The b-Branching Problem in Digraphs

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Abstract

In this paper, we introduce the concept of b-branchings in digraphs, which is a generalization of branchings serving as a counterpart of b-matchings. Here b is a positive integer vector on the vertex set of a digraph, and a b-branching is defined as a common independent set of two matroids defined by b: an arc set is a b-branching if it has at most $b(v)$ arcs sharing the terminal vertex v, and it is an independent set of a certain sparsity matroid defined by b. We demonstrate that b-branchings yield an appropriate generalization of branchings by extending several classical results on branchings. We first present a multi-phase greedy algorithm for finding a maximum-weight b-branching. We then prove a packing theorem extending Edmonds' disjoint branchings theorem, and provide a strongly polynomial algorithm for finding optimal disjoint b-branchings. As a consequence of the packing theorem, we prove the integer decomposition property of the b-branching polytope. Finally, we deal with a further generalization in which a matroid constraint is imposed on the $b(v)$ arcs sharing the terminal vertex v.

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1 Introduction

Since the pioneering work of Edmonds [12, 14], the importance of *matroid intersection* has been well appreciated. A special case of matroid intersection is *branchings* (or *arborescences*) in digraphs. Branchings have several good properties which do not hold for general matroid intersection. The objective of this paper is to propose a class of the matroid intersection problem which generalizes branchings and inherits those good properties of branchings.

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One of the good properties of branchings is that a maximum-weight branching can be found by a simple combinatorial algorithm [4, 6, 11, 23]. This algorithm is much simpler than general weighted matroid intersection algorithms, and is referred to as a "multi-phase greedy algorithm" in the textbook by Kleinberg and Tardos [35].

Another good property is the elegant theorem for packing disjoint branchings [13]. In terms of matroid intersection, this theorem says that, if there exist k disjoint bases in each of the two matroids, then there exist k disjoint common bases. This packing theorem leads to a proof that the branching polytope has the *integer decomposition property* (defined in Section 2).

In this paper, we propose *b-branchings*, a class of matroid intersection generalizing branchings, while maintaining the above two good properties. This offers a new direction of fundamental extensions of the classical theorems on branchings.

Let $D = (V, A)$ be a digraph and let $b \in \mathbb{Z}_{++}^V$ be a positive integer vector on V. For $v \in V$ and $F \subseteq A$, let $\delta_F^-(v)$ denote the set of arcs in F entering v, and let $d_F^-(v) = |\delta_F^-(v)|$. One matroid $\mathbf{M}_{\rm in}$ on A has its independent set family $\mathcal{I}_{\rm in}$ defined by

$$\mathcal{I}_{\rm in} = \{F \subseteq A \colon d_F^-(v) \leq b(v) \text{ for each } v \in V\}. \quad (1)$$

That is, $\mathbf{M}_{\rm in}$ is the direct sum of a uniform matroid on $\delta_A^-(v)$ of rank $b(v)$ for every $v \in V$. Hence, each vertex can have indegree at most $b(v)$, which can be more than one. Indeed, this is the reason why we refer to it as a b-branching, as a counterpart of a b-matching.

In order to make b-branchings a satisfying generalization of branchings, the other matroid should be defined appropriately. Our answer is a *sparsity matroid* determined by D and b, which is defined as follows. For $F \subseteq A$ and $X \subseteq V$, let $F[X]$ denote the set of arcs in F induced by X. Also, denote $\sum_{v \in X} b(v)$ by $b(X)$. Now define a matroid $\mathbf{M}_{\rm sp}$ on A with independent set family $\mathcal{I}_{\rm sp}$ by

$$\mathcal{I}_{\rm sp} = \{F \subseteq A \colon |F[X]| \leq b(X) - 1 \ (\emptyset \neq X \subseteq V)\}. \quad (2)$$

It is known that $\mathbf{M}_{\rm sp}$ is a matroid [20, Theorem 13.5.1], referred to as a *count matroid* or a *sparsity matroid*.

Now we refer to an arc set $F \subseteq A$ as a *b-branching* if $F \in \mathcal{I}_{\rm in} \cap \mathcal{I}_{\rm sp}$. It is clear that a branching is a special case of a b-branching where $b(v) = 1$ for each $v \in V$. We demonstrate that b-branchings yield a reasonable generalization of branching by proving that the two fundamental results on branchings can be extended. That is, we present a multi-phase greedy algorithm for finding a maximum-weight b-branching, and a theorem for packing disjoint b-branchings.

Our multi-phase greedy algorithm is an extension of the weighted branching algorithm [4, 6, 11, 23], and it has the following features. First, its running time is $\mathrm{O}(|V||A|)$, which is as fast as a simple implementation of the weighted branching algorithm [4, 6, 11, 23], and faster than the current best general weighted matroid intersection algorithm. Second, our algorithm also finds an optimal dual solution, which is integer if the arc weights are integer. Thus, the algorithm constructively proves the total dual integrality of the associated linear system. Finally, the algorithm leads to a characterization of the existence of a b-branching with prescribed indegree, which is a generalization of that for an arborescence [4, 11, 23].

This characterization theorem is extended to a theorem on packing disjoint b-branchings. Let k be a positive integer, and $b_1, \ldots, b_k$ be nonnegative integer vectors on V such that $b_i(v) \leq b(v)$ for each $v \in V$ and $b_i \neq b$ $(i = 1, \ldots, k)$. Note that, when there exists a b-branching B_i satisfying $d_{B_i}^-(v) = b_i(v)$ for each $v \in V$ $(i = 1, \ldots, k)$, these assumptions about b_i follow from the definition (1) and (2) of b-branchings. We provide a necessary and sufficient condition for D to contain k disjoint b-branchings $B_1, \ldots, B_k$ satisfying $d^-_{B_i}(v) = b_i(v)$ for every $v \in V$ and $i = 1, \ldots, k$, which extends Edmonds' disjoint branching theorem [13]. We then show such disjoint b-branchings $B_1, \ldots, B_k$ can be found in strongly polynomial time by at most $|A|$ times of submodular function minimization [28, 37, 46]. We further prove that, when the arc-weight vector $w \in \mathbb{R}^A_+$ is given, disjoint b-branchings $B_1, \ldots, B_k$ that minimize $w(B_1) + \cdots + w(B_k)$ can be found in strongly polynomial time by optimization over a submodular flow polyhedron [16, 21, 29, 30]. By utilizing our disjoint b-branchings theorem, we also prove the integer decomposition property of the b-branching polytope.

