On the Price of Independence for Vertex Cover, Feedback Vertex Set and Odd Cycle Transversal

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Abstract

Let $\mathrm{vc}(G)$, $\mathrm{fvs}(G)$ and $\mathrm{oct}(G)$ denote, respectively, the size of a minimum vertex cover, minimum feedback vertex set and minimum odd cycle transversal in a graph G. One can ask, when looking for these sets in a graph, how much bigger might they be if we require that they are independent; that is, what is the *price of independence*? If G has a vertex cover, feedback vertex set or odd cycle transversal that is an independent set, then we let, respectively, $\mathrm{ivc}(G)$, $\mathrm{ifvs}(G)$ or $\mathrm{ioct}(G)$ denote the minimum size of such a set. We investigate for which graphs H the values of $\mathrm{ivc}(G)$, $\mathrm{ifvs}(G)$ and $\mathrm{ioct}(G)$ are bounded in terms of $\mathrm{vc}(G)$, $\mathrm{fvs}(G)$ and $\mathrm{oct}(G)$, respectively, when the graph G belongs to the class of H-free graphs. We find complete classifications for vertex cover and feedback vertex set and an almost complete classification for odd cycle transversal (subject to three non-equivalent open cases).

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1 Introduction

We define a number of transversals of the vertex set of a graph G. A set $S \subseteq V(G)$ is a *vertex cover* if for every edge $uv \in E(G)$, at least one of u and v is in S, or, equivalently, if the graph $G - S$ contains no edges. A set $S \subseteq V(G)$ is a *feedback vertex set* if for every cycle in G, at least one vertex of the cycle is in S, or, equivalently, if the graph $G - S$ is a forest. A set $S \subseteq V(G)$ is an *odd cycle transversal* if for every cycle in G containing an odd number of vertices, at least one vertex of the cycle is in S, or, equivalently, if the graph $G - S$ is bipartite. For each of these transversals, one usually wishes to investigate how small they can be and there is a vast research literature on this topic.

One can add an additional constraint: require the transversal to be an *independent set*, that is, a set of vertices that are pairwise non-adjacent. It might be possible that no such transversal exists under this constraint. For example, a graph G has an independent vertex cover if and only if G is bipartite. We are interested in the following research question:

> *How is the minimum size of a transversal in a graph affected by adding the requirement that the transversal is independent?*

Of course, this question can be interpreted in many ways; for example, one might ask about the computational complexity of finding the transversals. In this paper, we focus on the following: for the three transversals introduced above, is the size of a smallest possible independent transversal (assuming one exists) bounded in terms of the minimum size of a transversal? That is, one might say, what is the *price of independence*?

To the best of our knowledge, the term price of independence was first used by Camby [4] in a recent unpublished manuscript. She considered dominating sets of graphs (sets of vertices such that every vertex outside the set has a neighbour in the set). As she acknowledged, though first to coin the term, she was building on past work. In fact, Camby and her co-author Plein had given a forbidden induced subgraph characterization of those graphs G for which, for every induced subgraph of G, there are minimum size dominating sets that are already independent [6], and there are a number of further papers on the topic of the price of independence for dominating sets (see the discussion in [4]).

We observe that this incipient work on the price of independence is a natural companion to recent work on the *price of connectivity*, investigating the relationship between minimum size transversals and minimum size connected transversals (which, in contrast to independent transversals, will always exist for the transversals we consider, assuming the input graph is connected). This work began with the work of Cardinal and Levy in their 2010 paper [8] and has since been taken in several directions; see, for example, [1, 5, 7, 9, 11, 12].

In this paper, as we broaden the study of the price of independence by investigating the three further transversals defined above, we will concentrate on classes of graphs defined by a single forbidden induced subgraph H, just as was done for the price of connectivity [1, 12]. That is, for a graph H, we ask what, for a given type of transversal, is the price of independence in the class of H-free graphs? The ultimate aim in each case is to find a dichotomy that allows us to say, given H, whether or not the size of a minimum size independent transversal can be bounded in terms of the size of a minimum transversal. We briefly give some necessary definitions and notation before presenting our results.

A *colouring* of a graph G is an assignment of positive integers (called *colours*) to the vertices of G such that if two vertices are adjacent, then they are assigned different colours. A graph is *k-colourable* if there is a colouring that only uses colours from the set $\{1, \ldots, k\}$. Equivalently, a graph is k-colourable if we can partition its vertex set into k (possibly empty) independent sets (called *colour classes* or *partition classes*). For $s, t \geq 0$, let $K_{s,t}$ denote the complete bipartite graph with partition classes of size s and t, respectively (note that $K_{s,t}$ is edgeless if $s = 0$ or $t = 0$). For $r \geq 0$, the graph $K_{1,r}$ is also called the $(r+1)$-vertex *star*; if $r \geq 2$ we say that the vertex in the partition class of size 1 is the *central vertex* of this star. For $n \geq 1$, let P_n and K_n denote the path and complete graph on n vertices, respectively. For $n \geq 3$, let C_n denote the cycle on n vertices. For $r \geq 1$, let $K_{1,r}^+$ denote the graph obtained from $K_{1,r}$ by subdividing one edge. The *disjoint union* $G + H$ of two vertex-disjoint graphs G and H is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. We denote the disjoint union of r copies of a graph G by rG.

The Price of Independence for Vertex Cover. As mentioned above, a graph has an independent vertex cover if and only if it is bipartite. For a bipartite graph G, let $\text{vc}(G)$ denote the size of a minimum vertex cover, and let $\text{ivc}(G)$ denote the size of a minimum independent vertex cover. Given a class $\mathcal{X}$ of bipartite graphs, we say that $\mathcal{X}$ is ivc-bounded if there is a function $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ such that $\text{ivc}(G) \leq f(\text{vc}(G))$ for every $G \in \mathcal{X}$ and we say that $\mathcal{X}$ is ivc-unbounded if no such function exists, that is, if there is a k such that for every $s \geq 0$ there is a graph G in $\mathcal{X}$ with $\text{vc}(G) \leq k$, but $\text{ivc}(G) \geq s$.

In our first main result, proven in Section 2, we determine for every graph H, whether or not the class of H-free bipartite graphs is ivc-bounded.

▶ **Theorem 1.** *Let H be a graph. The class of H-free bipartite graphs is* ivc*-bounded if and only if H is an induced subgraph of $K_{1,r} + rP_1$ or $K_{1,r}^+$ for some $r \geq 1$.*

The Price of Independence for Feedback Vertex Set. A graph has an independent feedback vertex set if and only if its vertex set can be partitioned into an independent set and a set of vertices that induces a forest; graphs that have such a partition are said to be *near-bipartite*. In fact, minimum size independent feedback vertex sets have been the subject of much research from a computational perspective: to find such a set is, in general, NP-hard, but there are fixed-parameter tractable algorithms and polynomial-time algorithms for certain graph classes; we refer to [2] for further details. For a near-bipartite graph G, let $\text{fvs}(G)$ denote the size of a minimum feedback vertex set, and let $\text{ifvs}(G)$ denote the size of a minimum independent feedback vertex set. Given a class $\mathcal{X}$ of near-bipartite graphs, we say that $\mathcal{X}$ is ifvs-bounded if there is a function $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ such that $\text{ifvs}(G) \leq f(\text{fvs}(G))$ for every $G \in \mathcal{X}$ and ifvs-unbounded otherwise.

