On Hadamard Series and Rotating $\mathbb{Q}$-Automata

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Abstract

In this paper, we study rotating $\mathbb{Q}$-automata, which are (memoryless) automata with weights in $\mathbb{Q}$, that can read the input tape from left to right several times. We show that the series realized by valid rotating $\mathbb{Q}$-automata are $\mathbb{Q}$-Hadamard series (which are the closure of $\mathbb{Q}$-rational series by pointwise inverse), and that every $\mathbb{Q}$-Hadamard series can be realized by such an automaton. We prove that, although validity of rotating $\mathbb{Q}$-automata is undecidable, the equivalence problem is decidable on rotating $\mathbb{Q}$-automata. Finally, we prove that every valid two-way $\mathbb{Q}$-automaton admits an equivalent rotating $\mathbb{Q}$-automaton. The conversion, which is effective, implies the decidability of equivalence of two-way $\mathbb{Q}$-automata.

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1 Introduction

Rotating automata are a natural model of automata. They were first considered as a restriction of two-way automata [8, 11]. In this model, the automaton reads the input from the left to the right, but when the right end of the input is reached, either the automaton stops, or the tape is rewinded back to the left end of the input. In the Boolean case, rotating automata are as expressive as NFA, but they have been studied since their size can be much smaller [8]. In particular they can compute the intersection of two regular languages with a linear number of states.

In the framework of weighted automata or transducers, rotating automata are more expressive than one-way automata or transducers [9]. In particular, in *rationally additive semirings* [5], they realize Hadamard series, which are the closure of rational series by Hadamard product and Hadamard inverse. This is a sound class of series, and in this framework, algorithms have been defined to convert rotating automata to expressions describing Hadamard series and to synthesize rotating automata from such expressions [4].

Hadamard series over a field have been studied for a long time [12] and it is not surprising that rotating automata can realize them. Nevertheless, fields are not rationally additive semirings, and the potentially infinite number of runs for some input must be handled in a more subtle way to translate rotating automata to Hadamard series. In this paper we prove that the set of Hadamard series over $\mathbb{Q}$ is exactly the behaviour of rotating $\mathbb{Q}$-automata.

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We also prove that two-way $\mathbb{Q}$-automata realize the same class of series. This is an extension of the work of Anselmo and Bertoni on probabilistic two-way automata [1]. In contrast, sweeping transducers with unary outputs are equivalent to rotating transducers and they are weaker than two-way transducers [7]. Notice that the algebraic characterization of series realized by weighted two-way automata is in general not achieved.

In Section 2, we briefly recall the definition of rational series and weighted automata. In Section 3, we define Hadamard series, which form a strictly larger class than rational series, and study different ways to describe them as well as the conversions between these descriptions. Finally we show that it is undecidable whether the description of a Hadamard series, called a Hadamard expression, is well defined, but in the case where two expressions actually define a Hadamard series, their equivalence is decidable.

In Section 4, we introduce rotating $\mathbb{Q}$-automata and we prove that they actually realize $\mathbb{Q}$-Hadamard series and that, conversely, a rotating $\mathbb{Q}$-automaton can be synthesized from any expression denoting a Hadamard series. In particular, rotating $\mathbb{Q}$-automata are strictly more expressive than one-way $\mathbb{Q}$-automata.

Finally, in Section 5 we formally define two-way $\mathbb{Q}$-automata and we show that they can be simulated by rotating $\mathbb{Q}$-automata. As a consequence, the equivalence of two-way $\mathbb{Q}$-automata is decidable.

2 Rational series and weighted automata

For every alphabet A, we denote A^* the free monoid generated by A. The set of formal power series over A^* with coefficients in $\mathbb{Q}$ is denoted $\mathbb{Q}\langle\langle A^*\rangle\rangle$. A *series* s in $\mathbb{Q}\langle\langle A^*\rangle\rangle$ is a mapping from A^* into $\mathbb{Q}$; the *coefficient* of a word w in s is denoted $\langle s, w\rangle$, the coefficient of the empty word is the *constant term* of the series, and s itself is denoted as a formal sum:

$$s = \sum_{w \in A^*} \langle s, w\rangle w. \tag{1}$$

The sum $s + t$ of two series is the pointwise sum. The *Cauchy product* is the extension of the usual polynomial product to series; the series 1 (where all the coefficients are 0 except the constant term equal to 1) is neutral for this product. The Cauchy product is associative and distributes over the sum. Hence $(\mathbb{Q}\langle\langle A^*\rangle\rangle, +, \cdot)$ is a semiring.

In a semiring, the *star* of an element is defined as the sum of the powers of the element, if it exists. Hence, in $(\mathbb{Q}\langle\langle A^*\rangle\rangle, +, \cdot)$, the star of s is defined if the constant term belongs to $]-1; 1[$; it is called the *Kleene star* of s and is denoted s^*. In contrast to the case of the Boolean semiring, the Kleene star is not idempotent in $\mathbb{Q}$, neither in series with coefficients in $\mathbb{Q}$.

The *support* of a series s is the set of words w such that $\langle s, w\rangle \neq 0$. A series with a finite support is a *polynomial*; if every word of A^* belongs to the support, we say that the series has a *full support*.

▶ **Definition 1.** The set $\mathbb{Q}\text{Rat}A^*$ of *rational series* is the closure of polynomials in $\mathbb{Q}\langle\langle A^*\rangle\rangle$ under sum, Cauchy product and Kleene star.

Likewise, the set of $\mathbb{Q}_+$-rational series is the closure of polynomials with positive coefficients under sum, Cauchy product and Kleene star.

The *behaviour* of an automaton is the language or series that describes the result of the automaton on every input. For instance, the behaviour of an NFA is the language of accepted words. The behaviour of a $\mathbb{Q}$-automaton is the series which maps every word to the weight of this word in the automaton.

▶ **Definition 2.** A $\mathbb{Q}$-automaton over an alphabet A is a tuple $\mathcal{A} = (Q, E, I, T)$, where
- Q is a finite set of states;
- I and T are, respectively, the initial and the final weight functions from Q into $\mathbb{Q}$;
- E is the transition weight function from $Q \times A \times Q$ into $\mathbb{Q}$.