We further deal with a generalized class of *matroid-restricted b-branchings*. This is a special case of matroid intersection in which $\mathbf{M}_{\text{in}}$ is the direct sum of an arbitrary matroid on $\delta^-_A(v)$ of rank $b(v)$ for all $v \in V$. Note that, in the class of b-branchings, the matroid $\mathbf{M}_{\text{in}}$ is the direct sum of a uniform matroid on $\delta^-_A(v)$ of rank $b(v)$. We show that our multi-phase greedy algorithm can be extended to this generalized class.

Let us conclude this section with describing related work. The weighted matroid intersection problem is a common generalization of various combinatorial optimization problems such as bipartite matchings, packing spanning trees, and branchings (or arborescences) in a digraph. The problem has also been applied to various engineering problems, e.g., in electric circuit theory [43, 44], rigidity theory [44], and network coding [8, 25]. Since 1970s, quite a few algorithms have been proposed for matroid intersection problems, e.g., [5, 18, 27, 37, 39, 40] (See [26] for further references). However, all known algorithms are not greedy, but based on augmentation; repeatedly incrementing a current solution by exchanging some elements.

The matroids in branchings are a partition matroid and a graphic matroid, which are interconnected by a given digraph. Such interconnection makes branchings more interesting. As mentioned before, branchings have properties that matroid intersection of an arbitrary pair of a partition matroid and a graphic matroid does not have. In particular, extending the packing theorem of branchings [13] is indeed a recent active topic. Kamiyama, Katoh, and Takizawa [33] presented a fundamental extension based on reachability in digraphs, which is followed by a further extension based on convexity in digraphs due to Fujishige [22]. Durand de Gevigney, Nguyen, and Szigeti [9] proved a theorem for packing arborescences with matroid constraints. Király [34] generalized the result of [9] in the same direction of [33]. A matroid-restricted packing of arborescences [3, 19] is another generalization concerning a matroid constraint. We remark that our packing and matroid restriction for b-branchings differ from the above matroidal extensions of packing of arborescences.

The organization of this paper is as follows. In Section 2, we review the literature of branchings and matroid intersection, including algorithmic, polyhedral, and packing results. In Section 3, we present a multi-phase greedy algorithm for finding a maximum-weight b-branching. Section 4 is devoted to proving a theorem on packing disjoint b-branchings. In Section 5, we extend the multi-phase greedy algorithm to matroid-restricted b-branchings. In Section 6, we conclude this paper with a couple of remarks.

2 Preliminaries

In this section, we review fundamental results on branchings and related theory of matroid intersection and polyhedral combinatorics. For more details, refer to [32, 36, 47].

In a digraph $D = (V, A)$, an arc subset $B \subseteq A$ is a *branching* if, in the subgraph (V, B), the indegree of every vertex is at most one and there does not exist a cycle in the undirected sense. In terms of matroid intersection, a branching is a common independent set of a partition matroid and a graphic matroid, i.e., intersection of

$$\{F \subseteq A \colon d_F^-(v) \leq 1 \text{ for each } v \in V\}, \tag{3}$$

$$\{F \subseteq A \colon |F[X]| \leq |X| - 1 \ (\emptyset \neq X \subseteq V)\}. \tag{4}$$

Recall that a branching is a special case of a b-branching where $b(v) = 1$ for each $v \in V$. Indeed, by putting $b(v) = 1$ for each $v \in V$ in (1) and (2), we obtain (3) and (4), respectively.

As stated in Section 1, a maximum-weight branching can be found by a multi-phase greedy algorithm [4, 6, 11, 23], which appears in standard textbooks such as [35, 36, 47]. To the best of our knowledge, we have no other nontrivial special case of matroid intersection which can be solved greedily. For example, intersection of two partition matroids is equivalent to bipartite matching. This seems the simplest nontrivial example of matroid intersection, but we do not know a greedy algorithm for finding a maximum bipartite matching.

Another important result on branchings is the disjoint branchings theorem by Edmonds [13], described as follows. For a positive integer k, the set of integers $\{1, \ldots, k\}$ is denoted by $[k]$. For $F \subseteq A$ and $X \subseteq V$, let $\delta_F^-(X) \subseteq A$ denote the set of arcs in F from $V \setminus X$ to X, and let $d_F^-(X) = |\delta_F^-(X)|$.

▶ **Theorem 1** (Edmonds [13]). *Let $D = (V, A)$ be a digraph and k be a positive integer, and $U_1, \ldots, U_k$ be subsets of V. Then, there exist disjoint branchings $B_1, \ldots, B_k$ such that $U_i = \{v \in V \colon d_{B_i}^-(v) = 1\}$ for each $i \in [k]$ if and only if*

$$d_A^-(X) \geq |\{i \in [k] \colon X \subseteq U_i\}| \quad (\emptyset \neq X \subseteq V).$$

From Theorem 1, we obtain a theorem on covering a digraph by branchings [17, 41].