In our second main result, proven in Section 3, we determine for every graph H, whether or not the class of H-free near-bipartite graphs is ifvs-bounded.

▶ **Theorem 2.** *Let H be a graph. The class of H-free near-bipartite graphs is* ifvs*-bounded if and only if H is isomorphic to $P_1 + P_2$, a star or an edgeless graph.*

The Price of Independence for Odd Cycle Transversal. A graph has an independent odd cycle transversal S if and only if it has a 3-colouring, since, by definition, we are requesting that S is an independent set of G such that $G - S$ has a 2-colouring. For a 3-colourable graph G, let $\text{oct}(G)$ denote the size of a minimum odd cycle transversal, and let $\text{ioct}(G)$ denote the size of a minimum independent odd cycle transversal. Given a class $\mathcal{X}$ of 3-colourable graphs, we say that $\mathcal{X}$ is ioct-bounded if there is a function $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ such that $\text{ioct}(G) \leq f(\text{oct}(G))$ for every $G \in \mathcal{X}$ and ioct-unbounded otherwise.

In our third main result, proven in Section 4, we address the question of whether or not, for a graph H, the class of H-free 3-colourable graphs is ifvs-bounded. Here, we do not have a complete dichotomy due to three missing cases, as we will discuss at the end of Section 4.

▶ **Theorem 3.** *Let H be a graph. The class of H-free 3-colourable graphs is* ioct*-bounded:*

- *if H is an induced subgraph of P_4 or $K_{1,3}+sP_1$ for some $s \geq 0$ and*
- *only if H is an induced subgraph of $K_{1,4}^+$ or $K_{1,4}+sP_1$ for some $s \geq 0$.*

Further Notation. Let G be a graph. For $x \in V(G)$, the *(open) neighbourhood* $N(v)$ of v is the set of vertices adjacent to v; the *closed neighbourhood* $N[v] = N(v) \cup \{v\}$. For $x \in V(G)$, let $\deg(v) = |N(v)|$ denote the *degree* of v. If $S \subseteq V(G)$, then $N(S) = \{v \in V(G) \setminus S \mid \exists u \in S, uv \in E(G)\}$ and $N[S] = N(S) \cup S$. For two vertices $x, y \in V(G)$, let $\text{dist}(x, y)$ denote the length of a shortest path from x to y (let $\text{dist}(x, y) = \infty$ if x and y are in different connected components of G). For $S \subseteq V(G)$, let $G[S]$ be the *induced subgraph of G on S*, that is, the graph with vertex set S, where two vertices in S are adjacent if and only if they are adjacent in G. For $S \subseteq V(G)$, let $G - S$ denote $G[V(G) \setminus S]$. A vertex $y \in V(G)$ is complete (resp. anti-complete) to a set $X \subseteq V(G)$ if y is adjacent (resp. not adjacent) to every vertex in X. A set $Y \subseteq V(G)$ is complete (resp. anti-complete) to a set $X \subseteq V(G)$ if every vertex of Y is complete (resp. anti-complete) to X. A vertex $v \in V(G)$ is *dominating* if it is complete to $V(G) \setminus \{v\}$. The *complement* $\overline{G}$ of a graph G has the same vertex set as G and an edge between two distinct vertices if and only if these vertices are not adjacent in G. A graph is *complete multi-partite* if its vertex set can be partitioned into independent sets that are complete to each other. For a set of graphs $\{H_1, \ldots, H_s\}$, a graph is said to be $(H_1, \ldots, H_s)$*-free* if it contains no induced subgraph isomorphic to a graph in the set.

2 Vertex Cover

We start with a useful lemma.

▶ **Lemma 4.** *Let $r, s \geq 1$. If G is a $(K_{1,r}+sP_1)$-free bipartite graph with bipartition (X, Y) such that $|X|, |Y| \geq rs + r - 1$, then either:*

- *every vertex of G has degree less than r or*
- *fewer than s vertices of X have more than $s-1$ non-neighbours in Y and fewer than s vertices of Y have more than $s-1$ non-neighbours in X.*

Proof. Let G be a $(K_{1,r}+sP_1)$-free bipartite graph with bipartition (X, Y) such that $|X|, |Y| \geq rs+r-1$. No vertex in X can have both r neighbours and s non-neighbours in Y, otherwise G would contain an induced $K_{1,r}+sP_1$. Therefore every vertex in X has degree either at most $r-1$ or at least $|Y|-(s-1) \geq rs+r-s$. By symmetry, we may assume that there is a vertex $x \in X$ of degree at least r. Suppose, for contradiction, that there is a set $X' \subseteq X$ of s vertices, each of which has more than $s-1$ non-neighbours in Y. Then every vertex of X' has degree at most $r-1$. Since $\deg(x) \geq rs+r-s = s(r-1)+r$, there must be a set $Y' \subseteq N(x)$ of r neighbours of x that have no neighbours in X'. Then $G[\{x\} \cup Y' \cup X']$ is a $K_{1,r}+sP_1$, a contradiction. It follows that fewer than s vertices in X have more than $s-1$ non-neighbours in Y. Since $|X| \geq r+(s-1)$, there is a set $X'' \subsetneq X$ of r vertices, each of which have at most $s-1$ non-neighbours in Y. Since $|Y| > r(s-1)$, there must be a vertex $y \in Y$ that is complete to X'', and therefore has $\deg(y) \geq r$. Repeating the above argument, it follows that fewer than s vertices of Y have more than $s-1$ non-neighbours in X. This completes the proof. ◀

Recall that a graph has an independent vertex cover if and only if it is bipartite.

▶ **Lemma 5.** *Let $r, s \geq 1$. If G is a $(K_{1,r}+sP_1)$-free bipartite graph, then* $\text{ivc}(G) \leq \text{vc}(G)r + rs$.

Proof. Let G be a $(K_{1,r} + sP_1)$-free bipartite graph. Fix a bipartition (X, Y) of G. Let S be a minimum vertex cover of G, so $|S| = \mathrm{vc}(G)$. We may assume that $\mathrm{vc}(G) \geq 2$, otherwise $\mathrm{ivc}(G) = \mathrm{vc}(G)$, in which case we are done. We may also assume that $|X|, |Y| > \mathrm{vc}(G)r+rs > rs + r - 1$, otherwise X or Y is an independent vertex cover of the required size, and we are done. If every vertex of G has degree at most $r - 1$, then $S' = (S \cap Y) \cup (N(S \cap X))$ is an independent vertex cover in G of size at most $\mathrm{vc}(G)(r - 1)$, and we are done. By Lemma 4, we may therefore assume that fewer than s vertices of X have more than $s-1$ non-neighbours in Y. We will show that this leads to a contradiction. Since $|X|, |Y| \geq \mathrm{vc}(G) + s$, there must be a set S' of $\mathrm{vc}(G) + 1$ vertices in X that have at least $\mathrm{vc}(G) + 1$ neighbours in Y. If a vertex $x \in V(G)$ has degree at least $\mathrm{vc}(G) + 1$, then $|N(x)| > |S|$, so $x \in S$. Therefore every vertex of S' must be in S, contradicting the fact that $|S'| = \mathrm{vc}(G) + 1 > \mathrm{vc}(G) = |S|$. ◀

▶ **Lemma 6.** *Let $r \geq 2$. If G is a $K^+_{1,r}$-free bipartite graph, then* $\mathrm{ivc}(G) \leq (\mathrm{vc}(G))^2(r - 1)$.

