Results on the Dimension Spectra of Planar Lines

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Abstract

In this paper we investigate the (effective) dimension spectra of lines in the Euclidean plane. The dimension spectrum of a line $L_{a,b}$, $\mathrm{sp}(L)$, with slope a and intercept b is the set of all effective dimensions of the points $(x, ax+b)$ on L. It has been recently shown that, for every a and b with effective dimension less than 1, the dimension spectrum of $L_{a,b}$ contains an interval. Our first main theorem shows that this holds for every line. Moreover, when the effective dimension of a and b is at least 1, $\mathrm{sp}(L)$ contains a *unit* interval.

Our second main theorem gives lower bounds on the dimension spectra of lines. In particular, we show that for every $\alpha \in [0,1]$, with the exception of a set of Hausdorff dimension at most α, the effective dimension of $(x, ax+b)$ is at least $\alpha + \frac{\dim(a,b)}{2}$. As a consequence of this theorem, using a recent characterization of Hausdorff dimension using effective dimension, we give a new proof of a result by Molter and Rela on the Hausdorff dimension of Furstenberg sets.

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1 Introduction

This paper is concerned with the algorithmic dimension of points on a given line in the Euclidean plane. The most well-studied algorithmic dimensions for a point $x \in \mathbb{R}^n$ are the *effective Hausdorff dimension*, $\dim(x)$, and its dual, the *effective packing dimension*, $\mathrm{Dim}(x)$ [3, 1]. Given the pointwise nature of effective dimension, it is natural to consider the dimension spectrum, $\mathrm{sp}(A)$, of a set $A \subseteq \mathbb{R}^n$, which is defined to be the set of $\dim(x)$ for all $x \in A$.

In this paper, we study the behavior of $\mathrm{sp}(L_{a,b})$, where $L_{a,b}$ is the line with slope a and intercept b. Turetsky [13] gave the first result on the dimension spectra of lines, showing that, for every $n \geq 2$, the set of all points in $\mathbb{R}^n$ with effective Hausdorff 1 is connected, implying that $1 \in \mathrm{sp}(L_{a,b})$. It was then asked by J. Lutz, with the expectation of a negative answer, if there were lines in the plane whose dimension spectrum was the singleton $\{1\}$. N. Lutz and Stull [9] showed that this cannot happen by proving the following theorem.

▶ **Theorem 1** (N. Lutz and Stull [9]). *For all $a, b, x \in \mathbb{R}$,*

$$\dim(x, ax+b) \geq \dim^{a,b}(x) + \min\{\dim(a,b), \dim^{a,b}(x)\}\,.$$

Theorem 1 implies that, when $\dim(a,b) < 1$, the dimension spectrum of $L_{a,b}$ contains the interval $[2\dim(a,b), \dim(a,b)+1]$. With this result, it is natural to conjecture that the dimension spectrum of every line $L_{a,b}$ contains an interval. Indeed, in a recent survey on effective dimension, N. Lutz [7] proposed the question of whether *every* line $L_{a,b}$ has a dimension spectrum containing a *unit* interval. Building upon the techniques of [9], N. Lutz and Stull [10] showed that this is the case for a restricted class of lines.

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Table 1 Previously known results about the dimension spectra of lines.

$\forall a, b$	$1 \in \mathrm{sp}(L_{a,b})$ [13]
$\dim(a,b) = 2$	$\mathrm{sp}(L_{a,b}) = [1,2]$
$\dim(a,b) \geq 1$	$\mathrm{sp}(L_{a,b})$ infinite [10]
$\dim(a,b) = d < 1$	$[2d, 1+d] \subseteq \mathrm{sp}(L_{a,b})$ [9]
$\dim(a,b) = 0$	$\mathrm{sp}(L_{a,b}) = [0,1]$
$\dim(a,b) = \mathrm{Dim}(a,b) = d$	$[\min\{1,d\}, 1+\min\{1,d\}] \subseteq \mathrm{sp}(L_{a,b})$ [10]

▶ **Theorem 2** (N. Lutz and Stull [10]).
1. *If* $\dim(a,b) = \mathrm{Dim}(a,b)$*, then* $\mathrm{sp}(L_{a,b})$ *contains a unit interval.*
2. *If* $\dim(a,b) \geq 1$*, then* $\mathrm{sp}(L_{a,b})$ *is infinite.*

The second item, combined with Theorem 1, shows that for every line $L_{a,b}$, the dimension spectrum of $L_{a,b}$ is infinite. Table 1 gives a summary of these results.

The question of whether the dimension spectrum of every line contains an interval has remained open. Our first main theorem settles this question.

▶ **Theorem 3.** *Let* $(a,b) \in \mathbb{R}^2$ *such that* $\dim(a,b) \geq 1$*. Then, for every real number* $d \in [0,1]$*, there is a point* x *such that*

$$\dim(x, ax+b) = 1 + d.$$

Our second main theorem deals with providing lower bounds on the dimension spectrum of a given line in the plane. The previously discussed theorems have all focused on results proving that the spectrum of a given line contains certain values. However, very little is known about the lower bound of $\mathrm{sp}(L_{a,b})$ for arbitrary lines $L_{a,b}$. Our second main theorem gives a lower bound of the spectrum of arbitrary lines, disregarding a set of small Hausdorff dimension.

▶ **Theorem 4.** *For every* $a, b \in \mathbb{R}$ *and* $\alpha \in (0,1)$*, the set*

$$A = \{x \mid dim(x, ax+b) \leq \alpha + \frac{\dim(a,b)}{2}\}$$

has Hausdorff dimension at most α*.*

Apart from being intrinsically interesting, the study of the effective dimension of points on a line has strong connections to important problems in the field of Fractal Geometry. This connection is mediated by the following theorem relating the two notions of the effective dimension of points with the Hausdorff and packing dimension of sets.