The set of initial (*resp.* final) states is the support of I (*resp.* T), *i.e.* the states p such that $I(p) \neq 0$ (*resp.* $T(p) \neq 0$). The set of transitions is the support of E.

As usual, a *run* with label $w = w_1 w_2 \ldots w_k$ in A^* is a sequence of states $(p_i)_{i \in [\![0;k]\!]}$ such that p_0 is an initial state, p_k is a final state, and for every i in $[\![1;k]\!]$, (p_{i-1}, w_i, p_i) is a transition. The *weight* of the run is equal to $I(p_0).\prod_{i=1}^{k} E(p_{i-1}, x_i, p_i).T(p_k)$. The weight $\mathcal{A}(w)$ of a word w in $\mathcal{A}$ is the sum of all runs with label w. The behaviour of $\mathcal{A}$ is the series $\sum_{w \in A^*} \mathcal{A}(w)w$. We say that $\mathcal{A}$ *realizes* the series s if s is the behaviour of $\mathcal{A}$. Like regular languages are the behaviour of NFA, by the Kleene-Schützenberger Theorem [15], $\mathbb{Q}$-rational series are the behaviour of one-way $\mathbb{Q}$-automata.

In the sequel, it will be useful to consider automata with particular properties.

▶ **Lemma 3.** *Every $\mathbb{Q}$-automaton is equivalent to an automaton with a single initial state with initial weight 1 and such that the weight of every transition is positive.*

Proof. Let $\mathcal{A} = (Q, E, I, T)$ be a $\mathbb{Q}$-automaton. We first show that $\mathcal{A}$ is equivalent to an automaton with a single initial state with initial weight 1. An initial state i is added; for every state q and every letter a, we extend E with $E(i, a, q) = \sum_p I(p).E(p, a, q)$, and T with $T(i) = \sum_p I(p).T(p)$. The automaton $\mathcal{A}' = (Q \cup \{i\}, E, \chi_i, T)$, where χ_i is the characteristic function of i, is then equivalent to $\mathcal{A}$.
We assume now that $\mathcal{A} = (Q, E, \chi_i, T)$ has a single initial state with initial weight 1. We define $\mathcal{B} = (Q \times \{-1, 1\}, E', \chi_{(i,1)}, T')$ such that, for every p, q in Q, every letter a, and every i, j in $\{-1, 1\}$,

$$E'((p, i), a, (q, j)) = \begin{cases} i.j.E(p, a, q) & \text{if } i.j.E(p, a, q) > 0, \\ 0 & \text{otherwise;} \end{cases} \qquad T'(p, i) = iT(p). \tag{2}$$

There is a natural bijection between runs of $\mathcal{A}$ and runs of $\mathcal{B}$ such that corresponding runs have the same weights, hence $\mathcal{B}$ is equivalent to $\mathcal{A}$. ◀

3 Hadamard series

3.1 Hadamard operations

We consider now the Hadamard product of two series: if s and t are two series in $\mathbb{Q}\langle\langle A^* \rangle\rangle$, then for every word w in A^*, $\langle s \odot t, w \rangle = \langle s, w \rangle.\langle t, w \rangle$. This product is also called the pointwise product. The neutral for this product is the series where the coefficient of every word is 1; this series is the characteristic series of A^* and is denoted itself A^*. Like in any commutative semiring, the Hadamard product preserves the rationality of series.

▶ **Proposition 4** ([16]). *The Hadamard product of two $\mathbb{Q}$-rational series is a $\mathbb{Q}$-rational series.*

This result is constructive (*cf.* for instance [13]): if two rational series are realized by two $\mathbb{Q}$-automata, their Hadamard product is realized by the *direct product* of the $\mathbb{Q}$-automata. As usual, the Hadamard power of a series can be defined from the Hadamard product. The 0-th Hadamard power of every series is the neutral for the Hadamard product, that is A^*. If

s is a series with each coefficient in $]-1;1[$, then the sum of the Hadamard powers of s is defined; it is called the Hadamard iteration of s and denoted $s^{\circledast}$.

Likewise, if the support of s is full, the Hadamard inverse of s is defined, and for every series t, the Hadamard quotient of t by s is defined as

$$\forall w \in A^*,\ \langle \odot\frac{t}{s}, w\rangle = \frac{\langle t, w\rangle}{\langle s, w\rangle}. \tag{3}$$

With this notation, the *Hadamard inverse* of s is $\odot\frac{A^*}{s}$.

▶ **Definition 5.** The set $\mathbb{Q}\mathrm{Had}A^*$ of Hadamard series over $\mathbb{Q}$ is the set of series which are equal to the Hadamard quotient of two rational series.

The set of Hadamard series is closed under sum and Hadamard product:

$$\odot\frac{t}{s} \odot \odot\frac{t'}{s'} = \odot\frac{t\odot t'}{s\odot s'}, \qquad \odot\frac{t}{s} + \odot\frac{t'}{s'} = \odot\frac{t\odot s' + t'\odot s}{s\odot s'}. \tag{4}$$

If x is a rational number in $]-1;1[$, the sum of powers of x is a rational number $x^* = (1-x)^{-1}$. Since the Hadamard product is a pointwise operation, this extends to formal power series: if every coefficient of a series s is in $]-1;1[$, then

$$s^{\circledast} = \odot\frac{A^*}{A^* - s}. \tag{5}$$

Therefore, series generated from rational series using sum, Hadamard product, and Hadamard iteration are Hadamard series.
Conversely, the Hadamard inverse can also be computed from the Hadamard iteration. To this end, for every rational number λ, we define $\mathsf{Geom}(\lambda)$ as the series over A^* such that, for every word w, $\langle \mathsf{Geom}(\lambda), w\rangle = \lambda^{|w|+1}$.

▶ **Lemma 6.** *Let t be a $\mathbb{Q}$-rational series such that, for every word w, $\langle t, w\rangle > 0$.*[1] *There exists a positive rational number λ such that $t' = A^* - t \odot \mathsf{Geom}(\lambda)$ is a $\mathbb{Q}_+$-rational series where every coefficient is in $[0;1[$.*

Proof. By Lemma 3, there exists a $\mathbb{Q}$-automaton $\mathcal{A} = (Q, E, I, T)$ that realizes t, with a single initial state i with weight 1 and positive transitions.