Proof. Clearly it is sufficient to prove the lemma for connected graphs G. Let G be a connected $K^+_{1,r}$-free bipartite graph. Fix a bipartition (X, Y) of G. Let S be a minimum vertex cover of G, so $|S| = \mathrm{vc}(G)$. We may assume that $\mathrm{vc}(G) \geq 2$, otherwise $\mathrm{ivc}(G) = \mathrm{vc}(G)$ and we are done. We may also assume that $|X|, |Y| > (\mathrm{vc}(G))^2(r - 1)$, otherwise X or Y is an independent vertex cover of the required size.

If there are two vertices $x, y \in X$ with $\mathrm{dist}(x, y) = 2$ and $\deg(x) \geq \deg(y) + (r - 1)$, then x, y, a common neighbour of x and y, and $r - 1$ vertices from $N(x) \setminus N(y)$ would induce a $K^+_{1,r}$ in G, a contradiction. Therefore, if $x, y \in X$ with $\mathrm{dist}(x, y) = 2$, then $|\deg(x) - \deg(y)| \leq r - 2$, so $|\deg(x) - \deg(y)| \leq (\frac{r-2}{2}) \mathrm{dist}(x, y)$. By the triangle inequality and induction, it follows that if $x, y \in X$, then $|\deg(x) - \deg(y)| \leq (\frac{r-2}{2}) \mathrm{dist}(x, y)$. Observe that $\mathrm{vc}(P_{2\,\mathrm{vc}(G)+2}) = \mathrm{vc}(G) + 1$, so G must be $P_{2\,\mathrm{vc}(G)+2}$-free. Since G is connected, it follows that if $x, y \in V(G)$, then $\mathrm{dist}(x, y) < 2\,\mathrm{vc}(G) + 1$. We conclude that if $x, y \in X$, then $|\deg(x) - \deg(y)| \leq \mathrm{vc}(G)(r - 2)$. Note that if a vertex $x \in V(G)$ has degree at least $\mathrm{vc}(G) + 1$, then $|N(x)| > |S|$ and so $x \in S$.

Since $|X| > (\mathrm{vc}(G))^2(r - 1) > \mathrm{vc}(G) = |S|$, there must be a vertex $y \in X \setminus S$. Since $y \in X \setminus S$, it follows that $\deg(y) \leq \mathrm{vc}(G)$. It follows that $\deg(x) \leq \deg(y) + \mathrm{vc}(G)(r - 2) \leq \mathrm{vc}(G)(r - 1)$ for all $x \in X$. We conclude that $S' = (S \cap Y) \cup (N(S \cap X))$ is an independent vertex cover in G of size at most $(\mathrm{vc}(G))^2(r - 1)$. This completes the proof. ◀

We are now ready to prove the main result of this section.

▶ **Theorem 1 (restated).** *Let H be a graph. The class of H-free bipartite graphs is* ivc*-bounded if and only if H is an induced subgraph of $K_{1,r} + rP_1$ or $K^+_{1,r}$ for some $r \geq 1$.*

Proof. If H is an induced subgraph of $K_{1,r} + rP_1$ or $K^+_{1,r}$ for some r, then Lemma 5 or 6, respectively, implies that the class of H-free bipartite graphs is ivc-bounded.

Now let H be a graph and suppose that there is a function $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ such that $\mathrm{ivc}(G) \leq f(\mathrm{vc}(G))$ for all H-free bipartite graphs G. We will show that H is an induced subgraph of $K_{1,r} + rP_1$ or $K^+_{1,r}$ for some r.

For $r \geq 1$, $s \geq 2$, let D^r_s denote the graph formed from $2K_{1,s}$ and P_{2r} by identifying the two end-vertices of the P_{2r} with the central vertices of the respective $K_{1,s}$'s (see also Figure 1; a graph of the form D^1_s is said to be a *double-star*). It is easy to verify that $\mathrm{vc}(D^r_s) = r + 1$ and $\mathrm{ivc}(D^r_s) = r + s$. Therefore, for every $r \geq 1$, $D^r_{f(r+1)}$ cannot be H-free by definition of f. Note that for $r, s, t \geq 1$, if $s \leq t$ then D^r_s is an induced subgraph of D^r_t. Therefore, for each $r \geq 1$, there must be an s such that D^r_s is not H-free. In other words, for each $r \geq 1$, H must be an induced subgraph of D^r_s for some s.

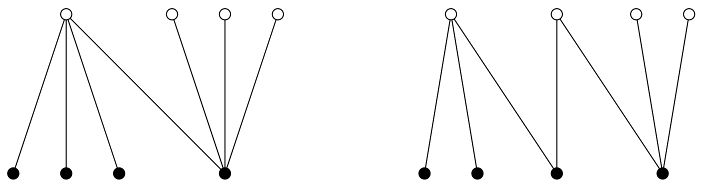

Figure 1 The graphs D_3^1 and D_2^2. The black vertices form a minimum independent vertex cover.

In particular, this means that we may assume that H is an induced subgraph of D_t^1 for some $t \geq 1$. If H contains at most one of the central vertices of the stars that form the D_t^1, then H is an induced subgraph of $K_{1,t} + tP_1$ and we are done, so we may assume H contains both central vertices. If one of these central vertices has at most one neighbour that is not a central vertex, then H is an induced subgraph of $K_{1,t+1}^+$, and we are done. We may therefore assume that H contains an induced D_2^1. However, for every $s \geq 1$, D_s^2 is D_2^1-free and therefore H-free. This contradiction completes the proof. ◀

3 Feedback Vertex Set

Recall that a graph has an independent feedback vertex set if and only if it is near-bipartite.

▶ **Lemma 7.** *If G is a $(P_1 + P_2)$-free near-bipartite graph, then* $\text{ifvs}(G) = \text{fvs}(G)$.

Proof. Let G be a $(P_1 + P_2)$-free near-bipartite graph. Note that $\overline{G}$ is a P_3-free graph, so $\overline{G}$ is a disjoint union of cliques. It follows that G is a complete multi-partite graph, say with a partition of its vertex sets into k non-empty independent sets $V_1, \ldots, V_k$. We may assume that $k \geq 2$, otherwise G is an edgeless graph, in which case $\text{ifvs}(G) = \text{fvs}(G) = 0$ and we are done. Since G is near-bipartite, it contains an independent set I such that $G - I$ is a forest. Note that $I \subseteq V_i$ for some $i \in \{1, \ldots, k\}$. Since near-bipartite graphs are 3-colourable, it follows that $k \leq 3$. Furthermore, if $k = 3$, then $|V_j| = 1$ for some $j \in \{1, 2, 3\} \setminus \{i\}$, otherwise $G - I$ would contain an induced C_4, a contradiction. In other words G is either a complete bipartite graph or the graph formed from a complete bipartite graph by adding a dominating vertex.