▶ **Theorem 5** (Point-to-set principle [4]). *Let* $n \in \mathbb{N}$ *and* $E \subseteq \mathbb{R}^n$*. Then*

$$\dim_H(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \dim^A(x), \text{ and}$$

$$\dim_P(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \mathrm{Dim}^A(x).$$

Recent work has used effective dimension and the point-to-set principle to prove new results in Fractal Geometry [6, 8]. In particular, the point-to-set principle combined with Theorem 1 gives improved lower bounds on the Hausdorff dimension of a certain class of *Furstenberg sets* [9], an important open problem in Fractal Geometry (see Section 5 for definitions). As our final result, we show that our second main theorem, Theorem 4, gives a new proof of a result by Molter and Rela [12] on the dimension of Furstenberg sets.

2 Preliminaries

2.1 Kolmogorov Complexity in Discrete Domains

The *conditional Kolmogorov complexity* of $\sigma \in \{0,1\}^*$ given $\tau \in \{0,1\}^*$ is

$$K(\sigma|\tau) = \min_{\pi \in \{0,1\}^*} \{\ell(\pi) : U(\pi, \tau) = \sigma\} ,$$

where U is a fixed universal prefix-free Turing machine and $\ell(\pi)$ is the length of π. Any π that achieves this minimum is said to *testify* to the value $K(\sigma|\tau)$. The *Kolmogorov complexity* of σ is $K(\sigma) = K(\sigma|\lambda)$, where λ is the empty string. An important property, due to Levin, of Kolmogorov complexity is the *symmetry of information*:

$$K(\sigma|\tau, K(\tau)) + K(\tau) = K(\tau|\sigma, K(\sigma)) + K(\sigma) + O(1) .$$

Kolmogorov complexity extends naturally to other discrete domains (e.g., integers, rationals, etc.) via standard binary encodings.

We will also frequently use *relativized Kolmogorov complexity*. Letting U be a universal *oracle* machine, we may *relativize* the definition in this section to an arbitrary oracle set $A \subseteq \mathbb{N}$. The definitions of $K^A(\sigma|\tau)$ and $K^A(\sigma)$ are then identical to those above, except that U is given oracle access to A.

2.2 Kolmogorov Complexity in Euclidean Spaces

In this section we show how to lift the definition of Kolmogorogov complexity to Euclideans spaces by introducing precision parameters [5, 4]. Let $x \in \mathbb{R}^m$, and let $r, s \in \mathbb{N}$.[1]

The *Kolmogorov complexity of x at precision r* is

$$K_r(x) = \min \{K(p) \,:\, p \in B_{2^{-r}}(x) \cap \mathbb{Q}^m\} .$$

The *conditional Kolmogorov complexity of x at precision r given $q \in \mathbb{Q}^m$* is

$$\hat{K}_r(x|q) = \min \{K(p|q) \,:\, p \in B_{2^{-r}}(x) \cap \mathbb{Q}^m\} .$$

The *conditional Kolmogorov complexity of x at precision r given $y \in \mathbb{R}^n$ at precision s* is

$$K_{r,s}(x|y) = \max \left\{\hat{K}_r(x|q) \,:\, q \in B_{2^{-r}}(y) \cap \mathbb{Q}^n\right\} .$$

We abbreviate $K_{r,r}(x|y)$ by $K_r(x|y)$.

We will frequently use the following lemma, which shows that increasing an estimate of a point is at most linearly correlated with the number of extra bits.

▶ **Lemma 6** (Case and J. Lutz [2]). *There is a constant $c \in \mathbb{N}$ such that for all $n, r, s \in \mathbb{N}$ and $x \in \mathbb{R}^n$,*

$$K_r(x) \leq K_{r+s}(x) \leq K_r(x) + K(r) + ns + a_s + c ,$$

where $a_s = K(s) + 2\log(\lceil \frac{1}{2} \log n \rceil + s + 3) + (\lceil \frac{1}{2} \log n \rceil + 3)n + K(n) + 2 \log n$.

In Euclidean spaces, we have a weaker version of symmetry of information.

[1] If we are given a nonintegral positive real as a precision parameter, we will always round up to the next integer. For example, $K_r(x)$ denotes $K_{\lceil r \rceil}(x)$ whenever $r \in (0, \infty)$.

▶ **Lemma 7** (J. Lutz and N. Lutz [4], N. Lutz and Stull [9])**.** *Let $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$. For all $r, s \in \mathbb{N}$ with $r \geq s$,*

1. $K_r(x, y) = K_r(x|y) + K_r(y) + O(\log r)$.
2. $K_r(x) = K_{r,s}(x|x) + K_s(x) + O(\log r)$.

As in the case of Kolmogorov complexity in discrete domains, we can relativize the definitions of this section. We define $K_r^A(x)$ and $K_r^A(x|y)$ as before, except we replace the unrelativized complexity K with K^A.

2.3 Effective Dimensions

Although effective Hausdorff dimension was initially developed by J. Lutz using generalized martingales [3], it was later shown by Mayordomo [11] that it may be equivalently defined using the Kolmogorov complexity of Euclidean points of the previous section. We will be using this Kolmogorov characterization here as a definition.

The *effective Hausdorff dimension* and *effective packing dimension* of a point $x \in \mathbb{R}^n$ are

$$\dim(x) = \liminf_{r \to \infty} \frac{K_r(x)}{r} \quad \text{and} \quad \mathrm{Dim}(x) = \limsup_{r \to \infty} \frac{K_r(x)}{r}.$$

Intuitively, these dimensions measure the density of algorithmic information in the point x. Recently, J. Lutz and N. Lutz [4], developed the *lower* and *upper conditional dimension* of points $x \in \mathbb{R}^m$ given $y \in \mathbb{R}^n$, defined by

$$\dim(x|y) = \liminf_{r \to \infty} \frac{K_r(x|y)}{r} \quad \text{and} \quad \mathrm{Dim}(x|y) = \limsup_{r \to \infty} \frac{K_r(x|y)}{r}.$$

Again, we can relativize the definitions of this section. We define $\dim^A(x)$, $\mathrm{Dim}^A(x)$ $\dim^A(x|y)$, and $\mathrm{Dim}^A(x|y)$ as before, except we replace the unrelativized complexity K_r with K_r^A.

Of particular importance in this paper is the complexity of a point x *relative to another point* y, written $K_r^y(x)$. This is achieved by encoding the binary expansion of y into an oracle $A_y \subseteq \mathbb{N}$ in the standard fashion. We then write $K_r^y(x)$ for $K_r^{A_y}(x)$. J. Lutz and N. Lutz showed that $K_r^y(x) \leq K_{r,t}(x|y) + K(t) + O(1)$ [4].