Let $M = \max(\max_{p,a} \sum_q E(p,a,q), \max_p T(p))$ and $\lambda = 1/M$. We define $\mathcal{B} = (Q \cup \{\bot\}, E', I, T')$, where $\bot$ is a fresh state and, for every p, q in $Q \cup \{\bot\}$ and every letter a,

$$E'(p,a,q) = \begin{cases} E(p,a,q).\lambda & \text{if } p,q \in Q,\\ 1 - \sum_{r\in Q} E(p,a,r).\lambda & \text{if } p \in Q \text{ and } q = \bot,\\ 1 & \text{if } p = q = \bot,\\ 0 & \text{otherwise.}\end{cases} \tag{6}$$

Depending on T', the automaton $\mathcal{B}$ may realize different series:

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[1] In particular, t has full support.

(a) $T'(p) = T(p).\lambda$ for every p in Q and $T'(\bot) = 0$; then there is a natural bijection between runs of $\mathcal{B}$ and runs of $\mathcal{A}$; the weight of each run with label w in $\mathcal{B}$ is equal to the weight of the corresponding run in $\mathcal{A}$ multiplied by $\lambda^{|w|+1}$. Hence $\langle \mathcal{B}, w\rangle = \langle t, w\rangle.\lambda^{|w|+1}$.

(b) $T'(p) = 1$ for every p in $Q \cup \{\bot\}$; for every letter a and every state p, the sum of weights of outgoing transitions from p with label a is equal to 1. As there is only one initial state with weight 1, the weight of every word in $\mathcal{B}$ is 1.

(c) $T'(p) = 1 - T(p).\lambda$ for every p in Q and $T'(\bot) = 1$. In this case, the behaviour is the difference between the two previous behaviours and $\langle \mathcal{B}, w\rangle = 1 - \langle t, w\rangle.\lambda^{|w|+1}$, which is in $[0; 1[$.

In the last case, every weight in $\mathcal{B}$ is positive; hence $\mathcal{B}$ is a $\mathbb{Q}_+$-automaton which realizes a $\mathbb{Q}_+$-rational series. ◀

▶ **Proposition 7.** *For every $\mathbb{Q}$-rational series s with full support, there exists a positive rational number λ such that the Hadamard iteration of $A^* - s \odot s \odot \mathsf{Geom}(\lambda)$ is defined and*

$$\odot\frac{A^*}{s} = (A^* - s \odot s \odot \mathsf{Geom}(\lambda))^{\circledast} \odot s \odot \mathsf{Geom}(\lambda). \tag{7}$$

Proof. If s is a $\mathbb{Q}$-rational series with full support, $t = s \odot s$ is also rational with positive coefficients. Hence, by Lemma 6, there exists λ such that $1 - \langle s, w\rangle^2.\lambda^{|w|+1}$ is in $[0; 1[$ for every word w, and

$$(A^* - t \odot \mathsf{Geom}(\lambda))^{\circledast} \odot s \odot \mathsf{Geom}(\lambda) = \odot\frac{A^*}{t \odot \mathsf{Geom}(\lambda)} \odot s \odot \mathsf{Geom}(\lambda) = \odot\frac{A^*}{s}. \tag{8}$$

◀

Finally, our definition of Hadamard series is consistent with the definition of [4]:

▶ **Corollary 8.** *$\mathbb{Q}\mathrm{Had}A^*$ is the closure of $\mathbb{Q}\mathrm{Rat}A^*$ under Hadamard product, sum, and Hadamard iteration.*

▶ **Proposition 9.** *There exist $\mathbb{Q}$-Hadamard series which are not rational.*

Proof. Let t be the series such that, for every k in $\mathbb{N}$, $\langle t, a^k\rangle = k + 1$. It is a rational series: $t = a^*.a^*$. Let s be the Hadamard inverse of t: for every k in $\mathbb{N}$, $\langle s, a^k\rangle = \frac{1}{k+1}$. The coefficients of a rational series over one variable satisfy a linear recurrence relation (*cf.* for instance [2]). Therefore s is not rational. ◀

▶ **Proposition 10.** *The set $\mathbb{Q}\mathrm{Had}A^*$ is not closed under Cauchy product.*

Proof. Let s be the series defined in the proof of Proposition 9. We consider the Cauchy product of s with itself; for large integers k, it holds:

$$\langle s.s, a^k\rangle = \frac{2}{k+2}\sum_{i=0}^{k}\frac{1}{i+1} \sim \frac{2\ln k}{k}. \tag{9}$$

If $s.s$ is a Hadamard series, it is the Hadamard quotient of two rational series x and y. If x (*resp.* y) is rational, its coefficients satisfy a linear recurrence relation. The ratio of $\langle x, a^k\rangle$ and $\langle y, a^k\rangle$ can not be equivalent to $\frac{2\ln k}{k}$. ◀

▶ Remark. By Lemma 6, if a series is the Hadamard quotient of two $\mathbb{Q}_+$-rational series, it is also in the closure of $\mathbb{Q}_+\mathrm{Rat}A^*$ under sum, Hadamard product and Hadamard iteration.

Nevertheless, it is unknown whether a series in this closure is always the Hadamard quotient of two $\mathbb{Q}_+$-rational series.

3.2 Validity and equivalence

Rational series over $\mathbb{Q}$ are represented by rational expressions. It is well known that such an expression may be non valid if the star operator is applied to a subexpression whose interpretation is a series s on which the star is not defined. Nevertheless, the star can be applied if and only if the constant term of s is in $]-1;1[$; this condition is decidable on rational expressions; thus it is decidable whether a rational expression is valid.

There are two ways to describe $\mathbb{Q}$-Hadamard series. First, a $\mathbb{Q}$-Hadamard series is a quotient of s by t where s and t are two $\mathbb{Q}$-rational series; hence, it can be described as a pair of rational expressions. Such a representation is valid if and only if both rational expressions are valid and t has full support.