First suppose that $k = 2$, so G is a complete bipartite graph. Without loss of generality assume that $|V_1| \geq |V_2| \geq 1$. Let S be a feedback vertex set of G. If there are two vertices in $V_1 \setminus S$ and two vertices in $V_2 \setminus S$, then these vertices would induce a C_4 in $G - S$, a contradiction. Therefore S must contain all but at most one vertex of V_1 or all but at most one vertex of V_2, so $\text{fvs}(G) \geq \min(|V_1| - 1, |V_2| - 1) = |V_2| - 1$. Let I be a set consisting of $|V_2| - 1$ vertices of V_2. Then I is independent and $G - I$ is a star, so I is an independent feedback vertex set. It follows that $\text{ifvs}(G) \leq |V_2| - 1$. Since $\text{fvs}(G) \leq \text{ifvs}(G)$, we conclude that $\text{ifvs}(G) = \text{fvs}(G)$ in this case.

Now suppose that $k = 3$, so G is obtained from a complete bipartite graph by adding a dominating vertex. Without loss of generality assume that $|V_1| \geq |V_2| \geq |V_3| = 1$. Let S be a feedback vertex set of G. By the same argument as in the $k = 2$ case, S must contain all but at most one vertex of V_1 or all but at most one vertex of V_2. If there is a vertex in $V_i \setminus S$ for all $i \in \{1, 2, 3\}$, then these three vertices would induce a C_3 in $G - S$, a contradiction. Therefore S must contain every vertex in V_i for some $i \in \{1, 2, 3\}$. Since $|V_1| \geq |V_2| \geq |V_3| = 1$, it follows that $|S| \geq \min(|V_2| - 1 + |V_3|, |V_2|) = |V_2|$. Therefore $\text{fvs}(G) \geq |V_2|$. Now V_2 is an independent set and $G - V_2$ is a star, so V_2 is an independent feedback vertex set. It follows that $\text{ifvs}(G) \leq |V_2|$. Since $\text{fvs}(G) \leq \text{ifvs}(G)$, we conclude that $\text{ifvs}(G) = \text{fvs}(G)$. ◀

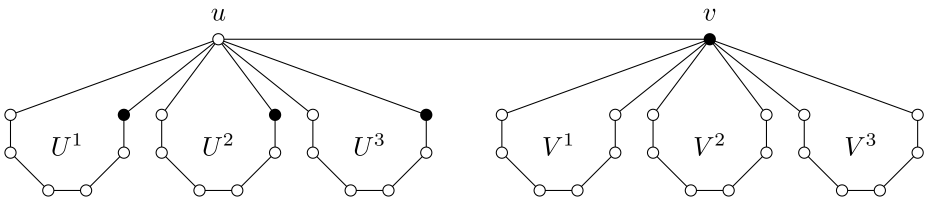

Figure 2 The graph S_3^3.

▶ **Lemma 8.** *If $r \geq 1$ and G is a $K_{1,r}$-free near-bipartite graph, then* $\text{ifvs}(G) \leq (2r^2 - 5r + 3)\,\text{fvs}(G)$*.*

Proof. Fix integers $k \geq 0$ and $r \geq 1$. Suppose G is a $K_{1,r}$-free near-bipartite graph with a feedback vertex set S such that $|S| = k$. Since G is near-bipartite, $V(G)$ can be partitioned into an independent set V_1 and a set $V(G) \setminus V_1$ that induces a forest in G. Since forests are bipartite, we can partition $V(G) \setminus V_1$ into two independent sets V_2 and V_3.

Suppose $x \in V_i$ for some $i \in \{1, 2, 3\}$. Then x has no neighbours in V_i since V_i is an independent set. For $j \in \{1, 2, 3\} \setminus \{i\}$, the vertex x can have at most $r - 1$ neighbours in V_j, otherwise G would contain an induced $K_{1,r}$. It follows that $\deg(x) \leq 2(r - 1)$ for all $x \in V(G)$.

Let $S' = S$. Let $F' = V(G) \setminus S'$, so $G[F']$ is a forest. To prove the lemma, we will iteratively modify S' until we obtain an independent feedback vertex set S' of G with $|S'| \leq (2r^2 - 5r + 3)|S|$. Every vertex $u \in S'$ has at most $2r - 2$ neighbours in F'. Consider two neighbours v, w of u in F'. As F' is a forest, there is at most one induced path in F' from v to w, so there is at most one induced cycle in $G[F' \cup \{u\}]$ that contains all of u, v and w. Therefore $G[F' \cup \{u\}]$ contains at most $\binom{2r-2}{2} = \frac{1}{2}(2r - 2)(2r - 2 - 1) = 2r^2 - 5r + 3$ induced cycles. Note that every cycle in G contains at least one vertex of V_1. Therefore, if $s \in S' \cap (V_2 \cup V_3)$, then we can find a set X of at most $2r^2 - 5r + 3$ vertices in $V_1 \setminus S'$ such that if we replace s in S' by the vertices of X, then we again obtain a feedback vertex set. Repeating this process iteratively, for each vertex we remove from $S' \cap (V_2 \cup V_3)$, we add at most $2r^2 - 5r + 3$ vertices to $S' \cap V_1$. We stop the procedure once $S' \cap (V_2 \cup V_3)$ becomes empty, at which point we have produced a feedback vertex set S' with $|S'| \leq (2r^2 - 5r + 3)|S|$. Furthermore, at this point $S' \subseteq V_1$, so S' is independent. It follows that $\text{ifvs}(G) \leq (2r^2 - 5r + 3)\,\text{fvs}(G)$. ◀

Note that all near-bipartite graphs are 3-colourable (use one colour for the independent set and the two other colours for the forest). We prove the following lemma.

▶ **Lemma 9.** *Let $k \geq 3$. The class of C_k-free near-bipartite graphs is* ifvs*-unbounded and* ioct*-unbounded.*

Proof. For $r, s \geq 2$, let S_s^r denote the graph constructed as follows (see also Figure 2). Start with the graph that is the disjoint union of $2s$ copies of P_{2r}, and label these copies $U^1, \ldots, U^s, V^1, \ldots, V^s$. Add a vertex u adjacent to both endpoints of every U^i and a vertex v adjacent to both endpoints of every V^i. Finally, add an edge between u and v.

Every induced cycle in S_s^r is isomorphic to C_{2r+1}, which is an odd cycle. Thus a set $S \subseteq V(S_s^r)$ is a feedback vertex set for S_s^r if and only if it is an odd cycle transversal for S_s^r. It follows that $\text{fvs}(S_s^r) = \text{oct}(S_s^r)$ and $\text{ifvs}(S_s^r) = \text{ioct}(S_s^r)$.

