leq \ell$.*

▶ **Lemma 3.** *Let $A,B,B' \subseteq V$ be non-empty subsets in a graph $G=(V,E)$, where $B' \subseteq B$, and k be a nonnegative integer. Then one can find in $2^{3k}(n+m)^{O(1)}$ time a family $\mathcal{Y}$ of at most 2^{3k} subsets of $N_G(B')$ with a property that $N_G(B') \cap V_1 \in \mathcal{Y}$ for any minimal (A,B)-cut (V_1,V_2) with size at most k.*

We will use $I=(G=(V,E),A,B)$ to denote an instance of the problem, where u_A, u_B and k are omitted since they remain unchanged throughout our argument. We use Z_A and Z_B to denote the sets of A-*unsatisfied vertices* and B-*unsatisfied vertices*, respectively, i.e., $Z_A \triangleq \{v \in V \mid \deg_G(v) > \mathrm{u}_A(v)\}$ and $Z_B \triangleq \{v \in V \mid \deg_G(v) > \mathrm{u}_B(v)\}$. We call I an *easy instance* if it holds that $Z_A \cup Z_B \subseteq A \cup B$, $N_G(Z_A \cap A) \subseteq A \cup B$, and $N_G(Z_B \cap B) \subseteq A \cup B$. We can see that an easy instance can be solved in polynomial time.

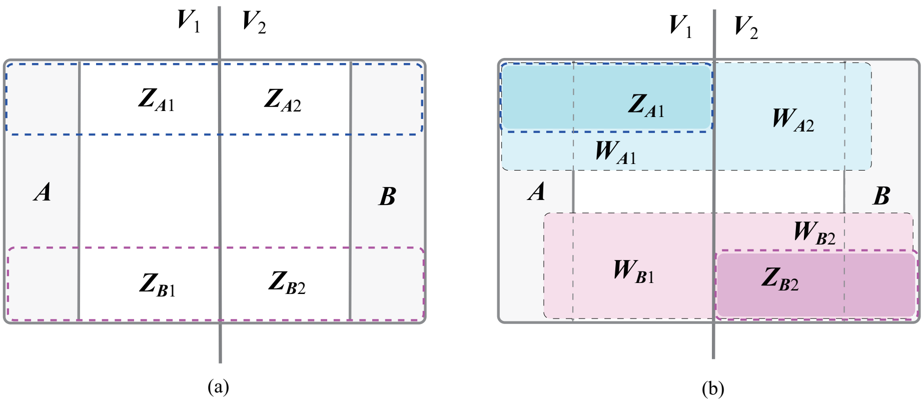

Figure 1 (a) An (A, B)-cut $\pi = (V_A, V_B)$ to I and the partitions $\{Z_{A1}, Z_{A2}\}$ of Z_A and $\{Z_{B1}, Z_{B2}\}$ of Z_B by π, where possibly $Z_A \cap Z_B \neq \emptyset$, (b) The partitions $\{W_{B1}, W_{B2}\}$ of $W_B = N_G(Z_{B2})$ and $\{W_{A1}, W_{A2}\}$ of $W_A = N_G(Z_{A1})$ by π, where possibly $W_A \cap W_B \neq \emptyset$.

For a given instance I with a feasible (A, B)-cut (V_1, V_2), we try to guess some subsets $V_1' \subseteq V_1 \setminus A$ and $V_2' \subseteq V_2 \setminus B$ so that the new instance $(G, A^* = A \cup V_1', B^* = B \cup V_2', \mathrm{u}_A, \mathrm{u}_B, k)$ remains feasible and is an easy. We will generate at most 2^{18k} easy instances.

3 Constructing Easy Instances

For a minimal (A, B)-cut $\pi = (V_1, V_2)$ (not necessarily feasible) in a given instance $I = (G = (V, E), A, B)$, we define the following notation on vertex subsets:
$Z_{Ai} \triangleq Z_A \cap V_i$ and $Z_{Bi} \triangleq Z_B \cap V_i$, $i = 1, 2$;
$W_A \triangleq N_G(Z_{A1})$ and $W_B \triangleq N_G(Z_{B2})$; $W_{Ai} \triangleq W_A \cap V_i$, and $W_{Bi} \triangleq W_B \cap V_i$, $i = 1, 2$;
$A_\pi \triangleq A \cup Z_{A1} \cup Z_{B1} \cup W_{A1} \cup W_{B1}$ and $B_\pi \triangleq B \cup Z_{A2} \cup Z_{B2} \cup W_{A2} \cup W_{B2}$.
See in Fig. 1 for an illustration on these subsets. Observe that the resulting instance (G, A_π, B_π) is an easy instance. The (A, B)-cut $\pi = (V_1, V_2)$ is feasible if and only if the corresponding instance (G, A_π, B_π) is feasible.