It is undecidable whether a $\mathbb{Q}$-rational series has full support (*cf.* [2, 13]). Notice that it is decidable whether a $\mathbb{Q}_+$-rational series has full support.

▶ **Proposition 11.** *It is undecidable whether the representation of a $\mathbb{Q}$-Hadamard series as a pair of $\mathbb{Q}$-rational series is valid.*
If the denominator series is $\mathbb{Q}_+$-rational, the validity of the representation is decidable.

Another description of $\mathbb{Q}$-Hadamard series consists in the application of pointwise operators (sum, Hadamard product, Hadamard iteration) to $\mathbb{Q}$-rational expressions. This leads to $\mathbb{Q}$-Hadamard expressions as defined in [4]. The validity of $\mathbb{Q}$-Hadamard expressions is undecidable, as well as the validity of $\mathbb{Q}_+$-Hadamard expressions.

▶ **Proposition 12.** *It is undecidable whether a $\mathbb{Q}_+$-Hadamard expression is valid.*

Proof. It is undecidable whether a probabilistic automaton with weights in $\{0; \frac{1}{2}; 1\}$ accepts some word with a probability larger than or equal to $\frac{1}{2}$ [10]. Let s be the $\mathbb{Q}_+$-rational series which is the behaviour of this automaton. It is undecidable whether $(2s)^{\circledast}$ is defined. ◀

Assume now that $h_1 = (s_1, t_1)$ and $h_2 = (s_2, t_2)$ are two valid representations of $\mathbb{Q}$-Hadamard series as pairs of $\mathbb{Q}$-rational series. Then h_1 and h_2 represent the same series if and only if $s_1 \odot t_2 = s_2 \odot t_1$. The Hadamard product of two $\mathbb{Q}$-rational series is a computable $\mathbb{Q}$-rational series (Prop. 4), and the equivalence of $\mathbb{Q}$-rational series is decidable (*cf.* [2, 13]).

If $\mathbb{Q}$-Hadamard series are described by $\mathbb{Q}$-Hadamard expressions, these descriptions can be converted into pairs of $\mathbb{Q}$-rational series. Actually, using Equations (4) and (5), every $\mathbb{Q}$-Hadamard expression can be turned into a pair of expressions which are combinations of $\mathbb{Q}$-rational expressions connected with sum and Hadamard product operators. Each of these expressions denotes a $\mathbb{Q}$-rational series.

▶ **Theorem 13.** *The equivalence of valid descriptions of $\mathbb{Q}$-Hadamard series is decidable.*

3.3 Extension to real and complex numbers

All results presented in this section apply directly to series over $\mathbb{R}$, up to the calculability of operations with real numbers.

To apply them on series over $\mathbb{C}$, we need to consider the complex conjugacy. The conjugacy commutes with rational operations; hence, if s is a $\mathbb{C}$-rational series, its complex conjugacy $\overline{s}$ is also a $\mathbb{C}$-rational series. Thus, Proposition 7 can be extended.

▶ **Proposition 14.** *For every $\mathbb{C}$-rational series s with full support, there exists a positive real number λ such that the Hadamard iteration of $A^* - s \odot \overline{s} \odot \mathsf{Geom}(\lambda)$ is defined and*

$$\odot\frac{A^*}{s} = (A^* - s \odot \overline{s} \odot \mathsf{Geom}(\lambda))^{\circledast} \odot \overline{s} \odot \mathsf{Geom}(\lambda). \tag{10}$$

Actually, $s \odot \overline{s}$ is a $\mathbb{C}$-rational series whose coefficients are positive real numbers, hence, it is a $\mathbb{R}$-rational series [6] and Lemma 6 applies.

4 Weighted rotating automata

A rotating $\mathbb{Q}$-automaton is a $\mathbb{Q}$-automaton that can read its input from left to right several times. To this end, it is endowed with transitions with a special label $\mathbf{r}$, which is not in A.

A run of a rotating $\mathbb{Q}$-automaton is accepting for a word w in A^* if the label of the run is in $(w\mathbf{r})^*w$.

For every word w, if the sum of the weights of the accepting runs for w is defined, the weight $\mathcal{A}(w)$ of w in $\mathcal{A}$ is equal to this sum. The automaton is *valid* if the weight of every word in A^* is defined.

▶ Remark. Every one-way $\mathbb{Q}$-automaton over an alphabet A can be considered as a rotating $\mathbb{Q}$-automaton without any transition with label $\mathbf{r}$. In this case, every run accepting a word w has label w, hence, its behaviour as a rotating $\mathbb{Q}$-automaton is the same as its behaviour as a one-way $\mathbb{Q}$-automaton.

▶ **Proposition 15.** *The behaviour of a valid rotating $\mathbb{Q}$-automaton is a $\mathbb{Q}$-Hadamard series.*

Proof. $\mathbb{Q}_+ \cup \{\infty\}$ is a *rationally additive semiring*: the star of every element is defined. By [9], every rotating $(\mathbb{Q}_+ \cup \{\infty\})$-automaton realizes a Hadamard series. Thus every valid rotating $\mathbb{Q}_+$-automaton realizes a $\mathbb{Q}$-Hadamard series. If $\mathcal{A}$ is a valid rotating $\mathbb{Q}$-automaton, its behaviour s can be split into $s = s_+ - s_-$, where s_+ and s_- are realized by rotating $\mathbb{Q}_+$-automata. This construction is similar to the one described in the proof of Lemma 3. Since s_+ and s_- are $\mathbb{Q}$-Hadamard series, the behaviour of $\mathcal{A}$ is a $\mathbb{Q}$-Hadamard series. ◀

▶ Remark. In the definition of validity, it is not assumed that the potentially infinite sums of weights are in $\mathbb{Q}$ (they might be in $\mathbb{R}$); it appears that these sums can be computed through the rational operations (sum, product and star), hence, if they are defined, they belong to $\mathbb{Q}$.

▶ Remark. Clearly, Proposition 15 extends to $\mathbb{R}$. It also holds for rotating $\mathbb{C}$-automata. Using a construction similar to the construction in the proof of Lemma 3 (with $Q' = Q \times \{-1, 1, i, -i\}$), one shows that every series s with coefficients in $\mathbb{C}$ can be split into four series with positive real coefficients: $s = s_{\text{re}+} - s_{\text{re}-} + i.s_{\text{im}+} - i.s_{\text{im}-}$.

