New Algorithms for Edge Induced König-Egerváry Subgraph Based on Gallai-Edmonds Decomposition

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Abstract

König-Egerváry graphs form an important graph class which has been studied extensively in graph theory. Much attention has also been paid on König-Egerváry subgraphs and König-Egerváry graph modification problems. In this paper, we focus on one König-Egerváry subgraph problem, called the Maximum Edge Induced König Subgraph problem. By exploiting the classical Gallai-Edmonds decomposition, we establish connections between minimum vertex cover, Gallai-Edmonds decomposition structure, maximum matching, maximum bisection, and König-Egerváry subgraph structure. We obtain a new structural property of König-Egerváry subgraph: every graph $G = (V, E)$ has an edge induced König-Egerváry subgraph with at least $2|E|/3$ edges. Based on the new structural property proposed, an approximation algorithm with ratio $10/7$ for the Maximum Edge Induced König Subgraph problem is presented, improving the current best ratio of $5/3$. To the best of our knowledge, this paper is the first one establishing the connection between Gallai-Edmonds decomposition and König-Egerváry graphs. Using $2|E|/3$ as a lower bound, we define the Edge Induced König Subgraph above lower bound problem, and give a kernel of at most $30k$ edges for the problem.

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1 Introduction

Given a graph G, a matching M in G is a set of vertex-disjoint edges. Matching problem is one of the fundamental problems in combinatorial optimization, and has wide applications in many fields. For several decades, much attention has been paid on matching and related problems.

The Vertex Cover problem is closely related to the matching problem, which is to decide, for a given graph $G = (V, E)$, whether there exists a subset $V' \subseteq V$ of at most k vertices such that each edge in G has at least one endpoint in V'. The Vertex Cover problem is one of the 21 NP-complete problems [19], and has been extensively studied in the field of parameterized complexity [8, 15, 22, 32, 36]. The current best parameterized algorithm for the Vertex Cover problem is of running time $O^*(1.2738^k)$ [8], where k is the size of vertex cover in given graph. Matching methods can also be applied to deal with the Vertex Cover problem. For the bipartite graphs, it is proved that the size of a minimum vertex cover is equal to the size of a maximum matching [1]. Thus, the Vertex Cover problem on bipartite graphs can be solved in polynomial time based on the algorithms of getting a maximum matching. For general graphs, based on the maximum matching, an approximation algorithm with ratio 2 can be obtained for the Vertex Cover problem, which is still the current best approximation ratio for the problem. By using matching number as a lower bound, a variant of the Vertex Cover problem, called Above-Guarantee Vertex Cover problem (given a graph G and parameter k, decide whether G has a vertex cover of size at most $|M| + k$, where M is a maximum matching in G) was first studied in [40]. Thereafter, several interesting results for the Above-Guarantee Vertex Cover problem have been obtained [9, 15, 25, 36, 37, 34].

The classical Gallai-Edmonds decomposition method provides an elegant structure for graphs based on matching. For any graph G, a Gallai-Edmonds decomposition of graph G can be obtained in polynomial time [27], which is a tuple (X, Y, Z), where X is the set of vertices in G which are not covered by at least one maximum matching of G, Y is $N(X)$ ($N(X)$ is the set of neighbors of the vertices in X with $N(X) \cap X = \emptyset$), and $Z = V(G) \backslash (X \cup Y)$. The application of Gallai-Edmonds decomposition has been paid lots of attention, and many problems were studied by applying Gallai-Edmonds decomposition from approximation algorithms and parameterized complexity points of view, such as approximation algorihtms [14, 28, 35], kernelizations [13, 21, 33], parameterized algorithms [7, 11, 15], etc. Gallai-Edmonds decomposition has also been applied to solve problems in many other fields [2, 3, 18, 38].

A graph G is a *König-Egerváry* graph (in short, König graph) if the size of a minimum vertex cover of G is equal to the size of a maximum matching of G. The structural properties of König graphs have been studied for a long time. Deming [10] studied the characterizations of König graphs through independence number of graphs, and proved that the König graphs can be recognized in polynomial time. Stersoul [39] studied the characterizations of König graphs through the structure of matchings in graphs. Lovász [26] studied König graphs with perfect matching, and gave the excluded subgraphs characterizations through matching-covered graphs. Bourjolly and Pulleyblank [5] studied the relation between König graphs and 2-bicritical graphs, and showed that the characterizations of König graphs can be used to obtain a structural characterization of 2-bicritical graphs. Korach, Nguyen, and Peis [20] studied subgraph characterizations of Red/Blue-Split graphs and König graphs, where Red/Blue-Split graphs are the generalizations of König graphs and Split graphs. Levit and Mandrescu [23] studied the relation between critical independent sets and König graphs.