Now $\{u, v\}$ is a minimum feedback vertex set of S_s^r, so $\text{fvs}(S_s^r) = \text{oct}(S_s^r) = 2$. However, any independent feedback vertex set S contains at most one vertex of u and v; say it does not contain u. Then it must contain at least one vertex of each U^i. It follows that $\text{ifvs}(S_s^r) = \text{ioct}(S_s^r) \geq s + 1$. Since for every $s \geq 2$, $k \geq 3$, the graph S_s^k is C_k-free. This completes the proof. ◀

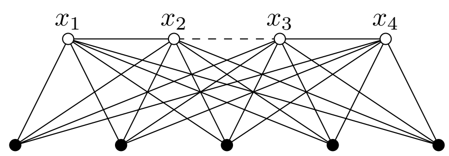

Figure 3 The graphs T_5 and T_5'. The edge x_2x_3 is present in T_5, but not in T_5'.

▶ **Theorem 2 (restated).** *Let H be a graph. The class of H-free near-bipartite graphs is* ifvs*-bounded if and only if H is isomorphic to $P_1 + P_2$, a star or an edgeless graph.*

Proof. If $H = P_1 + P_2$, then the theorem holds by Lemma 7. If H is isomorphic to a star or an edgeless graph, then H is an induced subgraph of $K_{1,r}$ for some $r \geq 1$. In this case the theorem holds by Lemma 8.

Now suppose that the class of H-free near-bipartite graphs is ifvs-bounded. By Lemma 9, H must be a forest. We will show that H is isomorphic to $P_1 + P_2$, a star or an edgeless graph.

We start by showing that H must be $(P_1+P_3, 2P_1+P_2, 2P_2)$-free. Let vertices x_1, x_2, x_3, x_4, in that order, form a path on four vertices. For $s \geq 3$, let T_s be the graph obtained from this path by adding an independent set I on s vertices (see also Figure 3) that is complete to the path and note that T_s is near-bipartite. Then $\{x_1, x_2, x_3\}$ is a minimum feedback vertex set in T_s. However, if S is an independent feedback vertex set, then S contains at most two vertices in $\{x_1, \ldots, x_4\}$. Therefore S must contain at least $s-1$ vertices of I, otherwise $T_s - S$ would contain an induced C_3 or C_4. Therefore $\text{fvs}(T_s) = 3$ and $\text{ifvs}(T_s) \geq s-1$. Note that T_s is $(P_1 + P_3, 2P_1 + P_2, 2P_2)$-free (this is easy to see by casting to the complement and observing that $\overline{T_s}$ is the disjoint union of a P_4 and a complete graph). Therefore H cannot contain $P_1 + P_3$, $2P_1 + P_2$ or $2P_2$ as an induced subgraph, otherwise T_s would be H-free, a contradiction.

Next, we show that H must be P_4-free. For $s \geq 3$ let T_s' be the graph obtained from T_s by removing the edge x_2x_3 (see also Figure 3). Then $\{x_1, x_2, x_3\}$ is a minimum feedback vertex set in T_s', so $\text{fvs}(T_s') = 3$. By the same argument as for T_s, we find that $\text{ifvs}(T_s') \geq s-1$ Now the complement $\overline{T_s'}$ is the disjoint union of a C_4 and a complete graph, so T_s' is P_4-free. Therefore H cannot contain P_4 as an induced subgraph.

We may now assume that H is a $(P_1+P_3, 2P_1+P_2, 2P_2, P_4)$-free forest. If H is connected, then it is a P_4-free tree, so it is a star, in which case we are done. We may therefore assume that H is disconnected. We may also assume that H contains at least one edge, otherwise we are done. Since H is $(2P_1 + P_2)$-free, it cannot have more than two components. Since H is $2P_2$-free, one of its two components must be isomorphic to P_1. Since H is a $(P_1 + P_3)$-free forest, its other component must be isomorphic to P_2. Hence H is isomorphic to $P_1 + P_2$. This completes the proof. ◀

4 Odd Cycle Transversal

Recall that a graph has an independent odd cycle transversal if and only if it is 3-colourable. We show the following two lemmas.

▶ **Lemma 10.** *If G is a P_4-free 3-colourable graph, then* $\text{ioct}(G) = \text{oct}(G)$.

Proof. Let G be a P_4-free 3-colourable graph. It suffices to prove the lemma component-wise, so we may assume that G is connected. Note that G cannot contain any induced odd cycles on more than three vertices, as it is P_4-free. Let (V_1, V_2, V_3) be a partition of $V(G)$ into independent sets. We may assume that G is not bipartite, otherwise $\text{ioct}(G) = \text{oct}(G) = 0$, in which case we are done. As G is connected, P_4-free and contains more than one vertex, its complement $\overline{G}$ must be disconnected. Therefore we can partition the vertex set of G into two parts X_1 and X_2 such that X_1 is complete to X_2. No independent set V_i can have vertices in both X_1 and X_2, so without loss of generality we may assume that $X_1 = V_1$ and $X_2 = V_2 \cup V_3$. Since $G[X_2]$ is a P_4-free bipartite graph, it is readily seen that it is a disjoint union of complete bipartite graphs.

Note that $G - X_1$ is a bipartite graph, so X_1 is an odd cycle transversal of G. Furthermore, X_1 is independent. Now let S be a minimum vertex cover of $G[X_2]$. Observe that $G - S$ is bipartite, so S is an odd cycle transversal of G. Since $G[X_2]$ is the disjoint union of complete bipartite graphs, for every component C of $G[X_2]$, S must contain one part of the bipartition of C, or the other; by minimality of S, it only contains vertices from one of the parts. It follows that S is independent.

We now claim that every minimum odd cycle transversal S of G contains either X_1 or a minimum vertex cover of $G[X_2]$, both of which we have shown are independent odd cycle transversals; by the minimality of S, this will imply that S is equal to one of them. Indeed, suppose for contradiction that there is a vertex $x \in X_1 \setminus S$ and two adjacent vertices $y, z \in X_2 \setminus S$. Then $G[\{x, y, z\}]$ is a C_3 in $G - S$. This contradiction completes the proof. ◀

▶ **Lemma 11.** *If G is a $K_{1,3}$-free 3-colourable graph, then $\text{ioct}(G) \leq 3\,\text{oct}(G)$.*

Proof. Fix an integer $k \geq 0$. Let G be a $K_{1,3}$-free 3-colourable graph with an odd cycle transversal S such that $|S| = k$. Fix a partition of $V(G)$ into three independent sets V_1, V_2, V_3. Without loss of generality assume that $|S \cap V_1| \geq |S \cap V_2|, |S \cap V_3|$, so $|S \cap (V_2 \cup V_3)| \leq \frac{2k}{3}$. Let $S' = S$ and note that $G - S'$ is bipartite by definition of odd cycle transversal. To prove the lemma, we will iteratively modify S' until we obtain an independent odd cycle transversal S' of G with $|S'| \leq 3k$.

Suppose $x \in V_i$ for some $i \in \{1, 2, 3\}$. Then x has no neighbours in V_i since V_i is an independent set. For $j \in \{1, 2, 3\} \setminus \{i\}$, the vertex x can have at most two neighbours in V_j, otherwise G would contain an induced $K_{1,3}$. It follows that $\deg(x) \leq 4$ for all $x \in V(G)$.