▶ **Theorem 2** ([17, 41]). *Let $D = (V, A)$ be a digraph and let k be a nonnegative integer. Then, the arc set A can be covered by k branchings if and only if*

$$d_A^-(v) \leq k \quad (v \in V),$$

$$|A[X]| \leq k(|X| - 1) \quad (\emptyset \neq X \subseteq V).$$

Theorem 2 leads to the *integer decomposition property* of the branching polytope. The *branching polytope* is a convex hull of the characteristic vectors of all branchings. It follows from the total dual integrality of matroid intersection [12] that the branching polytope is determined by the following linear system:

$$x(\delta^-(v)) \leq 1 \qquad (v \in V), \tag{5}$$

$$x(A[X]) \leq |X| - 1 \quad (\emptyset \neq X \subseteq V), \tag{6}$$

$$x(a) \geq 0 \qquad (a \in A). \tag{7}$$

▶ **Theorem 3** (see [47]). *The linear system* (5)–(7) *is totally dual integral.*

▶ **Corollary 4** (see [47]). *The linear system* (5)–(7) *determines the branching polytope.*

For a polytope P and a positive integer k, define $kP = \{x \colon \exists x' \in P, x = kx'\}$. A polytope P has the *integer decomposition property* if, for each positive integer k, any integer vector $x \in kP$ can be represented as the sum of k integer vectors in P. The integer decomposition property of the branching polytope is a direct consequence of Theorem 2 and Corollary 4.

▶ **Corollary 5** ([1]). *The branching polytope has the integer decomposition property.*

We remark that the integer decomposition property does not hold for an arbitrary matroid intersection polytope. Schrijver [47] presents an example of matroid intersection defined on the edge set of K_4 without integer decomposition property. Indeed, finding a class of polyhedra with integer decomposition property is a classical topic in combinatorics. Typical examples of polyhedra with integer decomposition property include polymatroids [1, 24], the branching polytope [1], and intersection of two strongly base orderable matroids [7, 42]. While there is some recent progress [2], the integer decomposition property of polyhedra is far from being well understood. In Section 4, we will prove that the b-branching polytope is a new example of polytopes with integer decomposition property.

3 Multi-phase greedy algorithm

3.1 Algorithm description

In this subsection, we present a multi-phase greedy algorithm for finding a maximum-weight b-branching by extending the one for branchings [4, 6, 11, 23]. Let $D = (V, A)$ be a digraph and $b \in \mathbb{Z}_{++}^V$ be a positive integer vector on V. Recall that an arc set $F \subseteq A$ is a b-branching if $F \in \mathcal{I}_{\rm in} \cap \mathcal{I}_{\rm sp}$, where $\mathcal{I}_{\rm in}$ and $\mathcal{I}_{\rm sp}$ are defined by (1) and (2), respectively.

We first show a key property of $\mathbf{M}_{\rm in}$ and $\mathbf{M}_{\rm sp}$, which plays an important role in our algorithm. Its proof can be found in the full version [31].

▶ **Lemma 6.** *An independent set F in $\mathbf{M}_{\rm in}$ is not independent in $\mathbf{M}_{\rm sp}$ if and only if (V, F) has a strong component X such that*

$$|F[X]| = b(X). \tag{8}$$

Moreover, for every strong component X in (V, F) satisfying (8), $F[X]$ is a circuit of $\mathbf{M}_{\rm sp}$.

Lemma 6 enables us to design the following multi-phase greedy algorithm for finding a maximum-weight b-branching:

- Find a maximum-weight independent set F in $\mathbf{M}_{\rm in}$.
- If (V, F) has a strong component X satisfying (8), then contract X, reset b and the weights of the remaining arcs appropriately, and recurse.

At the end of the algorithm, we expand every contracted component X in the following manner. Suppose that the solution F has an arc a' entering the vertex v_X created when contracting X. Denote the terminal vertex of a' before contracting X by $v' \in X$. In expanding X, we add $b(X) - 1$ arcs to F, consisting of $b(v)$ heaviest arcs among $\delta_A^-(v) \cap A[X]$ for each $v \in X \setminus \{v'\}$ and $b(v') - 1$ heaviest arcs among $\delta_A^-(v') \cap A[X]$. If F has no arc entering v_X, then we add $b(X) - 1$ arcs to F, consisting of $b(v)$ heaviest arcs among $\delta_A^-(v) \cap A[X]$ for each $v \in X$ except for the arc of minimum weight among those $b(X)$ arcs.

A formal description of the algorithm is as follows. We denote an arc $a \in A$ with initial vertex u and terminal vertex v by (u, v). We assume that the arc weights are nonnegative, which are represented by a vector $w \in \mathbb{R}_+^A$. For $F \subseteq A$, we denote $w(F) = \sum_{a \in F} w(a)$.

The complexity of Algorithm bB is analyzed as follows. It is clear that there are at most $|V|$ iterations. It is also straightforward to see that the i-th iteration requires $\mathrm{O}(|A^{(i)}|)$ time: Steps 2, 3, and 4 respectively require $\mathrm{O}(|A^{(i)}|)$ time. Thus, the total time complexity of the algorithms is $\mathrm{O}(|V||A|)$.

3.2 Optimality of the algorithm and totally dual integral system

In this subsection, we prove that the output of ALGORITHM bB is a maximum-weight b-branching by the following primal-dual argument. We first present a linear program describing

Algorithm 1 ALGORITHM bB.