3 Dimension Spectra of Lines of High Dimension

3.1 Approach and Previous Work

In this section we state the technical lemmas that underlie the proof of our first main theorem (Theorem 3). These lemmas were first stated and proved by N. Lutz and Stull [9, 10][2].

▶ **Lemma 8.** *Let $a, b, x \in \mathbb{R}$, $k \in \mathbb{N}$. Suppose that $r_1, \ldots, r_k \in \mathbb{N}$, $\delta \in \mathbb{R}_+$, and $\varepsilon, \eta_1, \ldots, \eta_k \in \mathbb{Q}_+$ satisfy the following conditions for every $1 \leq i \leq k$.*

1. $r_i \geq \log(2|a| + |x| + 6) + r_{i-1}$.
2. $K_{r_i}(a, b) \leq (\eta_i + \varepsilon) r_i$.
3. *For every $(u, v) \in \mathbb{R}^2$ such that $t = -\log \|(a, b) - (u, v)\| \in (r_{i-1}, r_i]$ and $ux + v = ax + b$, $K_{r_i}(u, v) \geq (\eta_i - \varepsilon) r_i + \delta \cdot (r_i - t)$.*

[2] Lemma 8 is stated here in a slightly stronger form than the version of [10]. The proof, however, is nearly identical. For completeness we give a proof in the Technical Appendix.

Then for every oracle set $A \subseteq \mathbb{N}$,

$$K^A_{r_k}(a,b,x \mid x, ax+b) \leq 2^k \left(K(\eta_1,\ldots,\eta_k) + K(\varepsilon) + \frac{4\varepsilon}{\delta} r_k + O(\log r_k) \right) .$$

We will briefly describe the intuition behind Lemma 8. For $k = 1$, Lemma 8 roughly states that, if x and (a,b) satisfy the following properties, then we can compute an approximation of (a,b) given an approximation of $(x, ax+b)$.

1. $K_r(a,b)$ is small.
2. For every (u,v) such that $ux + v = ax + b$ either
   - $K_r(u,v)$ is large, or
   - (u,v) is close to (a,b)

This follows outputting a (u,v) of low complexity such that $ux + v = ax + b$. Under the above assumptions, any such pair must be close to (a,b), and so we can recover (a,b) with a small amount of extra information. Roughly, when $k > 1$, we do this procedure iteratively. That is, we begin by computing (a,b) to precision r_1 in the manner described above. Having done so, we do the same procedure, except that we restrict to finding a pair (u,v) *within* 2^{-r_1} of (a,b), and so on. This is useful when we can only guarantee that $K_r(u,v)$ is large when (u,v) is somewhat close to (a,b), which is the case in the proof of Theorem 3.

The next two lemmas will ensure that item (1) and (2) hold for a given pair (a,b).

▶ **Lemma 9** (N. Lutz and Stull [9]). *Let $a,b,x \in \mathbb{R}$. For all $(u,v) \in \mathbb{R}^2$ such that $ux+v = ax+b$ and $t = -\log \|(a,b)-(u,v)\| \in (0,r]$,*

$$K_r(u,v) \geq K_t(a,b) + K^{a,b}_{r-t}(x) - O(\log r) .$$

▶ **Lemma 10** (N. Lutz and Stull [10]). *Let $z \in \mathbb{R}^n$, $\eta \in \mathbb{Q} \cap [0, \dim(z)]$, and $k \in \mathbb{N}$. For all $r_1,\ldots,r_k \in \mathbb{N}$, there is an oracle $D = D(r_1,\ldots,r_k,z,\eta)$ such that*

1. *For every $t \leq r_1$, $K^D_t(z) = \min\{\eta r_1, K_t(z)\} + O(\log r_k)$*
2. *For every $1 \leq i \leq k$,*

$$K^D_{r_i}(z) = \eta r_1 + \sum_{j=2}^{i} \min\{\eta(r_j - r_{j-1}), K_{r_j,r_{j-1}}(z \mid z)\} + O(\log r_k) .$$

3. *For every $t \in \mathbb{N}$ and $x \in \mathbb{R}$, $K^{z,D}_t(x) = K^z_t(x) + O(\log r_k)$.*

3.2 First Main Theorem

In this section we prove our first main theorem, Theorem 3. To do so, we will break the proof into two cases. In the first we assume that, for arbitrarily long intervals, $K_r(a,b)$ is arbitrarily close to 1. In this case, it is "locally" as if $\dim(a,b) = \mathrm{Dim}(a,b)$, and we can use a similar proof to that of Theorem 2 in [10]. This case is formalized in the following lemma, whose proof if deferred to the appendix.

▶ **Lemma 11.** *Let $(a,b) \in \mathbb{R}^2$ such that $\dim(a,b) \geq 1$. Assume that, for every $\tau > 0$ and every $M \in \mathbb{N}$ there are infinitely many $R \in \mathbb{N}$ such that*

$$K_s(a,b) \leq (1+\tau)s,$$

for every natural number $s \in [R, MR]$. Then, for every real number $d \in (0,1]$, there is a point x such that $\dim(x, ax+b) = 1 + d$.

Proof. Let $d \in (0,1]$. For every $n \in \mathbb{N}$, let $\tau_n = \frac{1}{n}$ and $M_n = \frac{2^n}{d}$. Let $R_1, R_2, \ldots$ be a sequence of natural numbers such that the following hold.

1. $2^{R_n} < R_{n+1}$.
2. For every n, R_n satisfies the hypothesis for the choices of τ_n and M_n.

We now define a real number x such that

$$\dim(x, ax+b) = 1 + d. \tag{1}$$

Let $y \in \mathbb{R}$ be a real number that is random relative to (a,b). That is, for every $r \in \mathbb{N}$,

$$K_r^{a,b}(y) \geq r - \log r.$$

For every $n \in \mathbb{N}$, define $h_n = \frac{(M_n - 1)R_n}{2}$. For every $r \in \mathbb{N}$, let

$$x[r] = \begin{cases} 0 & \text{if } \frac{r}{h_n} \in (d, 1] \text{ for some } n \in \mathbb{N} \\ y[r] & \text{otherwise} \end{cases}$$

where $x[r]$ is the rth bit of x. Define $x \in \mathbb{R}$ to be the real number with this binary expansion.