Levit and Mandrescu [24] studied maximum matchings in König graphs, and gave a new characterization through maximum matching. Bonomo et al. [4] presented a characterization of König graphs by forbidden subgraphs. Jarden et al. [17] further studied the relation between maximum independent sets, maximum matching, and König graphs, and gave two characterizations of König graphs. Cardoso et al. [6] gave some combinatorial and spectral properties of König graphs through Laplacian eigenvalues.

In this paper, we focus on the König-Egerváry subgraph problem, and study the problem from approximation algorithm and parameterized complexity points of view. For a graph $G = (V, E)$ and a subset $E' \subseteq E$, the subgraph induced by edges in E', denoted by $G[E']$, is the one that contains the endpoints of the edges in E', and contains the edges in E'. If the size of a minimum vertex cover is equal to the size of a maximum matching in $G[E']$, then $G[E']$ is called a *König subgraph*. We now give the definitions of the related problems.

Maximum Edge Induced König Subgraph:
Given a graph $G = (V, E)$, find a set $E' \subseteq E$ with maximum number of edges such that the edges in E' induce a König subgraph.

Edge Induced König Subgraph:
Given a graph $G = (V, E)$ and non-negative integer k, find a set E' of at least k edges in E such that the edges in E' induce a König subgraph, or report that no such set exists.

The Edge Induced König Subgraph problem is closely related to a graph modification problem, called König Edge Deletion problem, which is to delete at most k edges to turn a given graph into a König graph. For the NP-completeness, the Edge Induced König Subgraph problem and the König Edge Deletion problem are equivalent. However, the approximability and parameterized complexity of those two problems are totally different. For the Edge Induced König Subgraph problem, Mishra et al. [32] presented an approximation algorithm with ratio 5/3, and gave a parameterized algorithm of running time $O^*(2^k)$. For the König Edge Deletion problem, Mishra et al. [32] proved that this problem does not admit any constant-factor approximation algorithm unless UGC fails. As pointed out in [30, 31, 32], the parameterized complexity of the König Edge Deletion problem is still open. On the other hand, many other König subgraph and König graph problems have also been studied. Mishra et al. [30, 32] studied the Vertex Induced König Subgraph problem (given a graph G and non-negative integer k, decide whether there exists a set of at least k vertices that induces a König subgraph) and the König Vertex Deletion problem (given a graph G and non-negative integer k, decide whether there exists a set of at most k vertices whose deletion results in a König subgraph). For the Vertex Induced König Subgraph problem, Mishra et al. [32] proved that it is $W[1]$-hard. For the König Vertex Deletion problem, a series of parameterized algorithms have been proposed [9, 25, 30, 32]. As the generalizations of König graphs and Split graphs, Red/Blue-Split graph modification problems have also been studied [20, 29, 30].

In this paper, we study the Edge Induced König Subgraph problem from approximaition and parameterized algorithms points of view. The main contribution of this paper is that we present structural connections between minimum vertex cover, Gallai-Edmonds decomposition, maximum bisection, and König subgraphs, get a new structural property for the König subgraph of a given graph, and propose an improved approximation algorithm for the Edge Induced König Subgraph problem. To the best of our knowledge, this paper is the first one to establish connection between Gallai-Edmonds decomposition and the structures of König graphs.

We now point out the differences of our techniques and results in this paper with the ones in [31, 32].

(1) The 5/3-approximation algorithm for the Maximum Edge Induced König Subgraph problem in [31, 32] is based on an important property of König subgraph: every graph G has an edge induced König subgraph of at least $3m/5$ edges, where m is the number of edges in G, which is obtained in [31, 32] based on the maximum matching in G. In this paper, we exploit the connection between Gallai-Edmonds decomposition and König subgraphs, and present a new structural property of König subgraphs: every G has an edge induced König subgraph of at least $2m/3$ edges, which results in an improved approximation algorithm with ratio 10/7.

(2) For a Gallai-Edmonds decomposition (X, Y, Z) of given graph G, instead of directly applying the matching structure in the decomposition, we study the roles of factor-critical connected components of $G[X]$ to derive a König subgraph of G. For the connected components in $G[X]$, we use the "matching switching" strategy to analyze the number of edges from the connected components contained in the König subgraph, which is another key point to get the improved approximation algorithm for the problem.