Input. A digraph $D = (V, A)$, and vectors $b \in \mathbb{Z}_{++}^V$ and $w \in \mathbb{R}_+^A$.
Output. A b-branching $F \subseteq A$ maximizing $w(F)$.
Step 1. Set $i := 0$, $D^{(0)} := D$, $b^{(0)} := b$, and $w^{(0)} := w$.
Step 2. Define a matroid $\mathbf{M}_{\mathrm{in}}^{(i)} = (A^{(i)}, \mathcal{I}_{\mathrm{in}}^{(i)})$ accordingly to $D^{(i)}$ and $b^{(i)}$ by (1). Then, find $F^{(i)} \in \mathcal{I}_{\mathrm{in}}^{(i)}$ maximizing $w^{(i)}(F^{(i)})$.
Step 3. If $(V^{(i)}, F^{(i)})$ has a strong component X such that

$$|F^{(i)}[X]| = b^{(i)}(X), \tag{9}$$

then go to Step 4. Otherwise, let $F := F^{(i)}$ and go to Step 5.
Step 4. Denote by $\mathcal{X} \subseteq 2^{V^{(i)}}$ the family of strong components X in $(V^{(i)}, F^{(i)})$ satisfying (9). Execute the following updates to construct $D^{(i+1)} = (V^{(i+1)}, A^{(i+1)})$, $b^{(i+1)} \in \mathbb{Z}_{++}^{V^{(i+1)}}$, and $w^{(i+1)} \in \mathbb{R}_+^{A^{(i+1)}}$.

- For each $X \in \mathcal{X}$, execute the following updates. First, contract X to obtain a new vertex v_X. Then, for every arc $a = (z, y) \in A^{(i)}$ with $z \in V^{(i)} \setminus X$ and $y \in X$,

$$z' := \begin{cases} v_{X'} & (z \in X' \text{ for some } X' \in \mathcal{X}), \\ z & (\text{otherwise}), \end{cases}$$
$$a' := (z', v_X),$$
$$\Psi(a') := a,$$
$$w^{(i+1)}(a') := w^{(i)}(a) - w^{(i)}(\alpha(a, F^{(i)})) + w^{(i)}(a_X),$$

  where $\alpha(a, F^{(i)})$ is an arc in $\delta^-_{F^{(i)}}(y)$ minimizing $w^{(i)}$, and a_X is an arc in $F^{(i)}[X]$ minimizing $w^{(i)}$.[4]
- Define $b^{(i+1)} \in \mathbb{Z}_{++}^{V^{(i+1)}}$ by

$$b^{(i+1)}(v) := \begin{cases} 1 & (v = v_X \text{ for some } X \in \mathcal{X}), \\ b^{(i)}(v) & (\text{otherwise}). \end{cases}$$

Let $i := i + 1$ and go back to Step 2.
Step 5. If $i = 0$, then return F.
Step 6. For every strong component X in $(V^{(i-1)}, F^{(i-1)})$ such that (9) holds, apply the following update:

$$F := \begin{cases} ((F \setminus \{a'\}) \cup \{\Psi(a')\}) \cup (F^{(i-1)}[X] \setminus \{\alpha(\Psi(a'), F^{(i-1)})\}) & (\exists a' = (z, v_X) \in F), \\ F \cup (F^{(i-1)}[X] \setminus \{a_X\}) & (\text{otherwise}). \end{cases}$$

Let $i := i - 1$ and go back to Step 5.

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the maximum-weight b-branching problem. It is a special case of the linear program for weighted matroid intersection, and hence we already know that the linear system is endowed with total dual integrality. Here we show an algorithmic proof for the total dual integrality. That is, we show that, when w is an integer vector, integral optimal primal and dual solutions can be computed via ALGORITHM bB.

Consider the following linear program, in variable $x \in \mathbb{R}^A$, associated with the maximum-weight b-branching problem:

$$\begin{aligned}
&\text{maximize} && \sum_{a\in A} w(a)x(a) && && (10)\\
&\text{subject to} && x(\delta_A^-(v)) \le b(v) && (v \in V), && (11)\\
& && x(A[X]) \le b(X) - 1 && (\emptyset \ne X \subseteq V), && (12)\\
& && 0 \le x(a) \le 1 && (a \in A). && (13)
\end{aligned}$$

The constraints (11)–(13) are indeed a special case of a linear system describing the common independent sets in two matroids, which is totally dual integral (see [47]).

▶ **Theorem 7.** *The linear system* (11)–(13) *is totally dual integral. In particular, the linear system* (11)–(13) *determines the* b*-branching polytope.*

The dual problem of (10)–(13), in variable $p \in \mathbb{R}^{2^V}$ and $q \in \mathbb{R}^A$, is described as follows.

$$\begin{aligned}
&\text{minimize} && \sum_{v\in V} b(v)p(v) + \sum_{X:\, \emptyset\ne X\subseteq V} (b(X)-1)p(X) + \sum_{a\in A} q(a) && (14)\\
&\text{subject to} && p(v) + \sum_{X:\, a\in A[X]} p(X) + q(a) \ge w(a) \quad (a = uv \in A), && (15)\\
& && p(X) \ge 0 \quad (X \subseteq V), && (16)\\
& && q(a) \ge 0 \quad (a \in A). && (17)
\end{aligned}$$

Note that the dual variable $p(X)$ corresponds to the primal constraint (11) if $|X| = 1$, and to (12) if $|X| \ge 2$. The primal constraint (12) for X with $|X| = 1$ does not have a corresponding dual variable, since it is redundant in the linear problem (10)–(13).

An optimal solution (p^*, q^*) is computed via Algorithm bB in the following manner. At the beginning of Algorithm bB, set $w^\circ = w$, $p(X) = 0$ for each $X \subseteq V$, and $q(a) = 0$ for each $a \in A$. In Step 4 of Algorithm bB, for each strong component $X \in \mathcal{X}$, define $p^*(X) \in \mathbb{R}$ by

$$p^*(X) = \min\{\, \min\{w^\circ(\alpha^\circ(a)) - w^\circ(a) \colon a \in \delta^-_{A^{(i)}}(X)\}, \min\{w^\circ(a') \colon a' \in F^{(i)}[X]\}\},$$

where $\alpha^\circ(a)$ is the $b(y)$-th optimal arc with respect to w° among the arcs sharing the terminal vertex $y \in V$ with a in the original digraph D. Then for each arc $a \in A$ such that $a \in A^{(i)}[X]$ or a is deleted in the contraction of X' with $v_{X'}$ included in X, set $w^\circ(a) := w^\circ(a) - p^*(X)$. After the termination of Algorithm bB, let the value $p^*(v)$ be equal to the $b(v)$-th maximum value among $\{w^\circ(a) \colon a \in \delta_A^-(v)\}$ for each vertex $v \in V$. Finally, let $q^*(a) = \max\{w(a) - p^*(v) - \sum_{X:\, a\in A[X]} p^*(X), 0\}$. Observe that $w^\circ(a) \ge 0$ holds for $a \in F$, and $w^\circ(a) \le 0$ for $a \notin F$.