We first claim that the dimension of $(x, ax+b)$ is at most $1+d$. For every $n \in \mathbb{N}$, by our construction of x and choice of y,

$$\begin{aligned} K_{h_n}(x) &= K_{dh_n}(x) + O(\log h_n) \\ &= K_{dh_n}(y) + O(\log h_n) \\ &\leq dh_n + O(\log h_n). \end{aligned}$$

Therefore, by the above bound and Lemma 7,

$$\begin{aligned} \dim(x, ax+b) &= \liminf_{r\to\infty} \frac{K_r(x, ax+b)}{r} \\ &= \liminf_{r\to\infty} \frac{K_r(x) + K_r(ax+b \mid x) + O(\log r)}{r} \\ &\leq \liminf_{r\to\infty} \frac{K_r(x) + r + O(\log r)}{r} \\ &\leq \liminf_{n\to\infty} \frac{K_{h_n}(x) + h_n + O(\log h_n)}{h_n} \\ &= d + 1. \end{aligned}$$

To complete the proof, it suffices to show that, for every $\eta \in \mathbb{Q} \cap (0,1)$ and $\varepsilon \in \mathbb{Q}_+$,

$$\dim(x, ax+b) \geq \eta + d - \varepsilon. \tag{2}$$

To that end, let $\eta \in \mathbb{Q} \cap (0,1)$ and $\varepsilon \in \mathbb{Q}_+$. To prove inequality (2), we will partition $\mathbb{N}$ into intervals, and focus on the complexity of $(x, ax+b)$ at each precision r in these intervals. For every $n \in \mathbb{N}$, let $I_n = (dh_n, dh_{n+1}]$.

Fix $n \in \mathbb{N}$, and let $m = \frac{1-d}{1-\eta}$. We will first consider $r \in (dh_n, mh_n]$. Let $k = \frac{r}{dh_n}$, and define $r_i = idh_n$ for every $1 \leq i \leq k$. It is important to note that k is bounded by a constant depending only on η and d. In particular, this implies that $o(r_k)$ is sublinear for all r_i. Let $D_r = D(r_1, \ldots, r_k, a, b, \eta)$ be the oracle defined in Lemma 10. We first note that, by our assumption of (a,b) on the interval $[R_n, MR_n]$ and Lemma 7,

$$\begin{aligned} K_{r_i, r_{i-1}}(a, b \mid a, b) &= K_{r_i}(a,b) - K_{r_{i-1}}(a,b) - O(\log r_i) \\ &\geq r_i - o(r_i) - (1 + \frac{1}{n}) r_{i-1} - O(\log r_i) \\ &= r_i - r_{i-1} - \frac{r_{i-1}}{n} - o(r_i), \end{aligned}$$

for all sufficiently large r. Since k is bounded by a constant, for all sufficiently large n, we have

$$K_{r_i,r_{i-1}}(a,b\,|\,a,b) > \eta(r_i - r_{i-1}) - o(r_i).$$

Hence, by Lemma 10,

$$|K^{D_r}_{r_i}(a,b) - \eta r_i| < o(r_k), \tag{3}$$

for all $1 \leq i \leq k$.

We now show that the conditions of Lemma 8 are satisfied relative to D_r. Item 1 of Lemma 8 holds for all sufficiently large r. For item 2, by the construction of D_r, for every $1 \leq i \leq k$,

$$\begin{aligned} K^{D_r}_{r_i}(a,b) &= \eta r_1 + \sum_{j=2}^{i} \min\{\eta(r_j - r_{j-1}), K_{r_j,r_{j-1}}(z\,|\,z)\} + O(\log r_k) \\ &\leq \eta r_1 + \sum_{j=2}^{i} \eta(r_j - r_{j-1} + O(\log r_k) \\ &\leq \eta r_i + O(\log r_k) \\ &\leq (\eta + \varepsilon) r_i, \end{aligned}$$

for all sufficiently large r.

Let $\delta = 1 - \eta$. To see that item 3 of Lemma 8 is satisfied for $i = 1$, let $(u,v) \in B_1(a,b)$ such that $ux + v = ax + b$ and $t = -\log \|(a,b) - (u,v)\| \leq r_1$. Then, by Lemmas 9 and 10, and our construction of x,

$$\begin{aligned} K^{D_r}_{r_1}(u,v) &\geq K^{D_r}_t(a,b) + K^{D_r}_{r_1-t,r_1}(x|a,b) - O(\log r_1) \\ &\geq \min\{\eta r_1, K_t(a,b)\} + K_{r_1-t}(x) - o(r_k) \\ &\geq \min\{\eta r_1, t - o(t)\} + (\eta + \delta)(r_1 - t) - o(r_k) \\ &\geq \min\{\eta r_1, \eta t - o(t)\} + (\eta + \delta)(r_1 - t) - o(r_k) \\ &\geq \eta t - o(t) + (\eta + \delta)(r_1 - t) - o(r_k)\,, \end{aligned}$$

We conclude that $K^{D_r}_{r_1}(u,v) \geq (\eta - \varepsilon) r_1 + \delta(r_1 - t)$, for all sufficiently large r. To see that that item 3 is satisfied for $1 < i \leq k$, let $(u,v) \in B_{2^{-r_{i-1}}}(a,b)$ such that $ux + v = ax + b$ and $t = -\log \|(a,b) - (u,v)\| \leq r_i$. Since $(u,v) \in B_{2^{-r_{i-1}}}(a,b)$,

$$r_i - t \leq r_i - r_{i-1} = idh_j - (i-1)dh_j \leq dh_j + 1 \leq r_1 + 1\,.$$

Therefore, by Lemma 9, inequality (3), and our construction of x,

$$\begin{aligned} K^{D_r}_{r_i}(u,v) &\geq K^{D_r}_t(a,b) + K^{D_r}_{r_i-t,r_i}(x|a,b) - O(\log r_i) \\ &\geq \min\{\eta r_i, K_t(a,b)\} + K_{r_i-t}(x) - o(r_i) \\ &\geq \min\{\eta r_i, t - o(t)\} + (\eta + \delta)(r_i - t) - o(r_i) \\ &\geq \min\{\eta r_i, \eta t - o(t)\} + (\eta + \delta)(r_i - t) - o(r_i) \\ &\geq \eta t - o(t) + (\eta + \delta)(r_i - t) - o(r_i)\,. \end{aligned}$$