(3) In this paper, we exploit a connection between structures of the König subgraphs and the properties of the Maximum Bisection above tight lower bound problem (given a graph $G = (V, E)$ and a parameter k, decide whether V can be divided into two parts V_1, V_2 such that $||V_1| - |V_2|| \leq 1$, and the number of edges with one endpoint in V_1 and the other endpoint in V_2 is at least $\lceil |E|/2 \rceil + k$). The kernelization results of the Maximum Bisection above tight lower bound problem are applied to analyze the size of the König subgraphs.

(4) For the parameterized algorithm of the Edge Induced König Subgraph problem, since we can get that every graph has an edge induced König subgraph of at leat $2m/3$ edges, the parameter k in the given instance of the Edge Induced König Subgraph problem is large. By using $2m/3$ as a lower bound, we propose a variant of the Edge Induced König Subgraph problem, called Edge Induced König Subgraph above lower bound problem, and give a kernel of at most $30k$ edges for the problem.

2 Preliminaries

Given a graph $G = (V, E)$, for two vertices u, v in G, the edge between u and v if exists is denoted by uv. We say that edge uv is incident to u and v. For a vertex v in G, the degree of v denoted by $d(v)$ is the number of edges incident to v. For a subset $X \subseteq V$, $G[X]$ denotes the subgraph induced by the vertices in X. For a vertex v in X, $d_X(v)$ denotes the degree of v in the induced subgraph $G[X]$. For a matching M in G, let $V(M)$ be the set of vertices contained in M. The vertices in $V(M)$ are the vertices matched by M, and it is also called that the vertices in $V(M)$ are *saturated* by M. The vertices in $V - V(M)$ are called *unmatched vertices*, and the edges in M are called *matched edges*. A matching M in G is a *perfect matching* if all the vertices in V are matched vertices. For a graph G with n vertices, if every (vertex) induced subgraph with $n - 1$ vertices has a perfect matching, then G is called a *factor-critical* graph. For a matching M in graph $G = (V, E)$, if $V(M)$ contains $|V| - 1$ vertices, then M is called a *near-perfect matching* of G. A chord is an edge incident to two nonadjacent vertices in a cycle. A chordless cycle with at least four vertices is called a *hole*. For a subgraph C in G, let $V(C)$ and $E(C)$ denote the sets of vertices and edges contained in C, respectively. For two subsets $A, B \subseteq V$, $E(A, B)$ is the set of edges, each of which has one endpoint in A and the other endpoint in B. For a vertex u and a subset $B \subseteq V$, for simplicity, let $E(u, B) = E(\{u\}, B)$. For a partition (V_1, V_2) of V, (V_1, V_2) is called a *cut* in G, and an edge with one endpoint in V_1 and the other endpoint in V_2 is called a *cut edge* of (V_1, V_2). The size of cut (V_1, V_2) is the number of cut edges in $E(V_1, V_2)$. A cut (V_1, V_2) is called a *bisection* of G if $||V_1| - |V_2|| \leq 1$. A bisection with maximum number of cut edges is called a *maximum bisection.* A triangle is called a C_3.

▶ **Lemma 1** ([12, 27]). *For a given graph G, the Gallai-Edmonds decomposition (X, Y, Z) of G has the following properties:*
1. *the components of the subgraph induced by X are factor-critical,*
2. *the subgraph induced by Z has a perfect matching,*
3. *if M is any maximum matching of G, it contains a near-perfect matching of each component of $G[X]$, a perfect matching of each component of $G[Z]$, and matches all vertices of Y with vertices in distinct components of $G[X]$,*
4. *the size of a maximum matching is $\frac{1}{2}(|V| - \delta(G[X]) + |Y|)$, where $\delta(G[X])$ is the number of connected components in $G[X]$.*

3 New algorithms for Edge Induced König Subgraph

In this section, we give new structural properties of König subgraphs, and present an improved approximation algorithm for the Edge Induced König Subgraph problem. For a graph G, whether G is a König graph or not can be decided by the following lemma.

▶ **Lemma 2** ([30, 31, 32]). *A graph $G = (V, E)$ is a König graph if and only if there exists a cut (V_1, V_2) of V such that: (1) V_1 is a vertex cover of G; (2) there exists a matching across (V_1, V_2) saturating each vertex in V_1.*

We now give the relation between graphs with perfect matching and König graphs.

▶ **Lemma 3** ([31, 32]). *Let $G = (V, E)$ be a graph with a perfect matching M, where $|V| = n, |E| = m$. Then a König subgraph G' of G with at least $3m/4 + n/8$ edges can be found in $O(mn)$ time such that $|M'| = |M|$, where M' is a maximum matching in G'.*

Given an instance (G, k) of the Edge Induced König Subgraph problem, let (X, Y, Z) be a Gallai-Edmonds decomposition of G. By Lemma 3, we get the following result.