We can prove the optimality of F and (p^*, q^*) in the following way. We first show that F is a b-branching. It is straightforward to see that $F \in \mathcal{I}_{\text{in}}$. Then, it follows from Lemma 6 that $F \in \mathcal{I}_{\text{sp}}$ as well, since a strong component X satisfying (8) is always contracted in the algorithm and hence never exists in the output F. Next, the feasibility of (p^*, q^*) for (15)–(17) is obvious. Finally, we prove that the characteristic vector χ_F of the output F and (p^*, q^*) satisfy the complementary slackness condition as follows:

- Suppose $\chi_F(\delta_A^-(v)) < b(v)$ for $v \in V$. If v is not contained in a contracted vertex set, then, by Step 2 of Algorithm bB, $\delta_A^-(v)$ contains less than $b(v)$ arcs with positive weight, and hence $p^*(v) = 0$. If v is contained in a contracted vertex set $X \subsetneq V^{(i)}$, it follows that v is the terminal vertex of $a_X \in F^{(i)}[X]$ and $p^*(X) = w^\circ(a_X)$. Then $w^\circ(a_X)$ is the $b(v)$-th maximum value among $\{w^\circ(a)\colon a \in \delta_A^-(v)\}$ and becomes zero after contracting X, which implies that $p^*(v) = 0$.
- If $\chi_F(A[X]) < b(X) - 1$ for $X \subseteq V$ with $|X| \geq 2$, it follows that X is not contracted in the algorithm, and thus $p^*(X)$ is never changed from zero.
- If $\chi_F(a) > 0$ for $a \in A$, then it follows that $w^\circ(a) = w(a) - p^*(v) - \sum_{X\colon a \in A[X]} p^*(X) \geq 0$, and thus $q^*(a) = w(a) - p^*(v) - \sum_{X\colon a \in A[X]} p^*(X)$, implying the equality in (15).
- If $\chi_F(a) < 1$ for $a \in A$, then it follows that $w^\circ(a) = w(a) - p^*(v) - \sum_{X\colon a \in A[X]} p^*(X) \leq 0$ and thus $q^*(a) = 0$.

Therefore, F and (p^*, q^*) are optimal solutions for the linear programs (10)–(13) and (14)–(17), respectively. Moreover, (p^*, q^*) is integer if w is integer, which implies that (11)–(13) is totally dual integral.

3.3 Existence of a b-branching with prescribed indegree

Our algorithm leads to the following theorem characterizing the existence of b-branching with prescribed indegree, which is an extension of that for arborescences [4, 11, 23].

▶ **Theorem 8.** *Let $D = (V, A)$ be a digraph and $b \in \mathbb{Z}_{++}^v$ be a positive integer vector on V. Let $b' \in \mathbb{Z}_+^V$ be a nonnegative integer vector such that $b'(v) \leq b(v)$ for every $v \in V$ and $b' \neq b$. Then, D has a b-branching B such that $d_B^-(v) = b'(v)$ for each $v \in V$ if and only if*

$$d_A^-(v) \geq b'(v) \quad (v \in V), \tag{18}$$

$$d_A^-(X) \geq 1 \quad (\emptyset \neq X \subsetneq V,\ b'(X) = b(X) \neq 0). \tag{19}$$

Let $r \in V$ be a specified vertex. A characterization of the existence of an r-arborescence [4, 11, 23] is obtained as a special case of Theorem 8, by putting $b(v) = 1$ for every $v \in V$, $b'(v) = 1$ for every $v \in V \setminus \{r\}$, and $b'(r) = 0$.

Theorem 8 can be proved in two ways. The necessity of (18) and (19) is clear. One way to derive the sufficiency of (18) and (19) is Algorithm bB. Apply Algorithm bB to the case where $b = b'$ and $w(a) = 1$ for each $a \in A$. Then, (18) and (19) certify that $F^{(i)}$ found in Step 2 of Algorithm bB is always a base of $\mathbf{M}_{\text{in}}^{(i)}$. It thus follows that the output F of Algorithm bB is a b-branching with $d_F^- = b'$. An alternative proof for the sufficiency of (18) and (19) is implied by the proof for Theorem 10 in Section 4, which extends Theorem 8 to a characterization of the existence of disjoint b-branchings with prescribed indegree.

4 Packing disjoint b-branchings

In this section, we present a theorem on packing disjoint b-branchings $B_1, \ldots, B_k$ with prescribed indegree, which extends Theorem 1, as well as Theorem 8. We then show that such disjoint b-branchings can be found in strongly polynomial time. We further show that disjoint b-branchings $B_1, \ldots, B_k$ minimizing the weight $w(B_1) + \cdots + w(B_k)$ can be found in strongly polynomial time. Finally, as a consequence of our packing theorem, we prove the integer decomposition property of the b-branching polytope.

4.1 Characterizing theorem for disjoint b-branchings

Let $D = (V, A)$ be a digraph, $b \in \mathbb{Z}_{++}^V$ be a positive integer vector on V, and k be a positive integer. For $i \in [k]$, let $b_i \in \mathbb{Z}_+^V$ be a nonnegative integer vector such that $b_i(v) \leq b(v)$ for every $v \in V$ and $b_i \neq b$. We present a theorem for chracterizing whether D contains disjoint b-branchings $B_1, \ldots, B_k$ such that $d_{B_i}^- = b_i$ for each $i \in [k]$.