As $G - S'$ is a bipartite $K_{1,3}$-free graph, it is a disjoint union of paths and even cycles. Every vertex $u \in S'$ has at most four neighbours in $V(G) \setminus S'$. An induced odd cycle in $G - (S' \setminus \{u\})$ consists of the vertex u and an induced path P in $G - S'$ between two neighbours v, w of u such that $P \cap N(u)$ does not contain any vertices apart from v and w. If u has q neighbours in some component C of $G - S'$, then there can be at most q such paths P that lie in this component. It follows that there are at most four induced odd cycles in $G - (S' \setminus \{u\})$. Note that every induced odd cycle in G contains at least one vertex in each V_i. Therefore, if $s \in S' \cap (V_2 \cup V_3)$, then we can find a set X of at most four vertices in $V_1 \setminus S'$ such that if we replace s in S' by the vertices of X, then we again obtain an odd cycle transversal. Repeating this process iteratively, for each vertex we remove from $S' \cap (V_2 \cup V_3)$, we add at most four vertices to $S' \cap V_1$, so $|S'|$ increases by at most 3. We stop the procedure once $S' \cap (V_2 \cup V_3)$ becomes empty, at which point we have produced an odd cycle transversal S' with $|S'| \leq |S| + 3|S \cap (V_2 \cup V_3)| \leq k + 2 \times \frac{2k}{3} = 3k$. Furthermore, at this point $S' \subseteq V_1$, so S' is independent. It follows that $\text{ioct}(G) \leq 3\,\text{oct}(G)$. ◀

▶ **Lemma 12.** *Let $r, s \geq 1$. Suppose there is a function $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ such that $\text{ioct}(G) \leq f(\text{oct}(G))$ for every $K_{1,r}$-free 3-colourable graph G. Then $\text{ioct}(G) \leq \max(\text{oct}(G)r + r^2 + 3rs - 2r, f(\text{oct}(G)))$ for every $(K_{1,r} + sP_1)$-free 3-colourable graph G.*

Proof. Fix $r, s \geq 1$ and $k \geq 0$. Let G be a $(K_{1,r} + sP_1)$-free 3-colourable graph with a minimum odd-cycle transversal T on k vertices. Fix a partition of $V(G)$ into three independent sets V_1, V_2, V_3. We may assume that $\text{oct}(G) \geq 2$, otherwise $\text{ioct}(G) = \text{oct}(G)$ and we are done. If $|V_i| \leq \max(\text{oct}(G)r + r^2 + 3rs - 2r, f(\text{oct}(G)))$ for some $i \in \{1, 2, 3\}$, then deleting V_i from G yields a bipartite graph, so $\text{ioct}(G) \leq \max(\text{oct}(G)r + r^2 + 3rs - 2r, f(\text{oct}(G)))$ and we are done. We may therefore assume that $|V_i| > \max(\text{oct}(G)r + r^2 + 3rs - 2r, f(\text{oct}(G)))$ for all $i \in \{1, 2, 3\}$. If G is $K_{1,r}$-free, then $\text{ioct}(G) \leq f(\text{oct}(G))$, so suppose that G contains an induced $K_{1,r}$, say with vertex set X. Note that $|X| = r + 1$, and each V_i can contain at most r vertices of X, since every V_i is an independent set.

For every $i \in \{1, 2, 3\}$, there cannot be a set of s vertices in $V_i \setminus X$ that are anti-complete to X, otherwise G would contain an induced $K_{1,r} + sP_1$, a contradiction. For every $i \in \{1, 2, 3\}$, since $|V_i| > \text{oct}(G)r + r^2 + 3rs - 2r \geq r^2 + 3rs$, it follows that $|V_i \setminus X| \geq |V_i| - r > (s-1) + (r+1)(r-1) = (s-1) + |X|(r-1)$. Hence for every $i \in \{1, 2, 3\}$, there must be a vertex $x \in X$ that has at least r neighbours in V_i. Applying this for each i in turn, we find that at least two of the graphs in $\{G[V_1 \cup V_2], G[V_1 \cup V_3], G[V_2 \cup V_3]\}$ contain a vertex of degree at least r; without loss of generality assume that this is the case for $G[V_1 \cup V_2]$ and $G[V_1 \cup V_3]$. Let V_2' and V_3' denote the set of vertices in V_2 and V_3, respectively, that have more than $s - 1$ non-neighbours in V_1. By Lemma 4, $|V_2'|, |V_3'| \leq s - 1$.

Suppose a vertex $x \in V_2 \setminus V_2'$ is adjacent to a vertex $y \in V_3 \setminus V_3'$. By definition of V_2' and V_3', the vertices x and y each have at most $s - 1$ non-neighbours in V_1. Since $|V_1| - 2(s-1) \geq \text{oct}(G)+1$, it follows that $|N(x) \cap N(y) \cap V_1| \geq \text{oct}(G)+1$ so $N(x) \cap N(y) \cap V_1 \not\subseteq T$. We conclude that at least one of x or y must be in T. In other words, $T \cap ((V_2 \setminus V_2') \cup (V_3 \setminus V_3'))$ is a vertex cover of $G[(V_2 \setminus V_2') \cup (V_3 \setminus V_3')]$, of size at most $\text{oct}(G)$. Therefore $(T \cap ((V_2 \setminus V_2') \cup (V_3 \setminus V_3'))) \cup V_2' \cup V_3'$ is a vertex cover of $G[V_2 \cup V_3]$ of size at most $\text{oct}(G)+2(s-1)$. By Lemma 5, there is an independent vertex cover T' of $G[V_2 \cup V_3]$ of size at most $(\text{oct}(G)+2(s-1))r + rs = \text{oct}(G)r + 3rs - 2r$. Note that by definition of vertex cover, $(V_2 \cup V_3) \setminus T'$ is an independent set, and so $G - T'$ is bipartite. Therefore T' is an independent odd cycle transversal for G of size at most $\text{oct}(G)r + 3rs - 2r$. This completes the proof. ◀

The following result follows immediately from combining Lemmas 11 and 12.

▶ **Corollary 13.** *For* $s \geq 1$, $\text{ioct}(G) \leq 3\,\text{oct}(G) + 9s + 3$ *for every* $(K_{1,3} + sP_1)$*-free* 3*-colourable graph* G.

▶ **Lemma 14.** *The class of* $(P_1 + P_4, 2P_2)$*-free* 3*-colourable graphs is* ioct*-unbounded.*

Proof. Let $s \geq 2$. We construct the graph Q_s as follows (see also Figure 4). First, let A, B and C be disjoint independent sets of s vertices. Choose vertices $a \in A$, $b \in B$ and $c \in C$. Add edges so that a is complete to $B \cup C$, b is complete to $A \cup C$ and c is complete to $A \cup B$. Let Q_s be the resulting graph and note that it is 3-colourable with colour classes A, B and C.

Note that $\{a, b\}$ is a minimum odd cycle transversal of Q_s, so $\text{oct}(Q_s) = 2$.

Let S be a minimum independent odd cycle transversal. Then S contains at most one vertex in $\{a, b, c\}$, say S contains neither b nor c. If a vertex $x \in A$ is not in S, then $Q_s[\{x, b, c\}]$ is a C_3 in $Q_s - S$, a contradiction. Hence every vertex of A is in S, and so $\text{ioct}(Q_s) \geq s$.