The following proposition is a corollary of Proposition 15 and Theorem 13, since every rotating $\mathbb{Q}$-automaton can be turned into a $\mathbb{Q}$-Hadamard expression.

▶ **Proposition 16.** *The equivalence of valid rotating $\mathbb{Q}$-automata is decidable.*

We prove now that every $\mathbb{Q}$-Hadamard series can be realized by a rotating $\mathbb{Q}$-automaton. If two automata $\mathcal{A}$ and $\mathcal{B}$ respectively realize series s and t, it is straightforward that the union of $\mathcal{A}$ and $\mathcal{B}$ realizes the series $s + t$. Likewise, $s \odot t$ is realized by the automaton $\mathcal{A} \odot \mathcal{B}$ based on the union of $\mathcal{A}$ and $\mathcal{B}$, where

- for every final state p of $\mathcal{A}$ and every initial state q of $\mathcal{B}$ there is a transition with label $\mathbf{r}$ and weight $T_{\mathcal{A}}(p)I_{\mathcal{B}}(q)$;
- the initial function is restricted to states of $\mathcal{A}$ and the final function to states of $\mathcal{B}$.

In order to realize $s^{\circledast}$, a similar construction could be applied to $\mathcal{A}$ by adding transitions with label $\mathbf{r}$ from final states to initial states. This construction may lead to an infinite number of runs accepting a given word w, and, even if the weight of w is in $]-1;1[$, there is no guarantee that the sum of all these runs is defined. Therefore, we consider that $\mathbb{Q}$-Hadamard series are Hadamard quotients of $\mathbb{Q}$-rational series and we prove that such a quotient can be realized by a rotating $\mathbb{Q}$-automaton.

▶ **Proposition 17.** *Let s be a $\mathbb{Q}$-rational series with full support. There exists a valid rotating $\mathbb{Q}$-automaton $\mathcal{A}$ such that the behaviour of $\mathcal{A}$ is the Hadamard inverse of s.*

Proof. By Lemma 6, there exists a positive rational number λ such that $t = A^* - s \odot s \odot \mathsf{Geom}(\lambda)$ is a series with coefficients in $[0;1[$ that can be realized by a $\mathbb{Q}_+$-automaton $\mathcal{A} = (Q, E, I, T)$. Without loss of generality, we assume that $\mathcal{A}$ has a single initial state i with weight 1. Let $\mathcal{B}$ be the rotating $\mathbb{Q}$-automaton built from $\mathcal{A}$ by:

- adding a transition from each final state p to state i with label $\mathbf{r}$ and weight $T(p)$;
- adding a new state $\bot$, which is initial and final with weight 1, and such that there is a loop on the state with weight 1 for every label except $\mathbf{r}$.

Every run with label w in $\mathcal{B}$ is either a circuit in $\bot$ with weight 1, or a concatenation of runs in $\mathcal{A}$ (glued with $\mathbf{r}$-transitions). The weight of every run in $\mathcal{A}$ is positive and for every word w, the sum of the weights of all runs with label w in $\mathcal{A}$ is smaller than 1. Hence, the weight of w in $\mathcal{B}$ is defined: it is the star of the weight of w in $\mathcal{A}$. Therefore, $\mathcal{B}$ is valid and its behaviour is $(A^* - s \odot s \odot \mathsf{Geom}(\lambda))^{\circledast} = \odot\frac{A^*}{s \odot s \odot \mathsf{Geom}(\lambda)}$. It is then easy to build a rotating automaton that realizes $(A^* - s \odot s \odot \mathsf{Geom}(\lambda))^{\circledast} \odot s \odot \mathsf{Geom}(\lambda) = \odot\frac{A^*}{s}$. ◀

▶ **Theorem 18.** *The set of series which are behaviours of valid rotating $\mathbb{Q}$-automata is the set of $\mathbb{Q}$-Hadamard series.*

Rotating $\mathbb{Q}$-automata and $\mathbb{Q}$-Hadamard expressions are therefore equivalent. Proof of Proposition 11 applies on rotating $\mathbb{Q}$-automata and their validity is therefore undecidable.

▶ **Proposition 19.** *The validity of a rotating $\mathbb{Q}$-automaton is undecidable.*

A sweeping $\mathbb{Q}$-automaton is an automaton that can read its input from left to right and right to left, but can only change the direction of the reading head on one of the endmarkers. Every rotating $\mathbb{Q}$-automaton can be simulated by a sweeping $\mathbb{Q}$-automaton. Conversely, like in any commutative semiring, sweeping $\mathbb{Q}$-automata can be simulated by rotating $\mathbb{Q}$-automata.

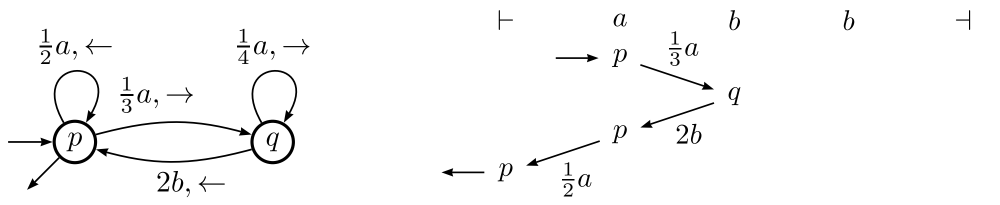
Hence, the validity of sweeping $\mathbb{Q}$-automata is undecidable, but the equivalence of valid sweeping $\mathbb{Q}$-automata is decidable.

5 From two-way to rotating automata

In this section, we formally define two-way $\mathbb{Q}$-automata and we show that they can be simulated by rotating $\mathbb{Q}$-automata. The proof presented here is inspired by the work in [1] on two-way probabilistic automata. Starting with a two-way $\mathbb{Q}$-automaton, for every word w, we define a matrix M_w such that the weight of w in the two-way $\mathbb{Q}$-automaton is an entry of M_w^*, the sum of iterated powers of M_w. We show that since M_w^* is the inverse of $\mathsf{Id} - M_w$, the weight of w can be computed as the ratio of an entry of the matrix of cofactors of $\mathsf{Id} - M_w$ by the determinant of $\mathsf{Id} - M_w$. We prove that both the entry of the matrix of cofactors and the determinant are rational power series (when w spans over A^*), therefore the behaviour of the two-way $\mathbb{Q}$-automaton is the Hadamard quotient of two rational series, *i.e.* a Hadamard series.