We begin with introducing a function which plays a key role in the sequel. Define a function $g: 2^V \to \mathbb{Z}_+$ by

$$g(X) = |\{i \in [k] \colon b_i(X) = b(X) \neq 0\}| \quad (X \subseteq V). \tag{20}$$

The following lemma is straightforward to observe. Its proof is described in the full version [31].

▶ **Lemma 9.** *The function g is supermodular.*

Our characterization theorem is described as follows.

▶ **Theorem 10.** *Let $D = (V, A)$ be a digraph, $b \in \mathbb{Z}_{++}^V$ be a positive integer vector on V, and k be a positive integer. For $i \in [k]$, let $b_i \in \mathbb{Z}_+^V$ be a nonnegative integer vector such that $b_i(v) \leq b(v)$ for every $v \in V$ and $b_i \neq b$. Then, D has disjoint b-branchings $B_1, \ldots, B_k$ such that $d_{B_i}^- = b_i$ for each $i \in [k]$ if and only if the following two conditions are satisfied:*

$$d_A^-(v) \geq \sum_{i=1}^{k} b_i(v) \quad (v \in V), \tag{21}$$

$$d_A^-(X) \geq g(X) \qquad (X \subseteq V). \tag{22}$$

We remark that Szegő's generalization of Theorem 1 for packing arc sets which cover some intersecting families [48] (see also [19, Theorem 10.3.2]) looks similar to Theorem 10, but it does not directly implies Theorem 10. In Theorem 10, every arc set should cover $\delta_A^-(v)$ multiple times (i.e., $\delta_A^-(v)$ should be covered $b_i(v)$ times by B_i), and this coverage is not immediately rephrased in the form of [48].

Below is a proof for Theorem 10, which extends the proof for Theorem 1 by Lovász [38].

Proof of Theorem 10. Necessity is clear. We prove sufficiency by induction on $\sum_{i=1}^k b_i(V)$. The case $\sum_{i=1}^k b_i(V) = 0$ is trivial; $B_i = \emptyset$ for each $i \in [k]$.

Without loss of generality, suppose $b_1(V) > 0$. Define a partition $\{V_0, V_1, V_2\}$ of V by $V_0 = \{v \in V \colon b_1(v) = 0\}$, $V_1 = \{v \in V \colon 0 < b_1(v) < b(v)\}$, and $V_2 = \{v \in V \colon b_1(v) = b(v)\}$. Then, it holds that

$$V_0 \cup V_1 \neq \emptyset, \tag{23}$$

$$V_0 \neq V, \tag{24}$$

which follow from $b_1 \neq b$ and $b_1(V) > 0$, respectively.

For $X \subseteq V$, define $g(X) = |\{i \in [k] \colon b_i(X) = b(X) \neq 0\}|$. Let $W \subseteq V$ be an inclusionwise minimal vertex subset satisfying

$$W \cap (V_0 \cup V_1) \neq \emptyset, \tag{25}$$

$$W \setminus V_0 \neq \emptyset, \tag{26}$$

$$d_A^-(W) = g(W). \tag{27}$$

Such W always exists, since $W = V$ satisfies (25)–(27): (25) follows from (23); (26) from (24); and (27) from $b_i \neq b$ ($i \in [k]$) and hence $g(V) = 0$. Let $W_j = W \cap V_j$ ($j = 0, 1, 2$).

▶ **Claim 1.** *There exists an arc* $(u, v) \in A$ *such that* $u \in W_0 \cup W_1$ *and* $v \in W_1 \cup W_2$.

Proof. First, suppose that $W_2 \neq \emptyset$. Then, it holds that $g(W_2) > g(W)$, since every $i \in [k]$ contributing to $g(W)$ also contributes to $g(W_2)$, and $i = 1$ does not contribute to $g(W)$ but to $g(W_2)$. Hence we obtain that

$$d_A^-(W_2) \geq g(W_2) > g(W) = d_A^-(W). \tag{28}$$

Now (28) implies that there exists an arc $(u, v) \in A$ such that $u \in W_0 \cup W_1$ and $v \in W_2$.

Next, suppose that $W_2 = \emptyset$. By (26), we have that $W_1 \neq \emptyset$. Then, it holds that

$$\begin{aligned} \sum_{v \in W_1} d_A^-(v) &\geq \sum_{i=1}^{k} b_i(W_1) \quad (\because (21)) \\ &> \sum_{i=2}^{k} b_i(W_1) \quad (\because b_1(W_1) > 0) \\ &\geq |\{i \in [k] \colon b_i(W) = b(W) \neq 0\}| \quad (\because b_1(W) \neq b(W)) \\ &= g(W) = d_A^-(W), \end{aligned}$$

implying that there exists an arc $(u, v) \in A$ such that $u \in W = W_0 \cup W_1$ and $v \in W_1$. ◀

Let $a = (u, v) \in A$ be an arc in Claim 1. We then show that resetting

$$A := A \setminus \{a\}, \tag{29}$$

$$b_1(v) := b_1(v) - 1 \tag{30}$$

maintains (21) and (22). (This resetting amounts to augmenting B_1 by adding a.)

It is straightforward to see that the resetting (29) and (30) maintain (21). To prove that it also maintain (22), suppose to the contrary that $X \subseteq V$ violates (22) after the resetting.

This violation implies that $d_A^-(X) = g(X)$ before the resetting, and $d_A^-(X)$ has decreased by one while $g(X)$ has remained unchanged by the resetting. It then follows that

$$u \in V \setminus X \quad \text{and} \quad v \in X. \tag{31}$$

It also follows that $i = 1$ does not contribute to $g(X)$, and hence before the resetting, it holds that

$$X \cap (V_0 \cup V_1) \neq \emptyset. \tag{32}$$

By (31), we have that $u \in W \setminus X$ and $v \in X \cap W$, and hence $\emptyset \neq X \cap W \subsetneq W$. Here we show that $X \cap W$ satisfies (25)–(27), which contradicts the minimality of W.