It remains to show that Q_s is $(P_1 + P_4, 2P_2)$-free. Consider a vertex $x \in A$. Then $Q_s - N[x]$ is an edgeless graph if $x = a$ and $Q_s - N[x]$ is the disjoint union of a star and an edgeless graph otherwise. It follows that $Q_s - N[x]$ is P_4-free. By symmetry, we conclude that Q_s is $(P_1 + P_4)$-free. Now consider a vertex $y \in N(a) \cap B$. Then $Q_s - N[\{a, y\}]$ is empty if $b = y$ and $Q_s - N[\{a, y\}]$ is an edgeless graph otherwise. It follows that $Q_s - N[\{a, y\}]$ is P_2-free. By symmetry, we conclude that Q_s is $2P_2$-free. This completes the proof. ◀

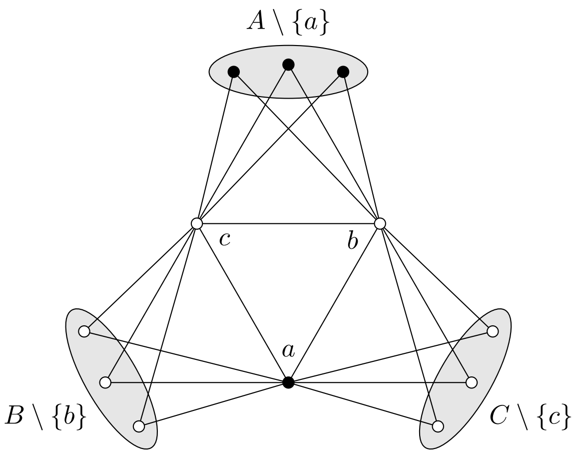

Figure 4 The graph Q_4.

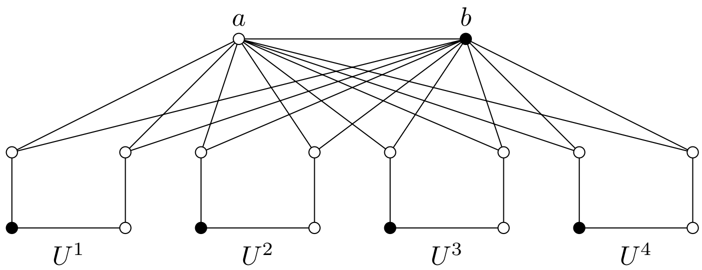

Figure 5 The graph Z_4.

▶ **Lemma 15.** *Let H be a graph with more than one vertex of degree at least 3. Then the class of H-free 3-colourable graphs is* ioct*-unbounded.*

Proof. Let $s \geq 1$. We construct the graph Z_s as follows (see also Figure 5). Start with the disjoint union of s copies of P_4 and label these copies $U^1, \ldots, U^s$. Add an edge ab and make a and b adjacent to the endpoints of every U^i. Let Z_s be the resulting graph and note that Z_s is 3-colourable (colour a and b with Colours 1 and 2, respectively, colour the endpoints of the U^is with Colour 3 and colour the remaining vertices of the U^is with Colours 1 and 2).

Note that $Z_s - \{a, b\}$ is bipartite, so $\{a, b\}$ is a minimum odd cycle transversal and $\text{oct}(Z_s) = 2$. However, any independent odd cycle transversal S contains at most one vertex of a and b; say it does not contain a. For every $i \in \{1, \ldots, s\}$, the graph $Z_s[U^i \cup \{a\}]$ is a C_5. Therefore S must contain at least one vertex from each U^i. It follows that $\text{ioct}(Z_s) \geq s$.

Let H be a graph with more than one vertex of degree at least 3. By Lemma 9, we may assume that H is a forest. It remains to show that Z_s is H-free. Suppose, for contradiction, that Z_s contains H as an induced subgraph and let x and y be two vertices that have degree at least 3 in H. Since H is a forest, x and y must each have three pairwise non-adjacent neighbours in Z_s. The endpoints of each U^i have exactly three neighbours, but two of them (a and b) are adjacent. Without loss of generality we may therefore assume that $x = a$ and $y = b$. Since x has degree at least 3 in H, the vertex x must have a neighbour $z \neq y$ in H and so z must be the endpoint of a U^i. Therefore x, y and z are pairwise adjacent, so $H[\{x, y, z\}]$ is a C_3, contradicting the fact that H is a forest. It follows that Z_s is H-free. This completes the proof. ◀

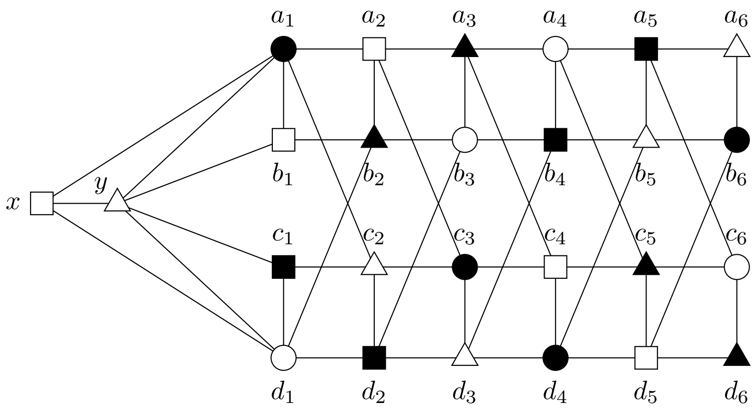

Figure 6 The graph Y_2. Different shapes show the unique 3-colouring of Y_2. Different colours show the 2-colouring of $Y_2 - \{x, y\}$.

▶ **Lemma 16.** *The class of $K_{1,5}$-free 3-colourable graphs is* ioct*-unbounded.*

Proof. Let $s \geq 1$. We construct the graph Y_s as follows (see also Figure 6).

1. Start with the disjoint union of four copies of P_{3s} and label the vertices of these paths $a_1, \ldots, a_{3s}$, $b_1, \ldots, b_{3s}$, $c_1, \ldots, c_{3s}$ and $d_1, \ldots, d_{3s}$ in order, respectively.
2. For each $i \in \{1, \ldots, 3s\}$ add the edges a_ib_i and c_id_i.
3. For each $i \in \{1, \ldots, 3s - 1\}$ add the edges a_ic_{i+1} and d_ib_{i+1}.
4. Finally, add an edge xy and make x adjacent to a_1 and d_1 and y adjacent to a_1, b_1, c_1 and d_1.

Let Y_s be the resulting graph.

First note that Y_s is $K_{1,5}$-free. The vertices y, a_1 and d_1 all have degree 5, but their neighbourhood is not independent, so they cannot be the central vertex of an induced $K_{1,5}$. All the other vertices have degree at most 4, so they cannot be the central vertex of an induced $K_{1,5}$ either. Therefore no vertex in Y_s is the central vertex of an induced $K_{1,5}$, so Y_s is $K_{1,5}$-free.

The graph $Y_s - \{x, y\}$ is bipartite with bipartition classes:

1. $\{a_i, c_i \mid 1 \leq i \leq 3s, i \equiv 1 \bmod 2\} \cup \{b_i, d_i \mid 1 \leq i \leq 3s, i \equiv 0 \bmod 2\}$ and
2. $\{a_i, c_i \mid 1 \leq i \leq 3s, i \equiv 0 \bmod 2\} \cup \{b_i, d_i \mid 1 \leq i \leq 3s, i \equiv 1 \bmod 2\}$.