▶ **Lemma 4.** *Let G_1 be the subgraph induced by vertices in Z, and M be a maximum matching in G. Then, there exists a König subgraph G'_1 in G_1 such that $|M'| = |E(G_1) \cap M|$, and $|E(G'_1)| \geq 3|E(G_1)|/4$, where M' is a maximum matching in G'_1.*

Since each connected component C of $G[X]$ is factor-critical, C contains an odd number of vertices. Based on the degrees of the vertices in X and a maximum matching M, we divide the vertices in X into the following groups.

$X_1 = \{v \in X \mid d_X(v) = 0\}$,
$X_2 = \{v \in X \mid d_X(v) \geq 1, \exists u \in Y, uv \in M\}$,
$X_3 = \{v \in X \mid d_X(v) \geq 1, v \notin V(M)\}$.

Based on X_1, X_2, and X_3, we divide the connected components of $G[X]$ into the following types.

(1) B_1: each connected component of B_1 is an isolated vertex from X_1;
(2) B_2: each connected component of B_2 contains a vertex from X_2;
(3) B_3: each connected component of B_3 contains a vertex from X_3, and has exactly three vertices;
(4) B_5: each connected component of B_5 contains a vertex from X_3, and has at least five vertices.

For each B_i ($i = 1, 2, 3, 5$), let $V(B_i)$ and $E(B_i)$ be the sets of vertices and edges of B_i, respectively. For each connected component C of B_3, let a, b, and c be the three vertices contained in C. By the definition of factor-critical, any two vertices from $\{a, b, c\}$ are adjacent. If $E(C, Y)$ is not empty, then arbitrarily choose any edge from $E(Y, C)$ (without loss of generality, assume that edge ua is chosen). Then, edge ua is called a *special edge*. Remark that any edge in $E(Y, C)$ can be viewed as special edge and only one edge from $E(Y, C)$ can be a special edge. For this case, if edge bc is in maximum matching M, then a is an unmatched vertex in C. We apply the strategy, called "*matching switching*", to deal with the edges in $M \cap E(C)$, i.e., we delete bc from M and add edge ab or ac to M. It is easy to see that the new M is still a maximum matching in G. After doing that, edge bc is not an edge in M, which is called a *candidate deleted edge*. Let SE be the set of special edges obtained by considering all connected components in B_3.

Given a graph G, we first give the relation between bisections and matchings in G.

▶ **Lemma 5.** [16] *Let G be a graph and M be a matching in G. Then G has a bisection of size at least $\lceil m/2 \rceil + \lfloor |M|/2 \rfloor$, which can be found in $O(m + n)$ time, where m, n are the number of edges and vertices in G, respectively.*

For simplicity, we assume that all the numbers in the following are divisible, without any floor and ceiling notations.

We now analyze the relation between subgraph $G[Y \cup V(B_1) \cup V(B_2)]$ and König subgraphs.

▶ **Lemma 6.** *Let G_2 be the graph constructed by the subgraph $G[Y \cup V(B_1) \cup V(B_2)]$ and edges in $E(Y, Z)$, $E(Y, V(B_3) \cup V(B_5)) \backslash SE$. Then, there exists a König subgraph G'_2 in G_2 such that $|M'| = |M \cap E(G_2)| = |Y| + |M \cap E(B_2)|$, and $|E(G'_2)| \geq 11|E(G_2)|/15$, where M' is a maximum matching in G'_2.*

Proof. Assume that B_2 is not empty. Let $B_2 = \{b^2_1, \ldots, b^2_{h_2}\}$. For each connected component b^2_i ($1 \leq i \leq h_2$), there must exist two vertices $u \in Y$ and $v \in V(b^2_i)$ such that edge uv is in M. Add u to a set U, which is initialized as an empty set. We need to consider the edges in $E(b^2_i)$, $E(u, Z)$, $E(u, X) \backslash SE$, and $E(u, Y)$. It is noted that for edges in $E(u, Y)$, there may exist another vertex u' in Y such that $E(u, Y) \cap E(u', Y) \neq \emptyset$. Therefore, in the process of analyzing the relation between $G[Y \cup V(B_1) \cup V(B_2)]$ and König subgraphs, we need to guarantee that each edge in $E(G[Y])$ can only be dealt with one time.