Before the resetting, it holds that

$$d_A^-(X \cap W) \leq d_A^-(X) + d_A^-(W) - d_A^-(X \cup W) \tag{33}$$

$$\leq g(X) + g(W) - g(X \cup W) \tag{34}$$

$$\leq g(X \cap W). \tag{35}$$

Indeed, (33) follows from submodularity of d_A^-. The inequality (34) follows from $d_A^-(X) = g(X)$, $d_A^-(W) = g(W)$, and $d_A^-(X \cup W) \geq g(X \cup W)$. Finally, (35) follows from Lemma 9.

Since $d_A^-(X \cap W) \geq g(X \cap W)$ by (22), all inequalities (33)–(35) hold with equality, and hence $d_A^-(X \cap W) = g(X \cap W)$ holds before the resetting.

Equality in (35) implies that $(X \cap W) \cap (V_0 \cup V_1) \neq \emptyset$. Indeed, we have that $W \cap (V_0 \cup V_1) \neq \emptyset$ because $u \in W \cap (V_0 \cup V_1)$, and hence $i = 1$ does not contribute to $g(W)$. Combined with (32), $i = 1$ contributes to none of $g(X)$, $g(W)$, and $g(X \cup W)$. Thus, by the equality in (35), $i = 1$ does not contribute to $g(X \cap W)$ as well, and hence $(X \cap W) \cap (V_0 \cup V_1) \neq \emptyset$ must hold.

We also have $(X \cap W) \setminus V_0 \neq \emptyset$, because $v \in (X \cap W) \setminus V_0$. Therefore, $X \cap W$ satisfies (25)–(27), contradicting the minimality of W. Thus, we have finished proving that resetting of (29) and (30) maintains (22).

Now we can apply induction to obtain disjoint b-branchings $B_1, \ldots, B_k$ in the digraph $(V, A \setminus \{a\})$ such that $d^-_{B_1} = b_1 - \chi_v$ and $d^-_{B_i} = b_i$ for $i = 2, \ldots, k$, where $\chi_v \in \mathbb{Z}^V$ is a vector defined by $\chi_v(v) = 1$ and $\chi_v(u) = 0$ for e very $u \in V \setminus \{v\}$. We complete the proof by showing that $B_1 \cup \{a\}$ is a b-branching.

In resetting, we always have $u \in W_0 \cup W_1$, which implies that the construction of B_1 begins with a vertex r with $b_1(r) < b(r)$ and the component in (V, B_1) containing a includes r. Thus, no $X \subseteq V$ comes to satisfy $|B_1[X]| = b(X)$. ◀

4.2 Algorithm for finding disjoint b-branchings

Let us discuss the algorithmic aspect of Theorem 10. First, we can determine whether (21) and (22) hold in strongly polynomial time. Condition (21) is clear. For (22), we have that $d^-_A(X)$ is submodular and $g(X)$ is supermodular (Lemma 9), and hence $d^-_A(X) - g(X)$ is submodular. Thus, we can determine whether there exists X with $d^-_A(X) - g(X) < 0$ by submodular function minimization, in strongly polynomial time [28, 37, 46].

Finding b-branchings $B_1, \ldots, B_k$ can also be done in strongly polynomial time. By the proof for Theorem 10, it suffices to find an arc $a \in A$ such that resetting

$$A := A \setminus \{a\}, \quad b_1(v) := b_1(v) - 1 \tag{36}$$

maintains (22). This can be done by determining whether there exists X with $d^-_A(X) - g(X) < 0$ after resetting (36) for each $a \in A$, i.e., at most $|A|$ times of submodular function minimization [28, 37, 46].

▶ **Theorem 11.** *Conditions* (21) *and* (22) *can be checked in strongly polynomial time. Moreover, if* (21) *and* (22) *hold, then disjoint b-branchings $B_1, \ldots, B_k$ such that $d^-_{B_i} = b_i$ for each $i \in [k]$ can be found in strongly polynomial time.*

Further, if a weight vector $w \in \mathbb{R}^A_+$ is given, we can find disjoint b-branchings $B_1, \ldots, B_k$ minimizing $w(B_1) + \cdots + w(B_k)$ in strongly polynomial time. Indeed, conditions (21) and (22) derive a totally dual integral system which determines a *submodular flow polyhedron*. A set family $\mathcal{C} \subseteq 2^V$ is called a *crossing family* if, for each $X, Y \in \mathcal{C}$ with $X \cup Y \neq V$ and $X \cap Y \neq \emptyset$, it holds that $X \cup Y, X \cap Y \in \mathcal{C}$. A function $f : \mathcal{C} \to \mathbb{R}$ defined on a crossing family $\mathcal{C} \subseteq V$ is called *crossing submodular* if, for each $X, Y \in \mathcal{C}$ with $X \cup Y \neq V$ and $X \cap Y \neq \emptyset$, it holds that $f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y)$. A function f is *crossing supermodular* if $-f$ is crossing submodular. A *submodular flow polyhedron* is a polyhedron described as

$$\begin{aligned} &x(\delta^-_A(X)) - x(\delta^+_A(X)) \leq f(X) && (X \in \mathcal{C}), \\ &l(a) \leq x(a) \leq u(a) && (a \in A) \end{aligned}$$

by some digraph (V, A), crossing submodular function f on a crossing family $\mathcal{C} \subseteq 2^V$, and vectors $l, u \in \mathbb{R}^A$, where $\delta^+_A(X)$ denotes the set of arcs in A from X to $V \setminus X$.