We conclude that $K^{D_r}_{r_i}(u,v) \geq (\eta - \varepsilon) r_i + \delta(r_i - t)$, for all sufficiently large r. Hence the conditions of Lemma 8 are satisfied. Therefore, by applying Lemma 8 and appealing to inequality (3),

$$\begin{aligned}
K_r(x, ax+b) \geq & K_r^{D_r}(x, ax+b) \\
& \geq K_r(a,b,x) - 2^k\left(K(\varepsilon) + K(\eta) + \frac{4\varepsilon}{1-\eta} r_k + O(\log r_k)\right) \\
& = K_r(a,b) + K_r(x \mid a,b) \\
& \quad - 2^k\left(K(\varepsilon) + K(\eta) + \frac{4\varepsilon}{1-\eta} r_k + O(\log r_k)\right) \\
& \geq dr + \eta r - 2^k\left(K(\varepsilon) + K(\eta) + \frac{4\varepsilon}{1-\eta} r_k + O(\log r_k)\right).
\end{aligned}$$

To complete the proof, we give lower bounds of $K_r(x, ax+b)$ for every $r \in [mh_n, dh_{n+1})$. By Lemma 7 and our construction of x,

$$\begin{aligned}
K_r(x) &= K_{r,h_n}(x \mid x) + K_{h_n}(x) - o(r) \\
&= r - h_n + dh_n - o(r) \\
&\geq \eta r - o(r).
\end{aligned}$$

The proof of Theorem 1 gives

$$\begin{aligned}
K_r(x, ax+b) &\geq K_r(x) + \dim(x) r - o(r) \\
&\geq \eta r + dr - o(r) \\
&\geq r(d+\eta) - \varepsilon r.
\end{aligned}$$

Putting together the lower bounds of $K_r(x, ax+b)$ on the intervals (dh_n, mh_n) and $[mh_n, dh_{n+1}]$ shows that

$$\dim(x, ax+b) \geq 1 + d,$$

and the proof is complete. ◀

If (a,b) is not of the first case, then there is a bound $(1+\tau) > 1$ so that $K_r(a,b) \geq r(1+\tau)$ for some r in every sufficiently large interval. This implies that, for almost every precision r, the conditional complexity $K_{s,r}(a,b \mid a,b) > s - r$, for some s at most a constant multiple of r. This fact allows us to use the procedure outlined in Section 3.1 at precision s. We will now formalize this intuition.

▶ **Theorem 3.** *Let $(a,b) \in \mathbb{R}^2$ such that $\dim(a,b) \geq 1$. Then, for every real number $d \in [0,1]$, there is a point x such that*

$$\dim(x, ax+b) = 1 + d.$$

Proof. Let $(a,b) \in \mathbb{R}^2$ such that $\dim(a,b) \geq 1$. For $d = 1$, we may choose an $x \in \mathbb{R}$ that is random relative to (a,b). That is, there is some constant $c \in \mathbb{N}$ such that for all $r \in \mathbb{N}$, $K_r^{a,b}(x) \geq r - c$. By Theorem 1,

$$\begin{aligned}
\dim(x, ax+b) &\geq \dim^{a,b}(x) + \min\{\dim(a,b), 1\} \\
&= \liminf_{r\to\infty} \frac{K_r(x)}{r} + 1 \\
&= 2,
\end{aligned}$$

and the conclusion holds. For $d = 0$, the conclusion follows from Turetsky's theorem [13]. We therefore assume that $d \in (0, 1)$.

If (a, b) satisfy the conditions of Lemma 11, then the conclusion is immediate. So assume that the conditions of Lemma 11 do not hold. Let $\tau > 0$ and $M > 0$ be constants such that, for almost every $R \in \mathbb{N}$,

$$K_s(a, b) > (1 + \tau)s,$$

for some $s \in [R, MR]$.

Let $y \in \mathbb{R}$ be random relative to (a, b). Define the sequence of natural numbers $\{h_n\}_{n\in\mathbb{N}}$ inductively as follows. Define $h_0 = 1$. For every $n > 0$, let

$$h_n = \min \left\{ h \geq 2^{h_{n-1}} : K_h(a, b) \geq \left(\mathrm{Dim}(a, b) - \frac{1}{n} \right) h \right\}.$$

Note that h_n always exists. For every $r \in \mathbb{N}$, let

$$x[r] = \begin{cases} 0 & \text{if } \frac{r}{h_n} \in (d, 1] \text{ for some } n \in \mathbb{N} \\ y[r] & \text{otherwise} \end{cases}$$

where $x[r]$ is the rth bit of x. Define $x \in \mathbb{R}$ to be the real number with this binary expansion. Then $K_{dh_n}(x) = dh_n + O(\log dh_n)$.

Claim 1: $\dim(x, ax + b) \leq 1 + d$.

Let $\eta \in \mathbb{Q} \cap (0, 1)$, $\varepsilon \in \mathbb{Q}$, $n \in \mathbb{N}$, and let $m = \frac{1-d}{1-\eta}$. We first give lower bounds of the complexity of $K_r(x, ax + b)$ on the interval (dh_n, mh_n). To begin, consider $r = h_n$. Let $k = \frac{r}{dh_n} = \frac{1}{d}$, and define $r_i = idh_n$ for every $1 \leq i \leq k$. As in the proof of Lemma 11, is important to note that k is bounded by a constant depending only on η and d.

Claim 2: $K_{h_n}(x, ax + b) \geq dh_n + \eta h_n - 2^k \Big(K(\varepsilon) + kK(\eta) + \frac{4\varepsilon}{1-\eta} h_n + O(\log h_n) \Big)$.