Since b^2_i is factor-critical, subgraph $G[V(b^2_i) \backslash \{v\}]$ has a perfect matching, and the number of edges of $G[V(b^2_i) \backslash \{v\}]$ contained in M is $(|V(b^2_i)| - 1)/2$. After dealing with all the connected components in B_2, U contains h_2 vertices. For each vertex u in U, there exists a connected component b^2_i ($1 \leq i \leq h_2$) in B_2 and a vertex v in b^2_i such that uv is in M. Let $Q^0_i = E(u, Z) \cup E(u, Y \backslash U) \cup E(u, X \backslash V(b^2_i)) \backslash SE$, $Q^1_i = E(v, V(b^2_i) \backslash \{v\})$, and $Q^2_i = E(G[V(b^2_i) \backslash \{v\}]) \backslash M$.

Based on the analysis of the edges in b^2_i and by Lemma 5, a bisection (A_1, A_2) of size at least $m'/2 + (E(b^2_i) \cap M)/2$ in graph $G[V(b^2_i) \backslash \{v\}]$ can be found in $O(m' + |V(b^2_i) \backslash \{v\}|)$ time, where m' is the number of edges in $G[V(b^2_i) \backslash \{v\}]$. Since $m' = |E(b^2_i) \cap M| + |Q^2_i|$, we get that the number of cut edges of bisection (A_1, A_2) is at least $|E(b^2_i) \cap M| + |Q^2_i|/2$. It is easy to get that $|E(G[A_1])| + |E(G[A_2])| \leq |Q^2_i|/2$. Based on the sizes of Q^0_i and Q^1_i, we now discuss how to delete edges to turn subgraph $G[V(b^2_i) \cup \{u\}]$ into a König subgraph.

Case 1. $|Q^0_i| \geq 3|Q^1_i|/8$. For this case, we put u into the minimum vertex cover of G. We will delete some edges in Q^1_i and Q^2_i to make $G[V(b^2_i) \cup \{u\}]$ be a König subgraph. We compare $|E(v, A_1)| + |E(G[A_1])|$ with $|E(v, A_2)| + |E(G[A_2])|$. Since $|E(v, A_1)| + |E(G[A_1])| + |E(v, A_2)| + |E(G[A_2])| \leq |Q^1_i| + |Q^2_i|/2$, one value of $|E(v, A_1)| + |E(G[A_1])|$

and $|E(v, A_2)| + |E(G[A_2])|$ is at most $(|Q_i^1| + |Q_i^2|/2)/2$. Without loss of generality, assume that $|E(v, A_2)| + |E(G[A_2])| \leq (|Q_i^1| + |Q_i^2|/2)/2$. We put the vertices in A_1 into the minimum vertex cover of G, and delete the edges in $E(v, A_2) \cup E(G[A_2])$ from subgraph $G[V(b_i^2) \cup \{u\}]$, and let G' be the resulted subgraph. Since $uv \in M$ and $|M \cap E(G[V(b_i^2) \cup \{u\}])| = (V(b_i^2) - 1)/2 + 1$, in the subgraph G', the size of minimum vertex cover is $|A_1| + 1 = (V(b_i^2) - 1)/2 + 1$. Thus, subgraph G' is a König subgraph. We now analyze the proportion of the deleted edges in Q_i^0 and $G[V(b_i^2) \cup \{u\}]$. Because vertex u is contained in the minimum vertex cover, all the edges incident to u are covered, i.e., the edges in Q_i^0 are covered by u. We get that

$$\begin{aligned} & \frac{|E(v, A_2)| + |E(G[A_2])|}{|Q_i^0| + |Q_i^1| + |Q_i^2| + |M \cap E(b_i^2)| + 1} \\ & \leq \frac{(|Q_i^1| + |Q_i^2|/2)/2}{|Q_i^0| + |Q_i^1| + |Q_i^2| + |M \cap E(b_i^2)|} \qquad (1) \end{aligned}$$

Since b_i^2 is factor-critical, we have $|M \cap E(b_i^2)| \geq |Q_i^1|/2$. Therefore, for inequality 1, we get that

$$\begin{aligned} & \frac{(|Q_i^1| + |Q_i^2|/2)/2}{|Q_i^0| + |Q_i^1| + |Q_i^2| + |M \cap E(b_i^2)|} \\ & \leq \frac{(|Q_i^1| + |Q_i^2|/2)/2}{|Q_i^0| + |Q_i^1| + |Q_i^2| + |Q_i^1|/2} \qquad (2) \\ & \leq \frac{(|Q_i^1| + |Q_i^2|/2)/2}{3|Q_i^1|/8 + |Q_i^1| + |Q_i^2| + |Q_i^1|/2} \qquad (3) \\ & \leq \frac{(|Q_i^1| + |Q_i^2|/2)/2}{15|Q_i^1|/8 + |Q_i^2|} \\ & = \frac{4|Q_i^1| + 2|Q_i^2|}{15|Q_i^1| + 8|Q_i^2|} \\ & \leq 4/15. \end{aligned}$$

From inequality 2 to inequality 3, we use the fact that $|Q_i^0| \geq 3|Q_i^1|/8$.