3.1 Partitioning Unsatisfied Vertices

For a minimal (A, B)-cut (V_1, V_2) to an instance I, let Z_{A1} and Z_{B2} be the subsets defined in the above. We observe that if the cut is feasible, then

$$|Z_{A1}|, |Z_{B2}| \leq k$$

since each vertex in $Z_{A1} \cup Z_{B2}$ has at least one incident edge included in $E_G(V_1, V_2)$ so that the degree constraint on the vertex holds.

By applying Lemma 2 to $(A, B, C = Z_A, k, \ell = k)$, we can construct in $2^{6k}(n + m)^{O(1)}$ time a family $\mathcal{X}_1$ of at most 2^{6k} subsets of Z_A such that $\mathcal{X}_1$ contains the set Z_{A1} defined to each feasible (A, B)-cut (V_1, V_2) in the instance $I = (G, A, B)$. Symmetrically it takes $2^{6k}(n + m)^{O(1)}$ time to find a family $\mathcal{X}_2$ of at most 2^{6k} subsets of Z_B such that $\mathcal{X}_2$ contains the set Z_{B2} defined to each feasible (A, B)-cut (V_1, V_2) in the instance $I = (G, A, B)$. Then the set $\mathcal{X}_{1,2}$ of all pairs (X_1, X_2) of disjoint sets $X_i \in \mathcal{X}_i$, $i = 1, 2$ contains the pair (Z_{A1}, Z_{B2}) defined to each feasible (A, B)-cut (V_1, V_2) in I. By noting that $|\mathcal{X}_{1,2}| \leq 2^{6k}2^{6k} = 2^{12k}$, we obtain the next.

▶ **Lemma 4.** *Given an instance $I = (G, A, B)$, one can construct in $2^{12k}(n+m)^{O(1)}$ time at most 2^{12k} new instances $I' = (G, A', B')$ with $Z_A \cup Z_B \subseteq A' \cup B'$, one of which is equal to $(G, A \cup Z_{A1} \cup Z_{B1}, B \cup Z_{A2} \cup Z_{B2})$ for each feasible (A, B)-cut (V_1, V_2) to I.*

3.2 Partitioning Neighbors of Unsatisfied Vertices

For a minimal (A, B)-cut (V_1, V_2) to an instance I, let W_{A2} and W_{B1} be the subsets defined in the above. We observe that if the cut is feasible, then

$$|W_{B1}|, |W_{A2}| \leq k$$

since each of $|N_G(Z_{B2}) \cap V_1|$ and $|N_G(Z_{A1}) \cap V_2|$ is at most $|E_G(V_1, V_2)| \leq k$ to the feasible (A, B)-cut (V_1, V_2).

By applying Lemma 3 to $(A \cup Z_{A1} \cup Z_{B1}, B \cup Z_{A2} \cup Z_{B2}, B' = Z_{B2}, k)$, we can construct in $2^{3k}n^{O(1)}$ time a family $\mathcal{Y}_1$ of at most 2^{3k} subsets of $N_G(Z_{B2})$ such that $\mathcal{Y}_1$ contains the set $W_{B1} = N_G(Z_{B2}) \cap V_1$ defined to each feasible (A, B)-cut (V_1, V_2) in the instance $I = (G, A, B)$. Symmetrically it takes $2^{3k}(n+m)^{O(1)}$ time to find a family $\mathcal{Y}_2$ of at most 2^{3k} subsets of $N_G(Z_{A1})$ such that $\mathcal{Y}_2$ contains the set $W_{A2} = N_G(Z_{A1}) \cap V_2$ defined to each feasible (A, B)-cut (V_1, V_2) in I. Then the set $\mathcal{Y}_{1,2}$ of all pairs (Y_1, Y_2) of disjoint sets $Y_i \in \mathcal{Y}_i$, $i = 1, 2$ contains the pair (W_{B1}, W_{A2}) defined to each feasible (A, B)-cut (V_1, V_2) in the instance $I = (G, A, B)$. By noting that $|\mathcal{Y}_{1,2}| \leq 2^{6k}$, we obtain the next.

▶ **Lemma 5.** *Given an instance $I = (G, A, B)$ and the subsets Z_{A1} and Z_{B2} defined to a feasible (A, B)-cut (V_1, V_2) in I, one can construct in $2^{6k}(n+m)^{O(1)}$ time at most 2^{6k} new easy instances $I' = (G, A', B')$, one of which is equal to (G, A_π, B_π) defined to the feasible (A, B)-cut $\pi = (V_1, V_2)$.*

By Lemmas 4 and 5, we obtain the next, which can imply Theorem 1.

▶ **Lemma 6.** *Given an instance $I = (G, A, B)$, one can construct in $2^{18k}(n+m)^{O(1)}$ time at most 2^{18k} new easy instances $I' = (G, A', B')$, one of which is equal to (G, A_π, B_π) for each feasible (A, B)-cut $\pi = (V_1, V_2)$ to I.*

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