With this bound on the complexity of $(x, ax + b)$ at precision h_n, we will use a symmetry of information argument to give a lower bound on the complexity at precision $r \in (dh_n, h_n)$. We defer the proof of this claim to the appendix.

Claim 3: For all $r \in [dh_n, h_n)$,

$$K_r(x, ax + b) \geq r(d + \eta) - 2^k \left(K(\varepsilon) + K(\eta) + \frac{4\varepsilon}{1 - \eta} h_n + O(\log h_n) \right).$$

Note that this lower bound is useful for $r \in (dh_n, h_n)$, since h_n is a fixed constant multiple of r.

We now turn to proving lower bounds for the complexity of $(x, ax + b)$ on the interval (h_n, mh_n). To do so, we will make use of our assumption that the complexity of $K_s(a, b)$ is at least $(1 + \tau)s$. In particular, this assumption implies that there is a *fixed* constant c such that $K_{ch_n,h_n}(a, b \mid a, b) \geq \eta(ch_n - h_n)$. Moreover, this constant is independent of η and ε. To see this, let $s > h_n$ be a precision such that $K_s(a, b) \geq (1 + \tau)s$. By Lemma 7 and our assumption of a, b,

$$\begin{aligned} K_{s,h_n}(a, b \mid a, b) &\geq K_s(a, b) - K_{h_n}(a, b) - O(\log s) \\ &\geq (1 + \tau)s - \mathrm{Dim}(a, b)h_n - O(\log s) \\ &\geq (1 + \tau)s - 2h_n - O(\log s). \end{aligned}$$

Thus,

$$K_{s,h_n}(a, b \mid a, b) \geq \eta(s - h_n),$$

for any such $s > ch_n$, for some fixed constant c depending only on τ. With this fact we are able to show the following, whose proof is deferred to the appendix.

Claim 4: There is a precision $h_n < j \leq ch_n$ such that

$$K_j(x, ax+b) \geq j - (h_n - dh_n) + \eta j - 2^k \left(K(\varepsilon) + kK(\eta) + \frac{4\varepsilon}{1-\eta} j + O(\log j)\right).$$

With this bound, we will again prove lower bounds at precisions $r \in (h_n, j)$ using symmetry of information arguments. While this is similar in spirit to the proof of Claim 3, there is an important difference. At precisions greater than h_n, the complexity of x begins increasing again. In particular, the construction of x and Lemma 6 implies the following.

$$K_j(x, ax+b) \leq K_r(x, ax+b) + 2(j-r).$$

We are still, however, able to achieve the required lower bounds on the complexity of $(x, ax+b)$ for all $r \in (h_n, j)$. The proof of this claim is deferred to the appendix.

Claim 5: For every $r \in (h_n, j)$,

$$K_r(x, ax+b) \geq r(d+\eta) - cr(1-\eta) - 2^k \left(K(\varepsilon) + K(\eta) + \frac{4\varepsilon}{1-\eta} cr + O(\log cr)\right).$$

This lower bound is useful since $j \leq ch_n$, and c is a constant depending only on τ. In particular, this allows us to make $\frac{cr(1-\eta)}{r}$ arbitrarily small by having η go to 1.

To complete the proof for the interval (dh_n, mh_n), we will apply the same method as in Claims 4 and 5, except that we use j instead of h_n. Specifically, we choose the first $j_2 > j$ such that

$$K_{j_2,j}(a, b \mid a, b) \geq \eta(j_2 - j),$$

and note that $j_2 \leq cj$. We then apply the proof of Claim 4 to $K_{j_2}(x, ax+b)$, and the proof of Claim 5 to the interval (j, j_2). We then repeat this argument until we have given the appropriate lower bound for all $r \in (h_n, mh_n)$.

Finally, taking Claims 1, 2, 3, 4 and 5 together yields the following. For every $r \in (dh_n, m_n)$,

$$K_r(x, ax+b) \geq r(d+\eta) - cr(1-\eta) - 2^k \left(K(\varepsilon) + K(\eta) + \frac{4\varepsilon}{1-\eta} cr + O(\log cr)\right). \quad (4)$$

To complete the proof, we bound $K_r(x, ax+b)$ for every $r \in [mh_n, dh_{n+1})$. By Lemma 7 and our construction of x,

$$\begin{aligned} K_r(x) &= K_{r,h_n}(x \mid x) + K_{h_n}(x) + o(r) \\ &= r - h_n + nh_n + o(r) \\ &\geq \eta r + o(r). \end{aligned}$$

The proof of Theorem 1 gives $K_r(x, ax+b) \geq K_r(x) + \dim(x) r - o(r)$, and so $K_r(x, ax+b) \geq r(d+\eta) - \varepsilon r$. Combined with inequality (4), for every $r \in (dh_n, dh_{n+1})$,

$$\begin{aligned} \frac{K_r(x, ax+b)}{r} &\geq r(d+\eta) - cr(1-\eta) - 2^k \left(K(\varepsilon) + K(\eta) + \frac{4\varepsilon}{1-\eta} cr + O(\log cr)\right) \\ &= d + \eta - 2^k \left(\frac{K(\varepsilon)}{r} + \frac{K(\eta)}{r} + \frac{4\varepsilon}{1-\eta} c + \frac{O(\log cr)}{r}\right) \\ &\geq d + \eta - 2^{\frac{m}{d}} \left(\frac{K(\varepsilon)}{r} + \frac{K(\eta)}{r} + \frac{4\varepsilon}{1-\eta} c + \frac{O(\log cr)}{r}\right) \end{aligned}$$

for all sufficiently large n. Since η and ε were chosen arbitrarily and independently,

$$\dim(x, ax+b) \geq d+1,$$

and the proof is complete. ◀

4 Lower Bounding the Dimension Spectrum of a Line

In this section we give the first nontrivial lower bounds of the dimension spectrum of an arbitrary line. For intuition behind the proof, first note the following simple observation.

▶ **Observation 12.** *For every $x, y, a, b \in \mathbb{R}$,*

$$K_r(x, y, a, b) \leq K_r(x, ax+b) + K_r(y, ay+b) + 2t,$$

where $t = -\log\|x-y\|$.