Case 2. $|Q_i^0| < 3|Q_i^1|/8$. For this case, we put v into the minimum vertex cover of G. We will delete all edges in Q_i^0 and some edges in Q_i^2 to make $G[V(b_i^2) \cup \{u\})]$ be a König subgraph. Since $|E(G[A_1])| + |E(G[A_2])| \leq |Q_i^2|/2$, one value of $|E(G[A_1])|$ and $|E(G[A_2])|$ is at most $|Q_i^2|/4$. Without loss of generality, assume that $|E(G[A_1])| \leq |Q_i^2|/4$. We put the vertices in A_2 into the minimum vertex cover of G, delete all the edges in Q_i^0, and delete the edges in $E(G[A_1])$ from subgraph $G[V(b_i^2) \cup \{u\}]$. Let G' be the new subgraph obtained. It is easy to see that the size of minimum vertex cover in G' is $|A_2|+1 = (V(b_i^2)-1)/2+1$. Since $uv \in M$ and $|M \cap E(G[V(b_i^2) \cup \{u\}])| = (V(b_i^2)-1)/2+1$, subgraph G' is a König subgraph. We now analyze the proportion of the deleted edges in Q_i^0 and $G[V(b_i^2) \cup \{u\}]$.

$$\begin{aligned} & \frac{|Q_i^0| + |E(G[A_1])|}{|Q_i^0| + |Q_i^1| + |Q_i^2| + |M \cap E(b_i^2)| + 1} \\ & \leq \frac{|Q_i^0| + |Q_i^2|/4}{|Q_i^0| + |Q_i^1| + |Q_i^2| + |Q_i^1|/2} \qquad (4) \\ & < \frac{3|Q_i^1|/8 + |Q_i^2|/4}{3|Q_i^1|/2 + |Q_i^2|} \qquad (5) \\ & = 1/4. \end{aligned}$$

From inequality 4 to inequality 5, we use the fact that $|Q_i^0| < 3|Q_i^1|/8$.
If $Y \backslash U$ is not an empty set, then for any vertex u' in $Y \backslash U$, an isolated vertex b^1 in B_1 can be found such that the edge formed by u' and b^1 is in maximum matching M. For this case, we put the vertices in $Y \backslash U$ into the minimum vertex cover. It is easy to get that the subgraph $G[V(B_1) \cup (Y \backslash U)]$ is a König subgraph.
For the case when B_2 is an empty set, it is obvious to get that $Y \backslash U$ is not empty, which can be handled as above.
Therefore, after dealing with all connected components of B_2 and B_1, a König subgraph G_2' in G_2 can be found such that $|M'| = |M \cap E(G_2)| = |Y| + |M \cap E(B_2)|$, and $|E(G_2')| \geq 11|E(G_2)|/15$. ◀

We now deal with the connected components of B_3.

▶ **Lemma 7.** *Let G_3 be the graph constructed by the subgraph $G[V(B_3)]$ and edges in SE. Then, a König subgraph G_3' can be obtained in G_3 such that $|M'| = |E(G_3) \cap M|$, and $|E(G_3')| \geq 2|E(G_3)|/3$, where M' is a maximum matching in G_3'. If graph G contains no C_3 as connected component, then $|E(G_3')| \geq 3|E(G_3)|/4$.*