5.1 Weighted two-way automata

There are different models of two-way automata. If reading the left and right endmarks is allowed, they are all equivalent; depending on the model, the computation can start (*resp.* stop) at the beginning, the middle or the end of the word, and the move of the reading head can be performed in each state or during each transition traversal.



(a) The automaton $\mathcal{A}_1$. **(b)** A run over the word abb.

Figure 1 A two-way automaton and one run over the word abb.

For convenience in the conversion from two-way $\mathbb{Q}$-automata to rotating $\mathbb{Q}$-automata, we use in this paper two-way $\mathbb{Q}$-automata where computations start and stop at the left end of the word.

Formally, if A is an alphabet, a two-way $\mathbb{Q}$-automaton over A is a tuple $\mathcal{A} = (Q, E, I, T)$, where

- Q is the finite set of *states*;
- E is the *transition* function: $Q \times (A \cup \{\vdash, \dashv\} \times Q \longrightarrow \mathbb{Q}$;
- I and T are respectively the initial and final functions in $Q \longrightarrow \mathbb{Q}$.

A transition is an element t of $Q \times (A \cup \{\vdash, \dashv\} \times Q$ such that $E(t)$ is different from 0. Likewise a state p is initial if $I(p) \neq 0$; it is final if $T(p) \neq 0$.

The value in $\{-1, +1\}$ on each transition shows the direction of the move of the head of the automaton on the current letter. If it is equal to $+1$, the head moves forward and we can denote it by $\rightarrow$, if it is equal to -1, the head moves backward and it can be denoted $\leftarrow$. There is no transition with backward move and label $\vdash$; likewise, there is no transition with forward move and label $\dashv$.

A *path* compatible with a word $w = w_1 \ldots w_k$ in A^* is a sequence of consecutive transitions $\rho = (p_{i-1}, x_i, m_i, p_i)_{i \in [\![1;\ell]\!]}$ such that there exists a function $\mathsf{pos} : [\![0;\ell]\!] \longrightarrow [\![0;k{+}1]\!]$ satisfying

- for every $i > 0$, $\mathsf{pos}(i) = \mathsf{pos}(i-1) + m_i$;
- for every $i \in [\![1;\ell]\!]$, if $\mathsf{pos}(i-1) = 0$, $x_i = \vdash$, and if $\mathsf{pos}(i-1) = k+1$, $x_i = \dashv$, otherwise $x_i = w_{\mathsf{pos}(i-1)}$.

If furthermore, p_0 is initial, $\mathsf{pos}(0) = 1$, p_ℓ is final, and $\mathsf{pos}(\ell) = 0$, then ρ is a *run* (over w).

The weight of such a run is $I(p_0).\left(\prod_{i=1}^{\ell} E(p_{i-1}, x_i, m_i, p_i)\right).T(p_\ell)$.

Notice that with this model, a run over the empty word w can exist; it contains at least one transition. For instance, if p is an initial state, q a final state and $(p, \dashv, \leftarrow, q)$ is a transition, there is a run over the empty word formed with this single transition.

The *weight* computed by $\mathcal{A}$ on a word w in A^* is the sum (if defined) of the weights of all runs over w. The automaton $\mathcal{A}$ is valid if for every word w, the sum of the weights of runs over w is defined.

▶ **Example 20.** Let $\mathcal{A}_1$ be the two-way automaton of Figure 1a. A run of this automaton over the word abb is described in Figure 1b; it is the path $(p, a, \rightarrow, q)(q, b, \leftarrow, p)(p, a, \leftarrow, p)$: it starts at position 1 in an initial state, and ends at position 0 in a final state.

We use a classical extension of the model of adjacency matrices for graphs. We suppose from now that $Q = [\![1;n]\!]$ where n is a positive integer. For every letter a in $A \cup \{\vdash, \dashv\}$ we define $F(a)$ (*resp.* $B(a)$) as the matrix of size $n \times n$ such that $F(a)_{p,q} = E(p, a, \rightarrow, q)$ (*resp.* $B(a)_{p,q} = E(p, a, \leftarrow, q)$).

These matrices represent paths of length 1. In graphs or one-way automata, paths of length k are represented by the k-th power of the adjacency matrix. In the case of a two-way automaton, the position of the head (given by the function pos) must be taken into account, and since there may exist runs with arbitrary large length for a given input, all the powers of the suitable matrix must be considered.

To this end, we define the star of a (square) matrix M as the (infinite) sum of its powers, if it is defined. It satisfies $M^* = \mathsf{Id} + M.M^*$, where Id is the identity matrix, and it can be inductively computed (cf [3]):

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^* = \begin{bmatrix} (A^*BD^*C)^*A^* & A^*(BD^*CA^*)^*BD^* \\ D^*C(A^*BD^*C)^*A^* & D^* + D^*CA^*(BD^*CA^*)^*BD^* \end{bmatrix}. \tag{11}$$

We fix from now w as a word with length k: $w = w_1 \dots w_k$. To study paths involved in the runs over w, we consider block matrices with dimension $(k+2)\times(k+2)$ where every entry is itself a $n \times n$ matrix. We assume that indices of blocks are integers in $[\![0; k+1]\!]$. If X is such a matrix, $X_{i,j}$ is a matrix and $(X_{i,j})_{p,q}$ represents some subpaths of runs over w which start in position i and state p, and end in position j and state q.