It follows that $\text{oct}(Y_s) = 2$.

Furthermore, Y_s is 3-colourable with colour classes:

1. $\{x\} \cup \{a_i, d_i \mid 1 \leq i \leq 3s, i \equiv 2 \bmod 3\} \cup \{b_i, c_i \mid 1 \leq i \leq 3s, i \equiv 1 \bmod 3\}$
2. $\{y\} \cup \{a_i, d_i \mid 1 \leq i \leq 3s, i \equiv 0 \bmod 3\} \cup \{b_i, c_i \mid 1 \leq i \leq 3s, i \equiv 2 \bmod 3\}$
3. $\{a_i, d_i \mid 1 \leq i \leq 3s, i \equiv 1 \bmod 3\} \cup \{b_i, c_i \mid 1 \leq i \leq 3s, i \equiv 0 \bmod 3\}$

In fact, we will show that this 3-colouring is unique (up to permuting the colours). To see this, suppose that $c : V(Y_s) \to \{1, 2, 3\}$ is a 3-colouring of Y_s. Since x and y are adjacent we may assume without loss of generality that $c(x) = 1$ and $c(y) = 2$. Since a_1 and d_1 are adjacent to both x and y, it follows that $c(a_1) = c(d_1) = 3$. Since b_1 is adjacent to y and a_1, it follows that $c(b_1) = 1$. By symmetry $c(c_1) = 1$.

We prove by induction on i that for every $i \in \{1, \ldots, 3s\}$, $c(a_i) = c(d_i) \equiv i + 2 \bmod 3$ and $c(b_i) = c(c_i) \equiv i \bmod 3$. We have shown that this is true for $i = 1$. Suppose that the claim holds for $i - 1$ for some $i \in \{2, \ldots, 3s\}$. Then $c(a_{i-1}) = c(d_{i-1}) \equiv (i - 1) + 2 \bmod 3$ and $c(b_{i-1}) = c(c_{i-1}) \equiv i - 1 \bmod 3$. Since b_i is adjacent to b_{i-1} and d_{i-1}, it follows that $c(b_i) \equiv i \bmod 3$. Since a_i is adjacent to b_i and a_{i-1}, it follows that $c(a_i) \equiv i + 2 \bmod 3$.

By symmetry $c(c_i) \equiv i \bmod 3$ and $c(d_i) \equiv i+2 \bmod 3$. Therefore the claim holds for i. By induction, this completes the proof of the claim and therefore shows that the 3-colouring of Y_s is indeed unique.

Furthermore, note that the colour classes in this colouring have sizes $4s+1$, $4s+1$ and $4s$, respectively. A set S is an independent odd cycle transversal of a graph if and only if it is a colour class in some 3-colouring of this graph. It follows that $\text{ioct}(Y_s) = 4s$. This completes the proof. ◀

We summarise the results of this section in the following theorem.

▶ **Theorem 3 (restated).** *Let H be a graph. The class of H-free 3-colourable graphs is* ioct*-bounded:*
- *if H is an induced subgraph of P_4 or $K_{1,3}+sP_1$ for some $s \geq 0$ and*
- *only if H is an induced subgraph of $K_{1,4}^+$ or $K_{1,4}+sP_1$ for some $s \geq 0$.*

Proof. If H is an induced subgraph of P_4 or $K_{1,3}+sP_1$ for some $s \geq 0$, then the class of H-free 3-colourable graphs is ioct-bounded by Lemma 10 or Corollary 13, respectively.

Now suppose that the class of H-free 3-colourable graphs is ioct-bounded. By Lemma 9, H must be a forest. By Lemma 16, H must be $K_{1,5}$-free. Since H is a $K_{1,5}$-free forest, it has maximum degree at most 4. By Lemma 14, H must be $(P_1+P_4, 2P_2)$-free.

First suppose that H is P_4-free, so every component of H is a P_4-free tree. Hence every component of H is a star. In fact, as H has maximum degree at most 4, every component of H is an induced subgraph of $K_{1,4}$. As H is $2P_2$-free, at most one component of H contains an edge. Therefore H is an induced subgraph of $K_{1,4}+sP_1$ for some $s \geq 0$ and we are done.

Now suppose that H contains an induced P_4, say on vertices x_1, x_2, x_3, x_4 in that order and let $X = \{x_1, \ldots, x_4\}$. Since H is a forest, every vertex $v \in V(H) \setminus X$ has at most one neighbour in X. A vertex $v \in V(H) \setminus X$ cannot be adjacent to x_1 or x_4, since H is $2P_2$-free. By Lemma 15, the vertices x_2 and x_3 cannot both have neighbours outside X; without loss of generality assume that x_3 has no neighbours in $V(H) \setminus X$. Since H is (P_1+P_4)-free, every vertex $v \in V(H) \setminus X$ must have at least one neighbour in X, so it must be adjacent to x_2. As H has maximum degree at most 4, it follows that H is an induced subgraph of $K_{1,4}^+$. This completes the proof. ◀

By Lemma 12, for $r, s \geq 1$ the class of $K_{1,r}$-free 3-colourable graphs is ioct-bounded if and only if the class of $(K_{1,r}+sP_1)$-free 3-colourable graphs is ioct-bounded. Therefore, Theorem 3 leaves three open cases, as follows:

▶ **Open Problem 17.** *Is the class of H-free 3-colourable graphs* ioct*-bounded when H is:*
1. *$K_{1,4}$ (or equivalently $K_{1,4}+sP_1$ for any $s \geq 1$).*
2. *$K_{1,3}^+$ or*
3. *$K_{1,4}^+$.*

5 Conclusions

To develop an insight into the price of independence for classical concepts, we have investigated whether or not the size of a minimum independent vertex cover, feedback vertex set or odd cycle transversal is bounded in terms of the minimum size of the not-necessarily-independent variant of each of these transversals for H-free graphs (that have such independent transversals). We have determined the answer for every graph H except for the three missing cases for odd cycle transversal listed in Open Problem 17. While we note that the bounds we give in some of our results (Lemmas 7 and 10) are tight, in this paper we were mainly concerned with obtaining the dichotomy results on whether there is a bound, rather than trying to find exact bounds.

As results for the price of connectivity implied algorithmic consequences [9, 13], it is natural to ask if our results for the price of independence have similar consequences. The problems Independent Vertex Cover, Independent Feedback Vertex Set and Independent Odd Cycle Transversal ask to determine the minimum size of the corresponding independent transversal. The first problem is readily seen to be polynomial-time solvable. The other two problems are NP-hard for H-free graphs whenever H is not a linear forest [3], just like their classical counterparts Feedback Vertex Set [16, 17] and Odd Cycle Transversal [10] (see also [14, 15]). The complexity of these four problems restricted to H-free graphs is still poorly understood when H is a linear forest. Our results suggest that it is unlikely that we can obtain polynomial algorithms for the independent variants based on results for the original variants, as the difference between $\text{ifvs}(G)$ and $\text{fvs}(G)$ and between $\text{ioct}(G)$ and $\text{oct}(G)$ can become unbounded quickly.

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