Proof. For the case when B_3 is an empty set, the correctness of the lemma is trivial. Assume that B_3 is not empty. Let $B_3 = \{b_1^3, \dots, b_{h_3}^3\}$. For each connected component b_i^3 $(1 \leq i \leq h_3)$ in B_3, if b_i^3 is a C_3 and a connected component in graph G, then no vertex in b_i^3 is connected to vertices in Y, and there exists a König subgraph in b_i^3 with $2|E(b_i^3)|/3$ number of edges. On the other hand, if b_i^3 is a C_3 in $G[X]$ and not a connected component in graph G, then there exists a special edge e in SE with one endpoint in b_i^3, and a candidate deleted edge is contained in b_i^3. In the process of dealing with the connected components of B_2, all the edges in $E(Y, b_i^3)$ except special edge e are handled, i.e., the edges in $E(Y, b_i^3) \backslash \{e\}$ are either covered by the vertices in Y, or not contained in the König subgraph. For this case, we delete the candidate deleted edge in b_i^3, and put the endpoint of special edge e in b_i^3 into the minimum vertex cover. Let G_i^3 be the graph constructed by the subgraph $G[V(b_i^3)]$ and special edge e. Thus, a König subgraph G' of graph G_i^3 can be obtained. The proportion of the deleted edges in G_i^3 to get the König subgraph G' is $1/4$. Thus, after dealing with all connected components of B_3, a König subgraph G_3' can be found in G_3 such that $|M'| = |E(G_3) \cap M|$, and $|E(G_3')| \geq 2|E(G_3)|/3$. If graph G contains no C_3 as connected component, then $|E(G_3')| \geq 3|E(G_3)|/4$. ◀

For any connected component C of B_3, assume that C is also a connected component in G. Then, C is a triangle in G. It is easy to see that two edges of C can be in edge induced König subgraph of C, and any edge of C can be deleted to get the edge induced König subgraph. Therefore, for any given instance $(G = (V, E), k)$ of the EDGE INDUCED KÖNIG SUBGRAPH problem, we can firstly deal with the C_3s in graph G, without having any impact on the approximation ratio of the problem. We now give a refined analysis for the results in Lemma 7.

▶ **Lemma 8.** *Let G_3 be the graph constructed by the subgraph $G[V(B_3)]$ and edges in SE, where no connected component in B_3 is a connected component in G. Then, a König subgraph G_3' can be obtained in G_3 such that $|M'| = |E(G_3) \cap M|$, and $|E(G_3')| \geq 3|E(G_3)|/4$, where M' is a maximum matching in G_3'.*

We now study the properties of the connected components of B_5. Assume that B_5 is not empty, and let $B_5 = \{b_1^5, \dots, b_{h_5}^5\}$.

▶ **Lemma 9.** *For any connected component b_i^5 $(1 \leq i \leq h_5)$ of B_5, if b_i^5 is a hole, then a König subgraph with at least $4|E(b_i^5)|/5$ edges can be obtained.*

Proof. Assume that b_i^5 is a hole. Since hole b_i^5 is factor-critical, it contains at least five edges. By deleting any edge in b_i^5, a König subgraph of b_i^5 can be obtained, and contains $|E(b_i^5)| - 1$ edges. Thus, if b_i^5 is a hole, then a König subgraph with at least $4|E(b_i^5)|/5$ edges can be obtained. ◀

▶ **Lemma 10.** *For any subgraph C in $G[X]$, if C is factor-critical, then each vertex in C has degree at least two in C.*

Proof. Assume that C is factor-critical. Then, for any vertex v in C, $C\backslash\{v\}$ has a perfect matching. It is easy to see that C contains no isolated vertex. Assume that there exists a vertex v with degree one, and u is the neighbor of v. If vertex u is deleted, then v becomes an isolated vertex in $C\backslash\{u\}$, contradicting with the fact that $C\backslash\{u\}$ has a perfect matching. Thus, if C is factor-critical, then each vertex in C has degree at least two. ◀

For each connected component b_i^5 of B_5, a vertex w with minimum degree in b_i^5 can be found. Assume that $M' \subseteq M$ is a matching in b_i^5. If w is a matched vertex, then we apply "matching switching" strategy to deal with the edges in M'. In other words, we find a perfect matching M'' in $G[V(b_i^5)\backslash\{w\}]$, and let $M = (M - M') \cup M''$. Let $W_i^1 = E(w, V(b_i^5)\backslash\{w\})$, and $W_i^2 = E(G[V(b_i^5)\backslash\{w\}]\backslash M$.