We first consider the matrix M_w that represents subpaths of runs over w with length 1:

$$M_w = \left[\begin{array}{c|c|cc|c} 0 & F(\vdash) & 0 & \dots & 0 \\ \hline B(w_1) & 0 & F(w_1) & \dots & \vdots \\ \hline 0 & B(w_2) & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & 0 & F(w_k) \\ \hline 0 & \dots & 0 & B(\dashv) & 0 \end{array}\right]. \tag{12}$$

During the computation, if the automaton reads the letter w_i (in position i) and follows a forward transition, the head moves to position $i+1$. Hence, for every i, the matrix $F(w_i)$ is the block $(i, i+1)$ of M_w; likewise the matrix $B(w_i)$ is the block $(i, i-1)$, $F(\vdash)$ is the block $(0,1)$, and $B(\dashv)$ is the block $(k+1, k)$.

We show that $(M_w^*)_{i,j}$ represent all the subpaths of runs over w which start in position i and end in position j.

For r in $[\![0; k+1]\!]$, let $C^{(r)}$ be the $(r+1)\times(r+1)$ block matrix where $C^{(r)}_{i,j}$ is a $n \times n$ matrix such that the entry at position (p,q) is the sum of the weights of the paths compatible with w from position i and state p to position j and state q with no position larger than r.

▶ **Lemma 21.** *With the notations above, $M_w^* = C^{(k+1)}$.*

The proof is by induction on k and on the star of the restriction of M_w to the $k+1$ first row blocks and the $k+1$ column blocks.

On a two-way automaton, the initial and final functions can respectively be seen as row and column vectors in $\mathbb{Q}^n$. We let L_i denote the $1 \times (k+1)$ block matrix where every block is null, except the i-th block, which is the identity matrix. Proposition 22 follows then from Lemma 21:

▶ **Proposition 22.** *The weight of w computed by $\mathcal{A}$ is equal to $I.L_2.M_w^*.{}^tL_1.T$.*

If $\mathcal{A}$ is valid, the star of M_w is defined for every word w, hence $\mathsf{Id} - M_w$ is invertible. Otherwise, there would exist a non zero vector v such that $M_w.v = v$; since M_w^* is defined, $M_w^*.v = (\mathsf{Id} + M_w^*.M_w).v = v + M_w^*.v$, and $v = 0$ which is a contradiction. Therefore, for every word w,

$$\langle \mathcal{A}, w\rangle = I.L_2.(\mathsf{Id} - M_w)^{-1}.{}^tL_1.T = \frac{1}{\det(\mathsf{Id} - M_w)} \sum_{p,q\in Q} I_p.(\operatorname{adj}(\mathsf{Id} - M_w))_{n+p,q}.T_q, \quad (13)$$

where $\operatorname{adj}(X)$ is the adjugate matrix of X and $\det(X)$ is the determinant of X. For every p, q in Q, let $\alpha_{p,q}$ be the series defined by $\langle \alpha_{p,q}, w\rangle = (\operatorname{adj}(\mathsf{Id} - M_w))_{n+p,q}$ and let δ be the series defined by $\langle \delta, w\rangle = \det(\mathsf{Id} - M_w)$. We show in the next part that all these series are $\mathbb{Q}$-rational.

5.2 Inductive computation of a determinant of a tridiagonal block matrix

Inductive computations of determinants of tridiagonal block matrices have already been studied [14]. We give here a new presentation of this computation in order to show that it can be realized by a (one-way) $\mathbb{Q}$-automaton.

Let n and k be two positive integers. We consider two families $(A_i)_{i\in[\![1;k]\!]}$ and $(A'_i)_{i\in[\![0;k]\!]}$ of matrices in $\mathbb{Q}^{n\times n}$.

For every r in $[\![0;k]\!]$, we consider the matrix $N^{(r)}$ in $\mathbb{Q}^{(r+1)n\times(r+2)n}$ defined as:

$$N^{(r)} = \begin{bmatrix} A'_r & \mathsf{Id}_n & A_r & & & 0 \\ 0 & A'_{r-1} & \mathsf{Id}_n & A_{r-1} & & \\ & & \ddots & \ddots & \ddots & \\ & & & A'_1 & \mathsf{Id}_n & A_1 \\ 0 & & & & A'_0 & \mathsf{Id}_n \end{bmatrix}, \quad (14)$$

where Id_n is the identity matrix with size n.

We introduce now some notations.

- For every set X and every positive integer i, $\mathcal{P}_iX$ denotes the set of subsets of X with i elements.
- If M is a matrix, the determinant of M is denoted $|M|$.
- If X is a set of indices, $\overline{X}$ is its complementary set, and ΣX is the sum of elements of X.
- If X and Y are two subsets of indices of a matrix M, $M_{X\times Y}$ is the restriction of M to rows in X and columns in Y.
- For every C in $\mathcal{P}_n[\![1;2n]\!]$, we let $G^{(r)}(C)$ denote the square matrix $N^{(r)}_{[\![1;(r+1)n]\!]\times\overline{C}}$.
- For every C, D in $\mathcal{P}_n[\![1;2n]\!]$,

$$K^{(r)}(C,D) = \begin{bmatrix} A'_r & \mathsf{Id}_n & A_r & 0 \\ 0 & A'_{r-1} & \mathsf{Id}_n & A_{r-1} \end{bmatrix}_{[\![1;2n]\!]\times(\overline{C}\cap[\![1;2n]\!])\cup(D+2n)}. \quad (15)$$

▶ **Lemma 23.** *For every $r \in [\![2;k]\!]$, for every C in $\mathcal{P}_n[\![1;2n]\!]$,*

$$\left|G^{(r)}(C)\right| = \sum_{D\in\mathcal{P}_n[\![1;2n]\!]} (-1)^{\mathsf{sig}(D+n)}.\left|K^{(r)}(C,D)\right|.\left|G^{(r-2)}(D)\right| \quad (16)$$

where $\mathsf{sig}(D+n) = \dfrac{n(n+1)}{2} + \Sigma D$.