Essentially, this is true since any two points identify a line, and this can be done in a computable way. The $2t$ extra information is due to the fact the precision which we can compute (a, b) to is linearly correlated to the distance between x and y. This immediately suggests an approach to give the lower bound

$$\dim(x) + \frac{\dim(a,b)}{2} \geq \dim(a, b).$$

While this observation is at the core of the proof of our second main theorem, it alone does not suffice. The principle issue is that the values of $K_r(x, ax+b)$, $K_r(y, ay+b)$ might be "out of phase"; that is, $K_r(x, ax+b)$ is small when $K_r(y, ay+b)$ is large, and vice versa. Our main theorem will show that the set of these points has low Hausdorff dimension.

Our first lemma builds upon Observation 12. In particular, it shows that, if $K_r(x, ax+b)$ is small, then every other y such that $K_r(y, ay+b)$ is small must satisfy certain properties.

▶ **Lemma 13.** *Let $\alpha \in (0,1)$, $x, a, b \in \mathbb{R}$, and $n, r \in \mathbb{N}$ such that $2r^{-\frac{1}{2}} < \frac{1}{n}$. Assume that $K_r(x, ax+b) < \alpha r + \frac{K_r(a,b)}{2} - \frac{r}{n}$, and $K_r^{a,b}(x) \geq \alpha r$. Then, for every $y \in \mathbb{R}$, if $K_r(y, ay+b) < \alpha r + \frac{K_r(a,b)}{2}$, at least one of the following holds.*

1. *$t := -\log\|x-y\| \leq r^{\frac{1}{2}}$.*
2. *$K_r(y \mid a, b, x) < \alpha r$.*

Proof. Assume the hypothesis, but assume that neither condition is satisfied for some y. Then,

$$\begin{aligned} K_r(a, b, x, y) &\leq K_r(x, ax+b) + K_r(y, ay+b) + 2t \\ &< 2\alpha r + K_r(a, b) - \frac{r}{n} + 2t. \end{aligned}$$

However, by our hypothesis and Lemma 7 we have

$$\begin{aligned} K_r(a, b, x, y) &\geq K_r(a, b) + K_r(x \mid a, b) + K_r(y \mid a, b, x) - \log r \\ &\geq K_r(a, b) + 2\alpha r - \log r. \end{aligned}$$

Since condition (1) was assumed to not hold, we see that

$$\frac{1}{n} < 2r^{-\frac{1}{2}},$$

a contradiction. ◀

For every $a, b \in \mathbb{R}$ and $\alpha \in (0,1)$, define the set

$$A(\alpha, a, b) = \{x \mid \dim(x, ax+b) < \alpha + \frac{\dim(a,b)}{2}\}.$$

▶ **Theorem 4.** *For every $a, b \in \mathbb{R}$ and $\alpha \in (0,1)$, $\dim_H(A(\alpha, a, b)) \leq \alpha$.*

Proof. Our goal is to show that $\dim_H(A(\alpha, a, b) \leq \alpha$. We will actually prove a stronger theorem. For every n, define

$$A_n(\alpha, a, b) = \{x \mid (\exists^\infty r)\, K_r(x, ax+b) < \alpha r + \frac{K_r(a,b)}{2} - \frac{r}{n}\}.$$

Note that to prove $\dim_H(A(\alpha, a, b) \leq \alpha$, it suffices to show that $\dim_H(A_n(\alpha, a, b)) \leq \alpha$ for every n. To see that $A(\alpha, a, b) \subseteq \cup_n A_n(\alpha, a, b)$, let $x \in A$. Then there is an $\epsilon > 0$ such that, for infinitely many r,

$$\begin{aligned} K_r(x, ax+b) &< \alpha r + \frac{\dim(a,b)}{2} r - \epsilon r \\ &< \alpha r + \frac{K_r(a,b) + g(r)}{2} - \epsilon r, \end{aligned}$$

where g is a sublinear function. Therefore, for sufficiently large n and r, $x \in A_n(\alpha, a, b)$. Since the Hausdorff dimension of a countable union of sets $\cup_n A_n$ is the supremum of $\dim_H(A_n)$, it suffices to show that, for every n, $\dim_H(A_n(\alpha, a, b)) \leq \alpha$.

Define the set

$$U = \{x \mid (\exists^\infty r)\, K_r^{a,b}(x) \leq \alpha r\}.$$

It is immediate that $\dim^{a,b}(x) \leq \alpha$, for all $x \in U$.

For every $r \in \mathbb{N}$, choose x_r such that

$$\begin{aligned} &K_r^{a,b}(x) \geq \alpha r \\ &K_r(x, ax+b) < \alpha r + \frac{K_r(a,b)}{2} - \frac{r}{n}, \end{aligned}$$

if such an x_r exists. To reduce the notational burden we will, without loss of generality, always assume that such an x_r does exist. We then define

$$V = \{y \mid (\exists^\infty r)\, y \in (x_r - 2^{-r^{\frac{1}{2}}}, x_r + 2^{-r^{\frac{1}{2}}})\}.$$

Define oracle $R \subseteq \mathbb{N}$ which encodes the sequence $x_1, x_2, \ldots$ in the standard manner. Let $y \in V$, and let $r \in \mathbb{N}$ such that $y \in (x_r - 2^{-r^{\frac{1}{2}}}, x_r + 2^{-r^{\frac{1}{2}}})$. Then,

$$\begin{aligned} K^R_{r^{\frac{1}{2}}}(y) &\leq O(\log r) \\ &= O(\log r^{\frac{1}{2}}). \end{aligned}$$

Thus

$$\dim^R(y) = 0.$$

Let $x_1, x_2, \ldots$ be the sequence chosen above. Define

$$W = \{y \mid (\exists^\infty r)\, K_r(y \mid a, b, x_r) < \alpha r\}.$$

Let $y \in W$, and let $r \in \mathbb{N}$ such that $K_r(y \mid a, b, x_r) < \alpha r$. Then we have

$$\begin{aligned} K_r^{R,a,b}(y) &\leq K_r(y \mid a, b, x_r) + O(\log r) \\ &< \alpha r + O(\log r). \end{aligned}$$

Thus

$$\dim^{R,a,b}(y) \leq \alpha.$$

We now show that $A_n(\alpha, a, b) \subseteq U \cup V \cup W$. Let $y \in A_n(\alpha, a, b)$, and assume that $y \notin U \cup V$. So then y has the following properties.