▶ **Lemma 12** ([45]). *For a digraph $D=(V,A)$, let $f\colon 2^V \to \mathbb{R}$ be a crossing supermodular function on $\mathcal{C} \subseteq 2^V$ and $u \in \mathbb{R}^A$. Then, a polyhedron determined by*

$$x(\delta_A^-(X)) \geq f(X) \quad (X \in \mathcal{C}),$$
$$0 \leq x(a) \leq u(a) \quad (a \in A)$$

is a submodular flow polyhedron.

By Lemma 12, the linear inequality system (21) and (22) determines a submodular flow polyhedron. Indeed, we can define a crossing supermodular function $f\colon 2^V \to \mathbb{R}$ by

$$f(X) = \begin{cases} \sum_{i=1}^{k} b_i(v) & (X = \{v\} \text{ for some } v \in V), \\ g(X) & (\text{otherwise}). \end{cases}$$

Since a submodular flow polyherdron is totally dual integral [15], an arc set $B \subseteq A$ with (21) and (22) minimizing $w(B)$ can be found by optimization over a submodular flow polyhedron, which can be done in strongly polynomial time [16, 21, 29, 30]. After that, we can partition B into b-branchings $B_1, \ldots, B_k$ with $d_{B_i}^- = b_i$ $(i \in [k])$ in the same manner as above.

▶ **Theorem 13.** *If* (21) *and* (22) *hold, disjoint b-branchings $B_1, \ldots, B_k$ such that $d_{B_i}^- = b_i$ for each $i \in [k]$ minimizing $w(B_1) + \cdots + w(B_k)$ can be found in strongly polynomial time.*

4.3 Integer decomposition property of the b-branching polytope

In this subsection we show another consequence of Theorem 10: the integer decomposition property of the b-branching polytope. First, Theorem 10 leads to the following min-max relation on covering by b-branchings. This is an extension of Theorem 2, the theorem on covering by branchings [17, 41]. Its proof appears in the full version [31].

▶ **Corollary 14.** *Let $D=(V,A)$ be a digraph, $b \in \mathbb{Z}_{++}^V$ be a positive integer vector on V, and k be a positive integer. Then, the arc set A can be covered by k b-branchings if and only if*

$$d_A^-(v) \leq k \cdot b(v) \quad (v \in V), \tag{37}$$
$$|A[X]| \leq k(b(X) - 1) \quad (\emptyset \neq X \subseteq V). \tag{38}$$

The integer decomposition property of the b-branching polytope is a direct consequence of Corollary 14. Its proof is described in the full version [31].

▶ **Corollary 15.** *The b-branching polytope has the integer decomposition property.*

5 Matroid-restricted b-branchings

Our multi-phase greedy algorithm (Algorithm bB) can be extended to a more generalized problem of finding a maximum-weight *matroid-restricted b-branching*. Let $D=(V,A)$ be a digraph and $b \in \mathbb{Z}_{++}^V$ be a positive integer vector on V. For each vertex $v \in V$, a matroid $\mathbf{M}_v = (\delta^-(v), \mathcal{I}_v)$ with rank $b(v)$ is attached. We denote the direct sum of $\mathbf{M}_v$ for every $v \in V$ by $\mathbf{M}_V = (A, \mathcal{I}_V)$. Now an arc set $F \subseteq A$ is an $\mathbf{M}_V$*-restricted b-branching* if $F \in \mathcal{I}_V \cap \mathcal{I}_{\mathrm{sp}}$. Note that a b-branching is a special case where $\mathbf{M}_v$ is a uniform matroid for each $v \in V$.

A maximum-weight $\mathbf{M}_V$-restricted b-branching can be found by a slight extension of Algorithm bB. The extended algorithm is described in the full version [31].

6 Concluding remarks

In this paper, we have proposed b-branchings, a generalization of branchings. In a b-branching, a vertex v can have indegree at most $b(v)$, and thus b-branchings serve as a counterpart of b-matchings for matchings.

It is somewhat surprising that, to the best of our knowledge, such a fundamental generalization of branchings has never appeared in the literature. The reason might be that, in order to obtain a reasonable generalization, it is far from being trivial how the other matroid (graphic matroid) in branchings is generalized. We have succeeded in obtaining a generalization inheriting the multi-phase greedy algorithm [4, 6, 11, 23] and the packing theorem [13] for branchings by setting a sparsity matroid defined by (2) as the other matroid.

An important property of the two matroids is Lemma 6, which says that an independent set of one matroid is decomposed into an independent set and some circuits in the other matroid. This plays an important role in the design of a multi-phase greedy algorithm: find an optimal independent F set in one matroid; contract the circuits in F with respect to the other matroid; and the optimal common independent set can be found recursively. We remark that the definitions (1) and (2) are essential to attain this property. For example, the property fails if the vector b is not identical in (1) and (2). It also fails if the sparsity matroid is defined by $|F[X]| \leq b(X) - k$ for $k \neq 1$.

Another remark is on the similarity of our algorithm and the blossom algorithm for non-bipartite matchings [10], where a factor-critical component can be contracted and expanded. In our b-branching algorithm, for each strong component $X \in \mathcal{X}$ and each $v^* \in X$, there exists an arc set $F_X \subseteq A[X]$ such that $d^-_{F_X}(v^*) = b(v^*) - 1$ and $d^-_{F_X}(v) = b(v)$ for each $v \in X \setminus \{v^*\}$. In the blossom algorithm for nonbipatite matchings, for each factor-critical component X and each vertex $v^* \in X$, there exists a matching exactly covering $X \setminus \{v^*\}$.

We finally remark that the problem of finding a maximum-weight b-branching is a special case of a modest generalization of the framework of the $\mathcal{U}$-feasible t-matching problem in bipartite graphs [49]. In [49], it is proved that the $\mathcal{U}$-feasible t-matching problem in bipartite graphs is efficiently tractable under certain assumptions on the family of excluded structures $\mathcal{U}$. The b-branching problem can be regarded as a new problem which falls in this tractable class of the (generalized) $\mathcal{U}$-feasible t-matching problem.

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