▶ **Lemma 11.** *For each connected component b_i^5 $(1 \le i \le h_5)$ of B_5, if b_i^5 is not a hole, then $|W_i^1| \le |W_i^2|$.*

▶ **Lemma 12.** *Let G_4 be the subgraph induced by vertices in B_5. Then, there exists a König subgraph G_4' such that $|M'| = |M \cap E(G_4')|$, and $|E(G_4')| \ge 7|E(G_4)|/10$, where M' is a maximum matching in G_4'.*

Proof. For any connected component b_i^5 $(1 \le i \le h_5)$ of B_5, if b_i^5 is a hole, then by Lemma 9, there exists a König subgraph G' in $G[V(b_i^5)]$ with $|E(G')| \ge 4|E(b_i^5)|/5$. Now assume that b_i^5 is not a hole. By Lemma 5, a bisection (A_3, A_4) of size at least $m''/2 + |E(b_i^5) \cap M|/2$ in subgraph $G[V(b_i^5)\backslash\{w\}]$ can be found in $O(m'' + |V(b_i^5)\backslash\{w\}|)$ time, where m'' is the number of edges in $G[V(b_i^5)\backslash\{w\}]$. Since $m'' = |E(b_i^5) \cap M| + |W_i^2|$, we get that the number of cut edges of bisection (A_3, A_4) is at least $|E(b_i^5) \cap M| + |W_i^2|/2$. It is easy to get that $|E(G[A_3])| + |E(G[A_4])| \le |W_i^2|/2$. We have $|E(w, A_3)| + |E(G[A_3])| + |E(w, A_4)| + |E(G[A_4])| \le |W_i^1| + |W_i^2|/2$, and one value of $|E(w, A_3)| + |E(G[A_3])|$ and $|E(w, A_4)| + |E(G[A_4])|$ is at most $(|W_i^1| + |W_i^2|/2)/2$. Without loss of generality, assume that $|E(w, A_4)| + |E(G[A_4])| \le (|W_i^1| + |W_i^2|/2)/2$. Delete the edges in $E(w, A_4) \cup E(G[A_4])$ from subgraph $G[V(b_i^5) \cup \{w\}]$, and let G' be the resulted subgraph, which is a König subgraph by Lemma 2. We now analyze the proportion of the deleted edges in b_i^5.

$$\begin{aligned}
&\frac{|E(w, A_4)| + |E(G[A_4])|}{|E(b_i^5)|} \\
&\le \frac{(|W_i^1| + |W_i^2|/2)/2}{|W_i^1| + |W_i^2| + |M \cap E(b_i^5)|} && (6)\\
&\le \frac{(|W_i^1| + |W_i^2|/2)/2}{|W_i^1| + |W_i^2| + |W_i^1|/2} && (7)\\
&= \frac{|W_i^1|/2 + |W_i^2|/4}{3|W_i^1|/2 + 3|W_i^2|/4 + |W_i^2|/6 + |W_i^2|/12} && (8)\\
&\le \frac{|W_i^1|/2 + |W_i^2|/4}{3|W_i^1|/2 + 3|W_i^2|/4 + |W_i^1|/6 + |W_i^2|/12} && (9)\\
&= 3/10.
\end{aligned}$$

From inequality 6 to inequality 7, we use the fact that $|M \cap E(b_i^5)| \geq |W_i^1|/2$. Inequality 9 is obtained from inequality 8 by Lemma 11. ◀

By Lemma 4, Lemma 6, Lemma 7, and Lemma 12, we get the following result.

▶ **Theorem 13.** *For a given graph $G = (V, E)$, there exists an edge induced König subgraph G' of G such that G' contains at least $2|E|/3$ edges.*

By Lemma 4, Lemma 6, Lemma 12, and Lemma 8, we get the following result.

▶ **Theorem 14.** *For the* Edge Induced König Subgraph *problem, an approximation algorithm with ratio $10/7$ can be obtained in polynomial time.*

4 Kernelization for Edge Induced König Subgraph above Lower Bound

For the Edge Induced König Subgraph problem, using the results in Theorem 13, it is easy to get a kernel with at most $3k/2$ edges for the problem. In other words, if $2m/3 > k$, then the given instance is a Yes-instance. Otherwise, we have $m \leq 3k/2$. Under this parameterization, k is not a small value. In this paper, we study the following problem.

Edge Induced König Subgraph above lower bound:
Given a graph $G = (V, E)$ and non-negative integer k, find a set of at least $\lceil 2m/3 \rceil + k$ edges that induce a König subgraph, or report that no such set exists, where m is the number of edges in G.

For a given instance (G, k) of the Edge Induced König Subgraph above lower bound problem, we give the following two reduction rules.

Rule 1. For each connected component C of G, if C is a C_3, then remove C from G.

Rule 2. For each connected component C of G, if C is a tree, then remove C, and $k = k - |E(C)|/3$.

▶ **Lemma 15.** *Rule 1 is correct and can be executed in polynomial time.*

▶ **Lemma 16.** *Rule 2 is correct and can be executed in polynomial time.*

▶ **Theorem 17.** *The* Edge Induced König Subgraph above lower bound *problem admits a kernel of $30k$ edges.*

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