1. For infinitely many r, $K_r(y, ay+b) < \alpha r + \frac{K_r(a,b)}{2} - \frac{r}{n}$
2. For almost every r, $K_r^{a,b}(y) > \alpha r$.
3. For almost every r, $y \notin (x_r - 2^{-r^{\frac{1}{2}}}, x_r + 2^{-r^{\frac{1}{2}}})$.

Let $r \in \mathbb{N}$ be a sufficiently large integer such that item (1) holds. Then by Lemma 13, we must have that

$$K_r(y \mid a, b, x_r) < \alpha r.$$

Therefore, $y \in W$, and $A_n(\alpha, a, b) \subseteq U \cup V \cup W$. Hence $\dim^{R,a,b}(y) \leq \alpha$, and the proof is complete. ◀

5 Applications to Furstenberg Sets

In this section we will use the point-to-set principle, Theorem 5, in conjunction with the theorem of the previous section to give a new proof of a result by Molter and Rela on Furstenberg sets. Let $\alpha \in [0,1]$. A *set of Furstenberg type* with parameter α is a set $E \subseteq \mathbb{R}^2$ such that, for every $e \in S^1$, there is a line ℓ_e in the direction e satisfying $\dim_H(E \cap \ell_e) \geq \alpha$. It is an important open problem in Fractal Geometry to find the minimum possible dimension of a set of Furstenberg type with paramater α.

Molter and Rela [12] have recently introduced a generalization of Furstenberg sets, by removing the restriction that *every* direction must intersect the set. A set $E \subseteq \mathbb{R}^2$ is in the class $F_{\alpha\beta}$ if there is some set $J \subseteq S^1$ such that

1. $\dim_H(J) \geq \beta$, and
2. for every $e \in J$, there is a line ℓ_e in the direction e satisfying $\dim_H(E \cap \ell_e) \geq \alpha$.

They proved the following lower bound on the dimension of such sets.

▶ **Theorem 14.** (Molter and Rela [12]) *For all $\alpha, \beta \in (0,1]$ and every set $E \in F_{\alpha\beta}$,*

$$\dim_H(E) \geq \alpha + \frac{\beta}{2}.$$

We will now give a new proof of this theorem, using the theorems of the previous section.

Proof of Theorem 14. Let $\alpha, \beta \in (0,1]$, $\epsilon > 0$, and $E \in F_{\alpha\beta}$. Let $A \subseteq \mathbb{N}$ be an oracle testifying to the Hausdorff dimension of E; i.e.,

$$\dim_H(E) = \sup_{z \in E} \dim^A(z).$$

Let $e \in S^1$ satisfy $\dim^A(e) > \beta - \epsilon$. Note that such a direction exists by the point-to-set principle. Let l_e be a line in direction e such that $\dim_H(l_e \cap E) \geq \alpha$. Let $a, b \in \mathbb{R}$ be the reals such that $L_{a,b} = l_e$. Note that $\dim^A(a,b) = \dim^A(e)$ because the mapping $e \mapsto a$ is

Table 2 Updated table of the dimension spectra of lines.

$\forall a, b$	$1 \in \mathrm{sp}(L_{a,b})$	
$\dim(a,b) = 2$	$\mathrm{sp}(L_{a,b}) = [1,2]$	
$\dim(a,b) \geq 1$	$[1,2] \subseteq \mathrm{sp}(L_{a,b})$	$\mathrm{sp}(L_{a,b}) \subseteq [1,2]$ Except for a set of dimension at most $\frac{1}{2}$
$\dim(a,b) = d < 1$	$[2d, 1+d] \subseteq \mathrm{sp}(L_{a,b})$,	$\mathrm{sp}(L_{a,b}) \subseteq [d, 1+d]$ Except a set of dimension at most $\frac{d}{2}$
$\dim(a,b) = 0$	$\mathrm{sp}(L_{a,b}) = [0,1]$	
$\dim(a,b) = \mathrm{Dim}(a,b)$	$[d, 1+d\}] \subseteq \mathrm{sp}(L_{a,b})$, $d = \min\{1, \dim(a,b)\}$	

computable and bi-Lipschitz in a neighborhood of e. Let $S = \{x \mid (x, ax+b) \in E \cap l_e\}$. Note that this implies that $\dim_H(S) \geq \alpha$. We then have that

$$\dim_H(E) \geq \sup_{x \in S} \dim^A(x, ax+b).$$

Therefore, to complete the proof, it suffices to show that there exists a point $x \in S$ such that

$$\dim^A(x, ax+b) \geq \alpha + \frac{\beta}{2} - \epsilon. \tag{5}$$

By Theorem 4, relativized to A, the set of all x such that $\dim^A(x, ax+b) \leq \alpha + \frac{\beta}{2}$ has Hausdorff dimension at most $\alpha - \epsilon$. Since $\dim_H(S) \geq \alpha$, this implies that there is a point $x \in S$ which satisfies (5), and the proof is complete. ◀

6 Conclusion and Future Directions

In this paper, we have given two new results on the dimension spectra of lines in the plane, summarized in Table 2.

The first gives a partial answer to the question posed by Lutz, asking whether, for every line $L_{a,b}$, $\mathrm{sp}(L_{a,b})$ contains a unit interval. We showed that if $\dim(a,b) \geq 1$, then this is true. Together with a previous result of N. Lutz and Stull, this implies the following.

▶ **Corollary 15.** *For every $a, b \in \mathbb{R}$, $\mathrm{sp}(L_{a,b})$ contains an interval.*

However we still do not have complete answer to Lutz's question. This is an important open problem, and one that seems to require new techniques to solve.

We have also given the first nontrivial lower bound on the dimension spectrum of a given line. An important open problem is to improve the bounds given here. This would be not only an intrinsically interesting result, but would likely give improved bounds on Furstenberg sets. Another interesting direction for future research is to construct lines with "many" points of small dimension. In particular, for a given $\alpha > 0$, is there a line $L_{a,b}$ such that

$$\dim_H(A(\alpha, a, b)) = \alpha?$$

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