The proof is an application of the Laplace expansion of the determinant:

$$\forall N \in \mathbb{Q}^{d\times d}, \forall X \subseteq [\![1;d]\!], \qquad |N| = \sum_{Y\in\mathcal{P}_{|X|}[\![1;d]\!]} (-1)^{\Sigma X+\Sigma Y} \left|N_{X\times Y}\right| . \left|N_{\overline{X}\times\overline{Y}}\right| . \tag{17}$$

We apply now Lemma 23 to the computation of the determinant of $\mathsf{Id} - M_w$. For every matrix X, we let ${}^{\circlearrowleft}X$ denote the rotation of X by half-turn; notice that $|X| = |{}^{\circlearrowleft}X|$. For every word $w = w_1 \dots w_k$, we set $A_0 = -{}^{\circlearrowleft}B(\dashv)$, $A'_{k+1} = -{}^{\circlearrowleft}F(\vdash)$, $A_i = -{}^{\circlearrowleft}B(w_i)$, and $A'_i = -{}^{\circlearrowleft}F(w_i)$, for every i in $[\![1;k]\!]$, then $G^{(k+1)}([\![1;n]\!])$ is equal to ${}^{\circlearrowleft}(\mathsf{Id} - M_w)$.

5.3 The transformation automaton

We describe the (one-way) automaton that computes $\det(\mathsf{Id} - M_w)$, based on the induction described in Lemma 23. The set of states is $\{i\} \cup \mathcal{P}_n[\![1;2n]\!] \cup A \times \mathcal{P}_n[\![1;2n]\!]$. The order of the induction is 2; there are two different kinds of initial states, depending on the parity of the length of the input.

- State i is initial with weight 1 and, for every D in $\mathcal{P}_n[\![1;2n]\!]$ and every a in A, there is a transition from i to D with label a and weight

$$\left| \begin{bmatrix} -{}^{\circlearrowleft}F(a) & \mathsf{Id}_n & -{}^{\circlearrowleft}B(a) \\ 0 & -{}^{\circlearrowleft}F(\vdash) & \mathsf{Id}_n \end{bmatrix}_{[\![1;2n]\!]\times\overline{D}} \right| . \tag{18}$$

- For every D in $\mathcal{P}_n[\![1;2n]\!]$, D is initial with weight $\left| \begin{bmatrix} -{}^{\circlearrowleft}F(\vdash) & \mathsf{Id}_n \end{bmatrix}_{[\![1;n]\!]\times\overline{D}} \right|$, and for every a in A, there is a transition from D to (a, D) with label a and weight $(-1)^{\mathsf{sig}(D+n)}$.
- For every C, D in $\mathcal{P}_n[\![1;2n]\!]$ and every a, b in A, there is a transition from (a, D) to C with label b and weight

$$\left| \begin{bmatrix} -{}^{\circlearrowleft}F(b) & I_n & -{}^{\circlearrowleft}B(b) & 0 \\ 0 & -{}^{\circlearrowleft}F(a) & \mathsf{Id}_n & -{}^{\circlearrowleft}B(a) \end{bmatrix}_{[\![1;2n]\!]\times(\overline{C}\cap[\![1;2n]\!])\cup(D+2n)} \right| . \tag{19}$$

- Every state (a, D) is final with weight $\left| \begin{bmatrix} \mathsf{Id}_n & -{}^{\circlearrowleft}B(\dashv) & 0 \\ -{}^{\circlearrowleft}F(a) & \mathsf{Id}_n & -{}^{\circlearrowleft}B(a) \end{bmatrix}_{[\![1;2n]\!]\times([\![1;n]\!]\cup D+n)} \right|$.

By Lemma 23, this automaton computes $\det(\mathsf{Id} - M_w)$ for every word w; it realizes the series δ which is thus ℚ-rational.

Likewise, for every p, q in $[\![1;n]\!]$, $(\mathrm{adj}(\mathsf{Id} - M_w))_{p+n,q}$ is equal to the determinant of $C_w^{(q,p+n)}$, which is the matrix $\mathsf{Id} - M_w$ where every coefficient of the q-th row and every coefficient of the $p+n$-th column is replaced by 0, except the coefficient in $(q, p+n)$ which is replaced by 1. The determinant of $C_w^{(q,p+n)}$ can be computed with the same induction as for $\det(\mathsf{Id} - M_w)$ with different initial conditions.

Hence, for every p, q in Q, the series $\alpha_{p,q}$ is rational; so is the series $\alpha = \sum_{p,q\in Q} I_p.\alpha_{p,q}.T_p$. Finally, the series realized by the two-way ℚ-automaton is the Hadamard quotient of two rational series. It is therefore a Hadamard series that can be realized by a rotating ℚ-automaton.

▶ **Theorem 24.** *The set of series realized by two-way ℚ-automata is exactly the set* $\mathbb{Q}\mathrm{Had}A^*$ *of series realized by rotating ℚ-automata.*

Notice that the conversion of two-way ℚ-automata to rotating ℚ-automata is effective, hence the decidability of equivalence of rotating ℚ-automata extends to two-way ℚ-automata.

▶ **Theorem 25.** *The equivalence of valid two-way ℚ-automata is decidable.*

6 Conclusion

The results presented in this paper can be extended to other fields. Section 5 actually shows that the behaviour of a two-way automaton over a field is the Hadamard quotient of two rational series. It extends to any field. Notice that the definition of the behaviour of a two-way automaton involves infinite sums; hence, a structure (topology for instance) is required to handle these sums, and this structure must transfer to matrices to define the star of a matrix.

To show that two-way automata are equivalent to rotating automata, it must be proved that every Hadamard quotient of two rational series can be realized by a rotating automaton (Theorem 18), and Proposition 7 is crucial in this proof. This proposition applies to $\mathbb{Q}$ and $\mathbb{R}$; it can also be used to prove Theorem 18 in $\mathbb{C}$. For other fields, dedicated proofs should be provided.

The conversion from two-way $\mathbb{Q}$-automata to rotating $\mathbb{Q}$-automata heavily relies on the inversion of a matrix describing the computations in the two-way $\mathbb{Q}$-automaton. This inversion is considered as the quotient between some coefficients of the adjugate matrix and the determinant and we exhibit one-way $\mathbb{Q}$-automata that realize this computation. It is still an open question to find a more combinatorial argument to the equivalence between the two models. It could help in characterizing semirings where two-way and rotating automata are equivalent. It could also lead to more efficient algorithms to convert two-way $\mathbb{Q}$-automata into rotating $\mathbb{Q}$-automata or to compute $\mathbb{Q}$-Hadamard expressions of the series they realize.

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